6.003
Signal Processing

Week 1, Lecture B:
Sinusoidal Functions,
Sampling, Aliasing

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Before We Begin: Logistics

- P-Set 1 is available from the web in its entirety
  - Lab checkoff due: today, 9pm
  - Everything else due: Tuesday, noon

- Office Hours have been scheduled:
  - Tuesday, Thursday, Sunday, 7-9pm in 34-501

Demo: submitting p-sets, lab checkoffs
Sinusoids

Focus on sinusoids, since sinusoids are prevalent in nature (and human perception), and because they have some nice mathematical properties.

In general, a CT sinusoid has the form:

\[ x(t) = A \cos(\omega t + \phi) \]

- \( A \) is referred to as the **amplitude**
- \( \omega \) is referred to as the **radian frequency**
- \( \phi \) is referred to as the **phase**
Check Yourself: Frequency

Consider \( f(t) = \cos(\omega_1 t) \), shown below:

What is the value of \( \omega_1 \)? What are its units?
What would a graph of \( \cos(1.5\omega_1 t) \) look like?
Check Yourself: Phase

For some positive value $a$, one of the following represents $\cos(\omega_1 t + a)$ and the other represents $\cos(\omega_1 t - a)$.

What is the value of $a$?

In which graph have we added $a$, versus subtracting it?
Check Yourself: Hertz

In general, a CT sinusoid has the form:

\[ x(t) = A \cos(\omega t + \phi) \]

We are often interested in representing sinusoids based on their frequency \( f \), measured in Hz, rather than by \( \omega \).

What value of \( \omega \) corresponds to a frequency \( f = 261 \text{Hz} \)?

What is the general relationship between \( \omega \) and \( f \)?
Check Yourself: Sine

In general, a CT sinusoid has the form:

\[ x(t) = A \cos(\omega t + \phi) \]

Can we represent \( \sin(\omega t) \) in this form? If so, how? If not, why not?
DT Sinusoids

In general, a DT sinusoid has the form:

\[ x[n] = A \cos(\Omega n + \phi) \]

- \( A \) is referred to as the \textit{amplitude}.
- \( \Omega \) is referred to as the \textit{discrete frequency}.
- \( \phi \) is referred to as the \textit{phase}.

Importantly, \( n \) is always an integer!
Check Yourself: Frequency

The following graph represents $x[n] = \cos(\Omega n)$ for some value of $\Omega$.

What is the value of $\Omega$? What are its units?
1 cycle \((2\pi \text{ radians})\) in 10 samples: \(\Omega = 0.2\pi\)
11 cycles ($22\pi$ radians) in 10 samples: $\Omega = 2.2\pi$
21 cycles \((42\pi \text{ radians})\) in 10 samples: \(\Omega = 4.2\pi\)
Aliasing

Because we only have values at integer multiples of $\Omega$, there are *multiple* $\Omega$ values that lead to the exact same set of discrete points!

This graph could have resulted from any of an infinite number of different $\Omega$ values! We refer to this phenomenon as **aliasing**: the same signal can be described by different “names” (or aliases).
Aliasing

In our example, we had:

\[
x_1[n] = \cos(0.2\pi n)
\]

\[
x_2[n] = \cos(2.2\pi n)
\]

\[
x_3[n] = \cos(4.2\pi n)
\]

These all represent the exact same signal! They are all \textit{aliases} for that signal.
Although there are multiple frequencies $\Omega$ that we could use to refer to this signal, it is difficult to “see” anything but $\Omega = 0.2\pi$ by looking at this graph.

We can remove the ambiguity of which frequency is represented by a set of samples by choosing the one in the range $0 \leq \Omega \leq \pi$.

We call that range of frequencies the **base band** of frequencies, and and the value of $\Omega$ that falls in that range is often referred to as the **principle alias**.
Check Yourself

Consider the following signal:

\[ x[n] = \cos \left( \frac{5}{4} \pi \right) \]

Is it possible to represent this as a sinusoid with a discrete frequency in the range \(0 \leq \Omega \leq \pi\)?

If so, what is that value?
If not, why not?
Maximum Frequency

If we limit our attention to frequencies in the base band, then there is a maximum possible discrete frequency $\Omega_{\text{max}} = \pi$.

Note this difference from CT, where there is no maximum frequency.
Relating CT and DT signals

Our goal is to develop signal processing tools that help us understand and manipulate the world.

The increasing power and decreasing cost of computation makes the use of computation increasingly attractive. However, many important signals are naturally described with continuous domain.
→ understand the relation between CT and sampled signals
We convert CT signals to DT signals by *sampling*.

\[ x[n] = x(nT) \]

\( T \) (seconds / sample) = sampling interval

\( f_s \) (samples / second) = sampling rate

We want to understand how sampling affects the information in a signal.
Sampling

We would like to sample in a way that preserves information. However, information is clearly lost in the sampling process.

Clearly, we ave no information about $x(t)$ between the samples. Worse, information can be distorted through this process!
Example: Consider sampling a popular song at the following sampling rates:

- $f_s = 44100$ Hz
- $f_s = 22050$ Hz
- $f_s = 11025$ Hz
- $f_s = 5512$ Hz
- $f_s = 2756$ Hz
- $f_s = 1378$ Hz
- $f_s = 689$ Hz
- $f_s = 344$ Hz
Check Yourself!

What causes these distortions?
Relating CT and DT Frequencies

If a CT signal $x(t) = \cos \omega t$ is sampled at times $t = nT$, the resulting DT signal is $x[n] = \cos \Omega n$ where

$$\Omega = \omega T$$

If we restrict DT frequencies to the range $0 \leq \Omega \leq \pi$, then the corresponding CT frequencies are in the range $0 \leq \omega \leq \omega_N$ where

$$\omega_N = \frac{\pi}{T}$$

and

$$f_N = \frac{\omega_N}{2\pi} = \frac{1}{2T} = \frac{1}{2}f_s$$

where $f_N$ is the **Nyquist** frequency, which is half of the sampling rate $f_s$. 
Effects of Sampling

If there are frequencies in the CT signal that are greater than the Nyquist frequency, they will alias to frequencies in the base band (that really don’t have anything to do with the original frequency!).

This is the main cause of the distortions from earlier.

We can employ anti-aliasing to prevent these distortions: remove frequencies above the Nyquist frequency before sampling.
Today’s Recitation

Today in recitation: more with sampling and aliasing!