Final Examination

Name:
Kerberos Username:

Enter all answers in the boxes provided.
Other work on pages with QR codes may be considered when assigning partial credit.

You have three hours.
This quiz is closed book, but you may use three 8.5 × 11 sheets of paper (six sides total).
No calculators, computers, cell phones, music players, or other electronic devices.
1. Spectra  [12 points]

Below are plots of 8 time-domain waveforms. In the box next to each, enter the letter of the corresponding spectrum on the following page.

\[ x_1(t) \]

\[ x_2(t) \]

\[ x_3(t) \]

\[ x_4(t) \]

\[ x_5(t) \]

\[ x_6(t) \]

\[ x_7(t) \]

\[ x_8(t) \]

spectrum:

F

D

H

B

A

G

C

E
2. Frequency Response  [16 points]
Let $X(\omega)$ represent the following Fourier Transform (CTFT) of the continuous-time signal $x(t)$.

$$X(\omega) = \begin{cases} 
e^{j\omega} & \text{if } -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

**Part 1.** Determine a closed-form expression for $x(t)$.

$$x(t) = \frac{\sin(\pi(t+1))}{\pi(t+1)}$$

Plot $x(t)$ on the axes below. Indicate the times and values of all key features of your plot.

![Graph of x(t) with key features indicated]
Part 2. Let $y[n]$ represent the discrete-time signal that results from sampling $x(t)$ once every $T = \frac{1}{2}$ seconds. Determine $Y(\Omega)$, which represents the discrete-time Fourier transform (DTFT) of $y[n]$. Plot the magnitude and angle of $Y(\Omega)$ on the axes below.


$$y[n] = \frac{\sin \frac{\pi (n+2)}{2}}{\frac{\pi (n+2)}{2}}$$
Part 4. Let \( z[n] \) represent the discrete-time signal that results from sampling \( x(t) \) once every \( T = 1 \) second. Determine \( Z(\Omega) \), which represents the discrete-time Fourier transform (DTFT) of \( z[n] \). Plot the magnitude and angle of \( Z(\Omega) \) on the axes below.

![Magnitude and Angle Plots](image)

Part 5. Determine an expression for \( z[n] \).

\[
z[n] = \frac{\sin(\pi(n+1))}{\pi(n+1)} = \delta[n + 1]
\]
3. Two-Dimensional Patterns  [12 points]

Match each of the eight 2D signals (each 32 × 32 pixels) shown in the top eight panels with the magnitude of its 2D DFT (lower panels A–H). Black represents 0. White represents the most positive value in that panel (not necessarily 1).

\[
\begin{align*}
|F_0[k_r, k_c]| &= \boxed{D} & |F_1[k_r, k_c]| &= \boxed{F} & |F_2[k_r, k_c]| &= \boxed{C} & |F_3[k_r, k_c]| &= \boxed{E} \\
|F_4[k_r, k_c]| &= \boxed{A} & |F_5[k_r, k_c]| &= \boxed{G} & |F_6[k_r, k_c]| &= \boxed{B} & |F_7[k_r, k_c]| &= \boxed{H}
\end{align*}
\]
Worksheet (intentionally blank)
4. Spectrogram \[12\text{ points}\]

A 0.75-second-long CT signal is sampled with a sampling rate of 20kHz to produce a DT signal. The image below shows a portion of the spectrogram of this signal, computed with a window size of 1000 samples and a step size of 100 samples, and using a Hann window:

In the box below, enter a closed-form expression for \(x[n]\) that is consistent with the spectrogram above.

\[
x[n] = \begin{cases} 
\cos(2.2 \times 10^{-5} n^2) & \text{if } n < 7000 \\
\cos\left(\frac{2\pi}{10} n\right) + \cos\left(\frac{2\pi}{15} n\right) & \text{if } n > 7000 
\end{cases}
\]
For the first half of the displayed time period (window numbers less than 70), there is a single tone whose frequency increases linearly with time. Over this interval, the signal $x(t)$ must have the form

$$x(t) = \cos(\alpha t^2)$$

where the phase $\alpha t^2$ increases with $t^2$ so that the instantaneous frequency

$$\omega(t) = \frac{d}{dt} (\alpha t^2) = 2\alpha t$$

increases linearly with time. Sampling $x(t)$ produces a DT signal

$$x[n] = \cos(\beta n^2)$$

with instantaneous frequency $2\beta n$ after 70 steps (each with step size $s = 100$), which is at time $n = 70 \times 100 = 7000$. The instantaneous frequency $2\beta n$ can be computed from the value of $k$ shown in the spectrogram:

$$k = 50 = \frac{2\beta n}{2\pi} N$$

where $N = 1000$ is the window size of the Fourier analysis. Solving for $\beta$ we get

$$\beta = \frac{2\pi 50}{2nN} = \frac{100\pi}{14,000,000} \approx 2.2 \times 10^{-5}.$$  

For the second half of the displayed time period, there are two tones. One has $k = 20$ and the other has $k$ just over 100. These correspond to discrete frequencies

$$\Omega_1 = \frac{2\pi 20}{N} = \frac{2\pi}{50}$$

and

$$\Omega_2 = \frac{2\pi 100}{N} = \frac{2\pi}{10}$$

Notice that neither the amplitudes nor the absolute phases of the three tones can be determined from the spectrogram.
Worksheet (intentionally blank)
5. Slowing Down  [16 points]

Let $x[n]$ represent a discrete time signal whose DTFT is given by

$$X(Ω) = \begin{cases} 
1 & \text{if } |Ω| < \frac{π}{5} \\
0 & \text{if } \frac{π}{5} < |Ω| < π
\end{cases}$$

and is periodic in $Ω$ with period $2π$ as shown below.

![DTFT of x[n]](image)

A new signal $y_0[n]$ is derived by stretching $x[n]$ as follows:

$$y_0[n] = \begin{cases} 
x\left[\frac{n}{2}\right] & \text{if } n \text{ is even} \\
0 & \text{otherwise}
\end{cases}$$

as illustrated below.

![Stretched signal y0[n]](image)

**Part 1.** Let $Y_0(Ω)$ represent the DTFT of $y_0[n]$. Sketch the magnitude and angle of $Y_0(Ω)$ on the axes below. Label all important parameters of your plots.

$$Y_0(Ω) = \sum_{n=-∞}^{∞} y_0[n]e^{-jΩn} = \sum_{n \text{ even}} x\left[\frac{n}{2}\right]e^{-jΩn} = X(2Ω)$$

Let $n = 2m$:

$$Y_0(Ω) = \sum_{m=-∞}^{∞} x[m]e^{jΩ2m} = X(2Ω)$$
Part 2. The $y_0[n]$ signal alternates between desired values and zeros. To reduce the effect of these zeros, we can convolve $y_0[n]$ with $h_1[n]$ given by

$$h_1[n] = \delta[n] + \delta[n - 1]$$

to generate a new signal

$$y_1[n] = (y_0 * h_1)[n].$$

Sketch the first 10 samples of $y_1[n]$ on the axes below.

Briefly describe the relation between $y_0[n]$ and $y_1[n]$.

Relation: The zero-values that were inserted into $y_0[n]$ have been converted to copies of the previous non-zero values – i.e., $y_1[n]$ is a zero-order hold version of $y_0[n]$.

Let $H_1(\Omega)$ represent the DTFT of $h_1[n]$. Sketch the magnitude and angle of $H_1(\Omega)$ on the axes below. Label all important parameters of your plots.

The effect of convolving $y_0[n]$ with $h_1[n]$ is to filter $Y_0(\Omega)$ by $H_1(\Omega)$. Briefly describe the effect of this filtering on $Y_0(\Omega)$.

Effect: When $\Omega$ is near $\pi$, the magnitude of $H_1(\Omega)$ is small. Thus $H_1(\Omega)$ reduces the high frequency components that were introduced by inserting zeros in $x[n]$.

$$H_1(\Omega) = \sum_{n=-\infty}^{\infty} h_1[n]e^{-j\Omega n} = 1 + e^{-j\Omega} = e^{-j\Omega/2} \left( e^{j\Omega/2} + e^{-j\Omega/2} \right) = 2e^{-j\Omega/2} \cos \frac{\Omega}{2}$$
Part 3. An even better way to smooth \(y_0[n]\) is to convolve it with \(h_2[n]\) given by
\[
h_2[n] = \frac{1}{2} \delta[n + 1] + \delta[n] + \frac{1}{2} \delta[n - 1]
\]
to generate a new signal
\[
y_2[n] = (y_0 * h_2)[n].
\]
Sketch the first 10 samples of \(y_2[n]\) on the axes below.

Briefly describe the relation between \(y_0[n]\) and \(y_2[n]\).

Relation: The zero-values that were inserted into \(y_0[n]\) have been replaced with the average of the samples on either side of them, as in piecewise linear interpolation.

Let \(H_2(\Omega)\) represent the DTFT of \(h_2[n]\). Sketch the magnitude and angle of \(H_2(\Omega)\) on the axes below. Label all important parameters of your plots.

The effect of convolving \(y_0[n]\) with \(h_2[n]\) is to filter \(Y_0(\Omega)\) by \(H_2(\Omega)\). Briefly describe differences between the filtering provided by \(H_1(\cdot)\) and \(H_2(\cdot)\).

Effect: When \(\Omega\) is near \(\pi\), the magnitude of \(H_2(\Omega)\) is small. Thus \(H_2(\Omega)\) reduces the high frequency components that were introduced by inserting zeros in \(x[n]\).
However, \(H_2(\Omega)\) introduces no phase shift, while \(H_1(\Omega)\) introduced a delay. Also, the amplitudes of frequency components near \(\pi\) are reduced more by \(H_2\) than by \(H_1\).

\[
H_2(\Omega) = \sum_{n=-\infty}^{\infty} h_2[n]e^{-j\Omega n} = 1 + \frac{1}{2}e^{-j\Omega} + \frac{1}{2}e^{j\Omega} = 1 + \cos(\Omega)
\]
6. Mystery Photograph  [12 points]

Below are shown the DFT magnitudes of an image of a single small white object photographed against a black background. The image had dimensions of 480 × 480 pixels and represented an area that was 1.8cm wide and 1.8cm tall.

The brightness of each pixel in the image below is proportional to the magnitude of the DFT at that \((k_r, k_c)\) value.

In the box below, sketch the rough shape and orientation of the object in the spatial domain. Include labels indicating the rough dimensions of the object, in terms of continuous lengths (meters).

Do not worry about correctly sketching the size of the object relative to the size of the image.
The 2D DFT has the form of a sinc function in both the $r$ and $c$ directions. The inverse transform of a 2D sinc is a rectangle. The first zero of the sinc function in the horizontal direction is at $k_c = 70$ pixels and corresponds to the field of view (1.8 cm) divided by the width $W$. Thus the width of the rectangle is

$$W = \frac{1.8 \text{ cm}}{70}.$$ 

Similarly the height of the rectangle is determined by the first zero of the sinc function in the vertical direction, which is $k_r = 160$. Thus the height of the rectangle is

$$H = \frac{1.8 \text{ cm}}{160}.$$
Worksheet (intentionally blank)
7. Convolution  [20 points]

Part 1. Consider three DT signals, $a[n]$, $b[n]$, and $c[n]$, shown below. Each of these signals is zero if $n < 0$ or $n > 15$.

Determine which of the following signals can be generated by convolving $a[n]$ or $b[n]$ or $c[n]$ with $a[n]$ or $b[n]$ or $c[n]$, and enter $a$ or $b$ or $c$ in the appropriate boxes. If no combination of $a[n]$, $b[n]$, and $c[n]$ can produce the signal, enter None in both boxes.

Note: the answers may not be unique. Only one correct answer is needed for full credit.

$$f_0[n] = a \ast b$$

$$f_1[n] = \text{None} \ast \text{None}$$

$$f_2[n] = a \ast a$$

$$f_3[n] = c \ast c$$

$$f_4[n] = b \ast c$$

$$f_5[n] = \text{None} \ast \text{None}$$
**Part 2.** Two $7 \times 7$ images are circularly convolved by computing the 2D iDFT of the product of their 2D DFTs, where both the iDFT and DFTs are of length $N = 7$. The resulting image is one of the eight images shown below.

![Images A to H](image)

For each of the following parts, determine which of A-H results.

![Convolution Results](image)
Worksheet (intentionally blank)
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