

6.3000: Signal Processing

Superposition and Convolution

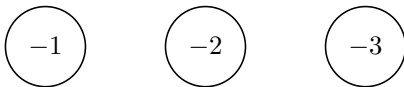
$$y(t) = (h * x)(t) = \int h(\tau)x(t - \tau) d\tau$$

$$y[n] = (h * x)[n] = \sum_m h[m]x[n - m]$$

Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

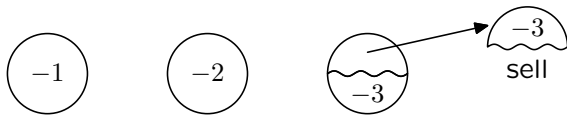
Start with 3 barrels of wine: newest at left, oldest at right.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

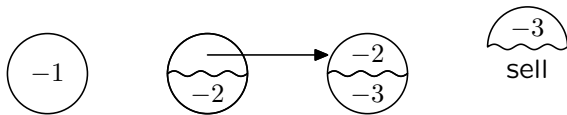
Sell half of the oldest stock.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

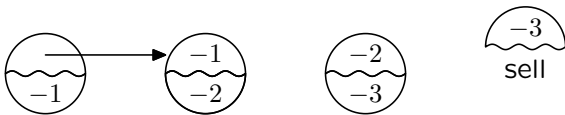
Refill oldest barrel from next-to-oldest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

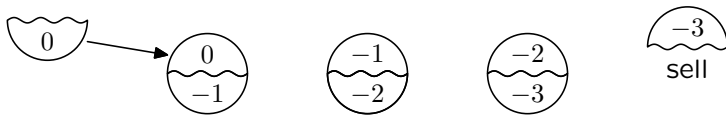
Refill next-to-oldest barrel from youngest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

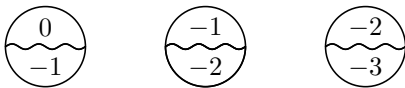
Refill youngest barrel with this year's harvest.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

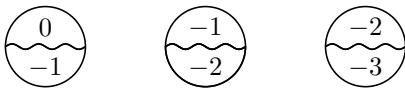
Old and new contents mix; ready for next year.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.



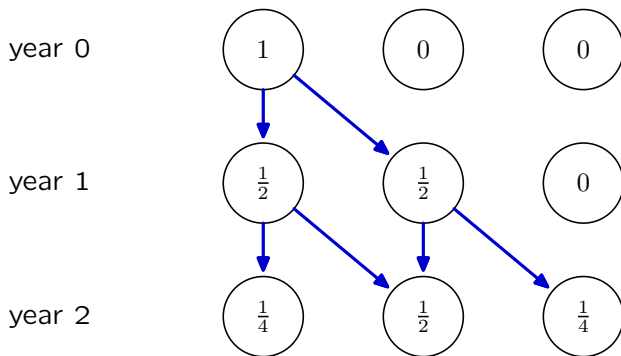
Properties of solera process:

- Mixing produces a more uniform product.
- Mitigates worst-case results of one bad year.
- Blends wines from MANY previous years.

Solera Analysis

We can analyze these effects with a tracer experiment.

Add 1 unit of tracer to new crop; track tracer through the system.

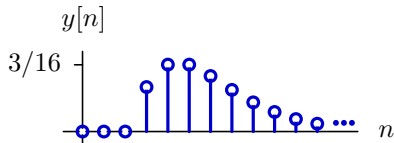
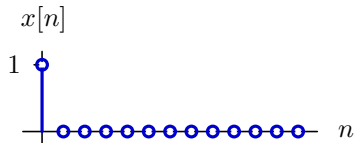


How much tracer will be in each barrel at the end of year 3?

Solera Analysis

Add 1 unit of tracer to new crop; track tracer through the system.

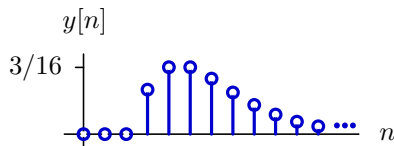
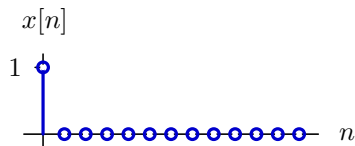
Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64



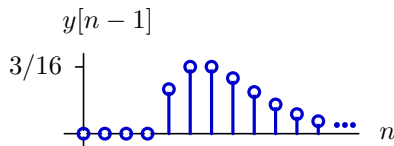
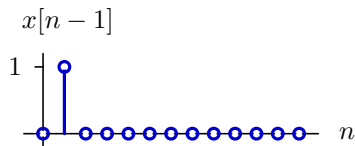
Solera Analysis

How would results change if tracer were added in year 1 (not 0)?

Original response:



Delayed input $\rightarrow$ delayed output:

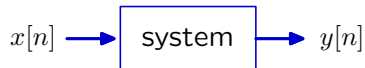


Delaying the input by a year simply delays the outputs by one year.

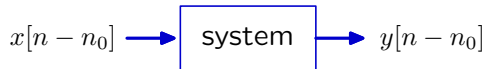
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given



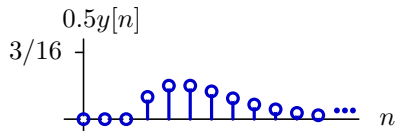
the system is **time invariant** if



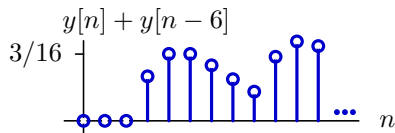
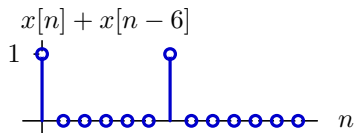
is true for all n_0 .

Solera Analysis

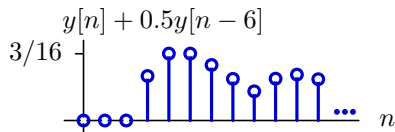
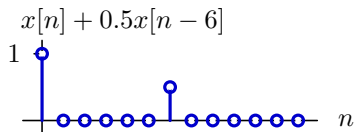
Scaling the input amplitudes:



Adding two inputs:



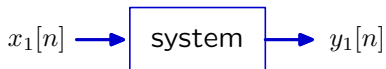
Linearly combining two inputs:



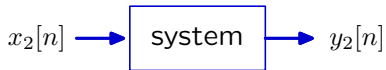
Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

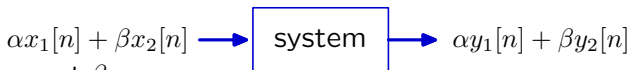
Given



and



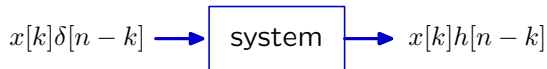
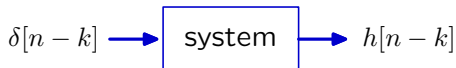
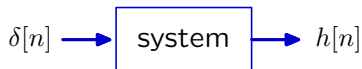
the system is linear if



is true for all α and β .

Convolution

If a system is linear and time invariant, the response to an arbitrary input is the convolution of that input with the unit-sample response.



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k] \longrightarrow \text{system} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?

1. $\frac{21}{32}$
2. $\frac{1}{2}$
3. $\frac{3}{16}$
4. $\frac{9}{16}$
5. none of above

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

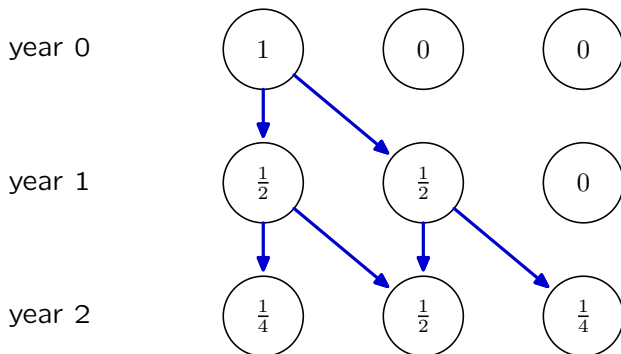
How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2? 2

1. $\frac{21}{32}$ 2. $\frac{1}{2}$ 3. $\frac{3}{16}$ 4. $\frac{9}{16}$ 5. none of above

Divide and Conquer

The content of barrel #3 has no direct dependence on barrel #1.

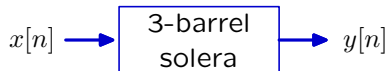
The new content of barrel #3 depends only on itself and barrel #2. All dependence on barrel #1 is through barrel #2.



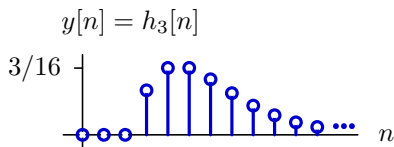
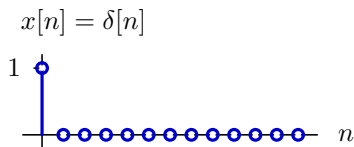
Since barrel #3 depends only on barrel #2, and barrel #2 depends only on barrel #1, the three barrel system is equivalent to the cascade of three one barrel systems!

Divide and Conquer

Making a three-barrel system by cascading three one-barrel systems.

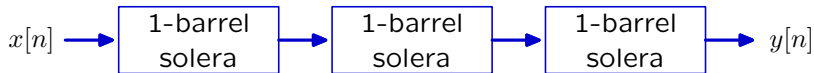


Year n	Tracer in $x[n] = \delta[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n] = h_3[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64



Divide and Conquer

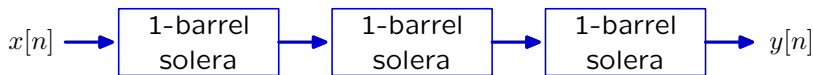
Find the unit-sample response $h_1[n]$ for a 1-barrel solera.



Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.

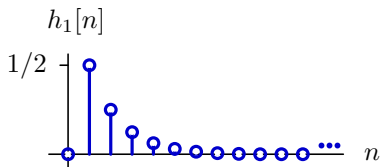
Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.



Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$



No tracer leaves barrel #1 on the year the tracer is added ($n = 0$).

Half leaves the following year.

Half of the remainder leaves on each subsequent year.

The sum of all that leaves (from $n = 0$ to ∞) is 1 (all of it).

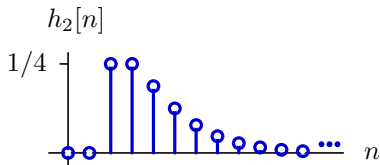
Divide and Conquer

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1]$$

$$\begin{aligned} h_2[n] &= (h_1 * h_1)[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_1[n-m] \\ &= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] \left(\frac{1}{2}\right)^{n-m} u[n-m-1] \end{aligned}$$

The value being summed is zero unless $m - 1 \geq 0$ and $n - m - 1 \geq 0$. Therefore $1 < m < n - 1$ and $n > 2$:

$$h_2[n] = \sum_{m=1}^{n-1} \left(\frac{1}{2}\right)^n = (n-1) \left(\frac{1}{2}\right)^n u[n-2]$$



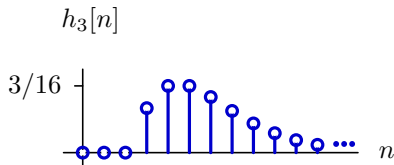
Divide and Conquer

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] \quad \text{and} \quad h_2[n] = (n-1) \left(\frac{1}{2}\right)^n u[n-2]$$

$$\begin{aligned} h_3[n] &= (h_1 * h_2)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_2[n-m] \\ &= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] (n-m-1) \left(\frac{1}{2}\right)^{n-m} u[n-m-2] \end{aligned}$$

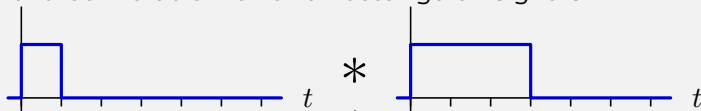
The value being summed is zero unless $m-1 \geq 0$ and $n-m-2 \geq 0$.
Therefore $1 \leq m \leq n-2$ and $n \geq 3$:

$$h_3[n] = \sum_{m=1}^{n-2} (n-m-1) \left(\frac{1}{2}\right)^n = \frac{(n-1)(n-2)}{2} \left(\frac{1}{2}\right)^n u[n-3]$$

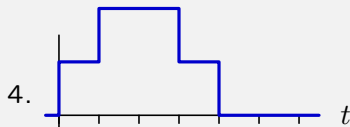
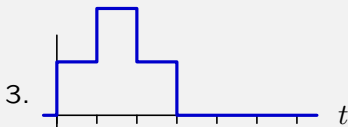
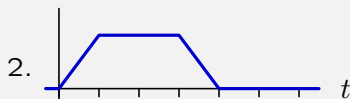
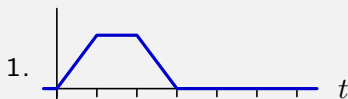


Check Yourself

Consider the convolution of two rectangular signals:



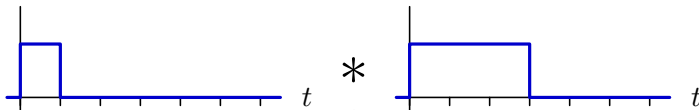
Which plot shows the result of the convolution above?



5. none of the above

Check Yourself

Which plot shows the result of the following convolution?



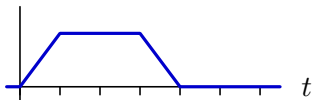
Start by flipping the first function about $t = 0$.

Multiply by the second function. No overlap $\rightarrow 0$.

As the flipped function slides right, the overlap increases linearly for first unit.

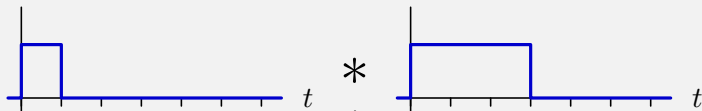
Then there is constant overlap for 2 units.

Then the overlap linearly decreases for the next unit till it is back to zero.

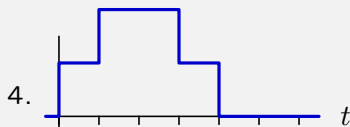
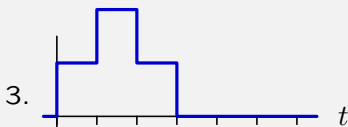
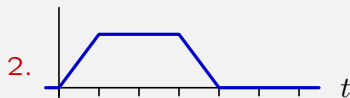
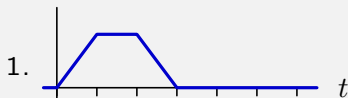


Check Yourself

Consider the convolution of two rectangular signals:



Which plot shows the result of the convolution above? **2**



5. none of the above

Check Yourself

Match expressions on the left with functions on the right where

$$f(t) = e^{-t} u(t)$$

$$g(t) = e^t u(-t)$$

$$(f * f)(t)$$

☐

$$(g * g)(t)$$

☐

$$(f * g)(t)$$

☐

$$(g * f)(t)$$

☐

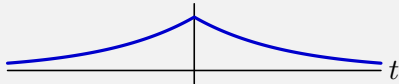
A



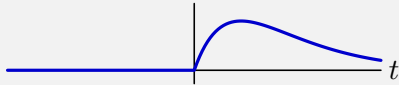
B



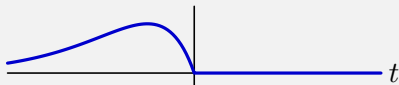
C



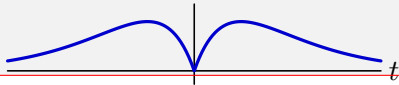
D



E



F



CT Convolution

Let

$$f(t) = e^{-t} u(t)$$

$$g(t) = e^t u(-t)$$

Find $(f * f)(t)$, $(g * g)(t)$, $(f * g)(t)$, and $(g * f)(t)$.

$$(f * f)(t) = \int_{-\infty}^{\infty} f(\tau) f(t - \tau) d\tau = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) e^{-(t-\tau)} u(t - \tau) d\tau$$

The integrand is zero unless $\tau > 0$ and $t - \tau > 0$.

Therefore $0 < \tau < t$ and $t > 0$.

$$(f * f)(t) = e^{-t} \int_0^t d\tau u(t) = t e^{-t} u(t)$$

Answer: D

CT Convolution

Let

$$f(t) = e^{-t} u(t)$$

$$g(t) = e^t u(-t)$$

Find $(f * f)(t)$, $(g * g)(t)$, $(f * g)(t)$, and $(g * f)(t)$.

Since $g(t) = f(-t)$,

$$(g * g)(t) = \int_{-\infty}^{\infty} g(\tau)g(t - \tau) d\tau = \int_{-\infty}^{\infty} f(-\tau)f(-t + \tau) d\tau$$

Now let $\lambda = -\tau$:

$$\begin{aligned}(g * g)(t) &= \int_{\infty}^{-\infty} f(\lambda)f(-t - \lambda) (-d\lambda) \\ &= \int_{-\infty}^{\infty} f(\lambda)f(-t - \lambda) d\lambda \\ &= (f * f)(-t)\end{aligned}$$

Answer: E

CT Convolution

Let

$$f(t) = e^{-t} u(t)$$

$$g(t) = e^t u(-t)$$

Find $(f * f)(t)$, $(g * g)(t)$, $(f * g)(t)$, and $(g * f)(t)$.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) e^{t-\tau} u(\tau - t) d\tau$$

The integrand is zero unless $\tau > 0$ and $\tau > t$. Therefore $\tau > \max(0, t)$.

$$(f * g)(t) = \begin{cases} e^t \int_0^{\infty} e^{-2\tau} d\tau = \frac{1}{2} e^t & \text{if } t < 0 \\ e^t \int_t^{\infty} e^{-2\tau} d\tau = \frac{1}{2} e^{-t} & \text{if } t > 0 \end{cases}$$

$$= \frac{1}{2} e^{-|t|}$$

$$= (g * f)(t) \quad \text{since convolution is commutative}$$

Answer: C

Check Yourself

Match expressions on the left with functions on the right where

$$f(t) = e^{-t} u(t)$$

$$g(t) = e^t u(-t)$$

$$(f * f)(t)$$

D

$$(g * g)(t)$$

E

$$(f * g)(t)$$

C

$$(g * f)(t)$$

C

A



B



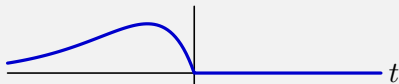
C



D



E



F

