

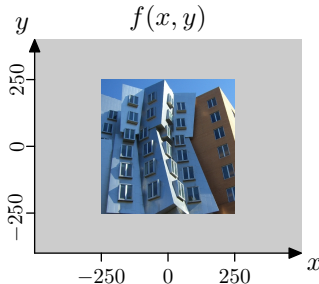
6.3000: Signal Processing

Fourier Series Trig Form

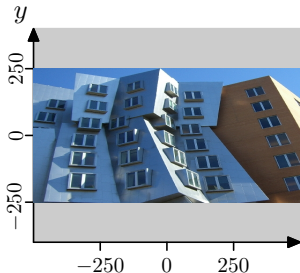
Representing Signals as Fourier Series

- Synthesis: making a signal from components
- Analysis: finding the components

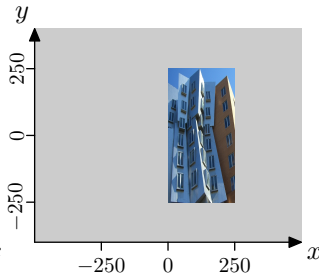
Start With Some Basic Transformations



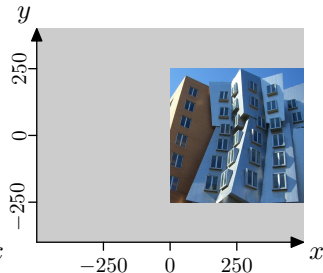
How many images match the expressions beneath them?



$$f_1(x, y) = f(2x, y) ?$$

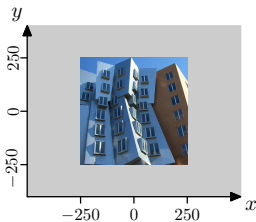


$$f_2(x, y) = f(2x - 250, y) ?$$

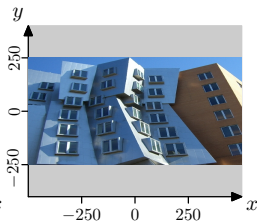


$$f_3(x, y) = f(-x - 250, y) ?$$

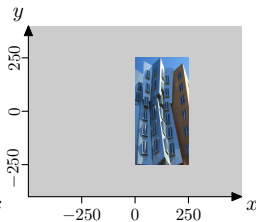
Start With Some Basic Transformations



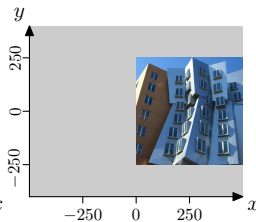
$f(x, y)$



$f_1(x, y) = f(2x, y) ?$



$f_2(x, y) = f(2x - 250, y) ?$



$f_3(x, y) = f(-x - 250, y) ?$

$$x = 0 \rightarrow f_1(0, y) = f(0, y) \quad \checkmark$$

$$x = 250 \rightarrow f_1(250, y) = f(500, y) \quad \times$$

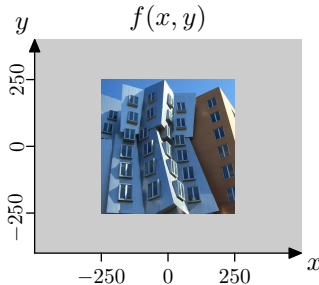
$$x = 0 \rightarrow f_2(0, y) = f(-250, y) \quad \checkmark$$

$$x = 250 \rightarrow f_2(250, y) = f(250, y) \quad \checkmark$$

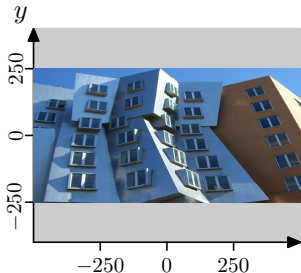
$$x = 0 \rightarrow f_3(0, y) = f(-250, y) \quad \times$$

$$x = 250 \rightarrow f_3(250, y) = f(-500, y) \quad \times$$

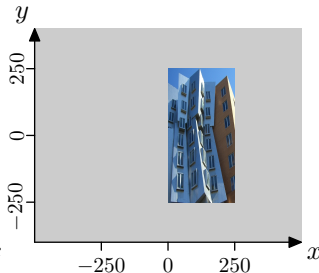
Start With Some Basic Transformations



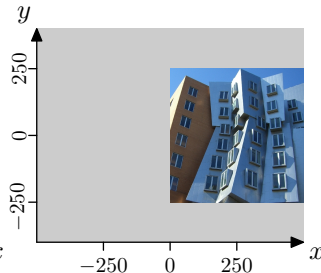
How many images match the expressions beneath them? 1



~~$f_1(x, y) = f(2x, y) ?$~~



$f_2(x, y) = f(2x - 250, y) ?$



~~$f_3(x, y) = f(x - 250, y) ?$~~

Fourier Series

Fourier representations are a major theme of this subject.

The basic ideas were described in lecture:

Synthesis Equation (making a signal from components):

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

Analysis Equations (finding the components):

$$c_0 = \frac{1}{T} \int_T f(t) dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi kt}{T}\right) dt ; \quad k \geq 1$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\left(\frac{2\pi kt}{T}\right) dt ; \quad k \geq 1$$

Warm Up

Find the Fourier series coefficients (c_k and d_k) for

$$f(t) = \cos(t)$$

Warm Up

Find the Fourier series coefficients (c_k and d_k) for

$$f(t) = \cos(t)$$

We can find c_k and d_k directly from the synthesis equation:

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

The function $f(t)$ is periodic in time with period

$$T = 2\pi.$$

The coefficients can be found by matching the expression on the left with that on the right:

$$c_k = \begin{cases} 1 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$d_k = 0$$

There is a single non-zero Fourier coefficient: $c_1 = 1$.

Warm Up

Alternatively, we can calculate c_k and d_k from the analysis equations:

$$f(t) = \cos(t)$$

$$c_0 = \frac{1}{2\pi} \int_T f(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t) dt = \frac{1}{2\pi} \sin(t) \Big|_{-\pi}^{\pi} = 0$$

For $k > 0$:

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi kt}{T}\right) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \cos(kt) dt$$

$$c_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2t) \right) dt = \frac{1}{\pi} \left(\frac{t}{2} + \frac{1}{4} \sin(2t) \right) \Big|_{-\pi}^{\pi} = 1$$

For $k > 1$:

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos((k+1)t) + \cos((k-1)t) \right) dt \\ &= \frac{1}{2\pi} \left[\frac{\sin((k+1)t)}{k+1} + \frac{\sin((k-1)t)}{k-1} \right]_{-\pi}^{\pi} = 0 \end{aligned}$$

$$d_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \sin(kt) dt = 0 \quad (\text{integrand is anti-symmetric})$$

Fourier Series Coefficients

How many of the following functions have **exactly one** non-zero Fourier series coefficient?

- $f_1(t) = \cos^2 t$
- $f_2(t) = \sin t \cos t$
- $f_3(t) = 4 \cos^3 t - 3 \cos t$
- $f_4(t) = \cos(12t) \cos(4t) \cos(2t)$

Fourier Series Coefficients

How many of the following functions have **exactly one** non-zero Fourier series coefficient? 2: $f_2(t)$ and $f_3(t)$

$$f_1(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2} \cos(2t)$$

→ 2 non-zero components: c_0 and c_1 . (could this also be c_0 and c_2 ?)

$$f_2(t) = \sin(t) \cos(t) = \frac{1}{2} \sin(2t) + \frac{1}{2} \sin(0) = \frac{1}{2} \sin(2t)$$

→ 1 non-zero component: d_1 .

$$\begin{aligned} f_3(t) &= 4 \cos^3(t) - 3 \cos(t) = \cos(t) (4 \cos^2(t) - 3) \\ &= \cos(t) (2 \cos(2t) - 1) = \cos(t) + \cos(3t) - \cos(t) = \cos(3t) \end{aligned}$$

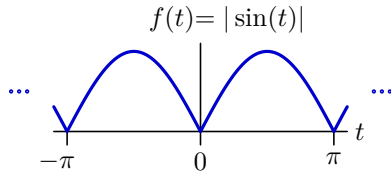
→ 1 non-zero component: c_1 .

$$\begin{aligned} f_4(t) &= \cos(12t) \cos(4t) \cos(2t) = \cos(12t) \left(\frac{1}{2} \cos(6t) + \frac{1}{2} \cos(2t) \right) \\ &= \frac{1}{4} \cos(18t) + \frac{1}{4} \cos(6t) + \frac{1}{4} \cos(14t) + \frac{1}{4} \cos(10t) \end{aligned}$$

→ 4 non-zero components: c_3 , c_5 , c_7 , and c_9 .

Rectified Sine Wave

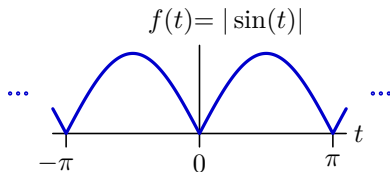
Consider a Fourier series representation of the following function.



- What is the approximate value of c_0 ?
- Which non-DC Fourier coefficient has the largest absolute value? What's the sign of that coefficient?
- Determine an expression for the Fourier coefficients of $f(t)$.
- Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

Rectified Sine Wave

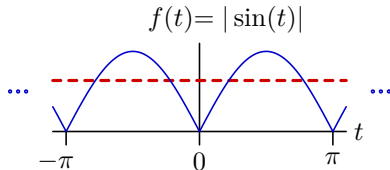
Consider a Fourier series representation of the following function.



Q: What is the approximate value of c_0 ?

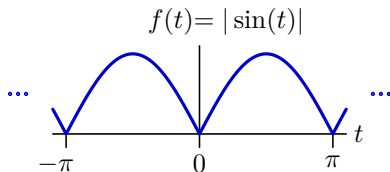
A: c_0 is the average value, which is clearly greater than $\frac{1}{2}$ but less than 1.

$$\text{More exactly, } c_0 = \frac{1}{T} \int_T f(t) dt = \frac{1}{\pi} \int_0^\pi \sin(t) dt = -\frac{\cos(t)}{\pi} \Big|_0^\pi = \frac{2}{\pi} \approx 0.64$$



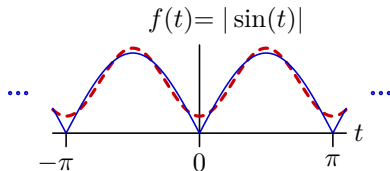
Rectified Sine Wave

Consider a Fourier series representation of the following function.



Q: Which non-DC Fourier coefficient has the largest absolute value?
What's the sign of that coefficient?

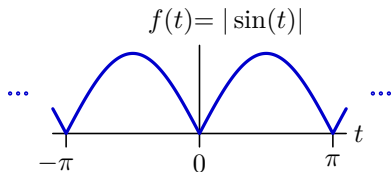
A: biggest deviations from mean at $t = 0$ and $t = \frac{\pi}{2} \rightarrow -\cos(t)$.



$$\begin{aligned} c_1 &= \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi t}{T}\right) dt = \frac{2}{\pi} \int_0^\pi \sin(t) \cos(2t) dt = \frac{1}{\pi} \int_0^\pi (\sin(3t) - \sin(t)) \\ &= \frac{1}{\pi} \left[-\frac{\cos(3t)}{3} + \cos(t) \right]_0^\pi = -\frac{4}{3\pi} \approx -0.42 \end{aligned}$$

Rectified Sine Wave

Consider a Fourier series representation of the following function.



Determine an expression for the Fourier coefficients of $f(t)$.

The function is symmetric about $t = 0$, so $d_k = 0$ for all k .

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \sin(t) dt = -\frac{\cos(t)}{\pi} \Big|_0^{\pi} = \frac{2}{\pi}$$

$$\begin{aligned} c_k &= \frac{2}{\pi} \int_0^{\pi} \sin(t) \cos(2kt) dt \quad ; k \geq 1 \\ &= \frac{1}{\pi} \int_0^{\pi} \left(\sin((2k+1)t) - \sin((2k-1)t) \right) dt \\ &= \frac{1}{\pi} \left[-\frac{\cos((2k+1)t)}{2k+1} + \frac{\cos((2k-1)t)}{2k-1} \right]_0^{\pi} = \frac{-4/\pi}{4k^2 - 1} \end{aligned}$$

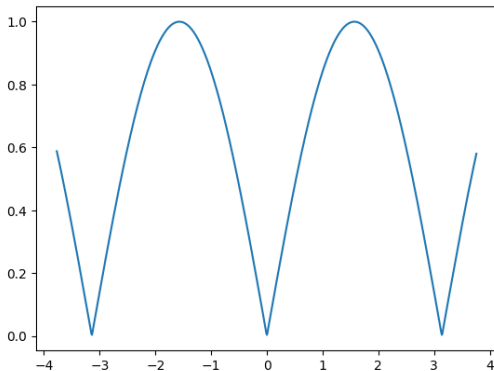
Verify Fourier Series of Rectified Sine Wave Numerically

Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

Verify Fourier Series of Rectified Sine Wave Numerically

Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

```
from math import cos, pi
from matplotlib.pyplot import plot, show
ff = []
tt = []
t = -1.2*pi
while t<1.2*pi:
    ff.append(2/pi+sum([-4/pi/(4*k*k-1)*cos(2*k*t) for k in range(1,100)]))
    tt.append(t)
    t += 0.01
plot(tt,ff)
show()
```



Trig Table

$$\sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b)$$

$$\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b)$$

$$\tan(a+b) = (\tan(a)+\tan(b))/(1-\tan(a) \tan(b))$$

$$\tan(a-b) = (\tan(a)-\tan(b))/(1+\tan(a) \tan(b))$$

$$\sin(A) + \sin(B) = 2 \sin((A+B)/2) \cos((A-B)/2)$$

$$\sin(A) - \sin(B) = 2 \cos((A+B)/2) \sin((A-B)/2)$$

$$\cos(A) + \cos(B) = 2 \cos((A+B)/2) \cos((A-B)/2)$$

$$\cos(A) - \cos(B) = -2 \sin((A+B)/2) \sin((A-B)/2)$$

$$\sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b)$$

$$\sin(a+b) - \sin(a-b) = 2 \cos(a) \sin(b)$$

$$\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2 \sin(a) \sin(b)$$

$$2 \cos(A) \cos(B) = \cos(A-B) + \cos(A+B)$$

$$2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B)$$

$$2 \sin(A) \cos(B) = \sin(A+B) + \sin(A-B)$$

$$2 \cos(A) \sin(B) = \sin(A+B) - \sin(A-B)$$