

6.3000: Signal Processing

Discrete Fourier Transform

analysis

synthesis

DFT:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi k}{N} n}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n}$$

DTFS:

$$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n}$$

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$$

DTFT:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega$$

Analyzing Frequency Content of Arbitrary Signals

Why use a DFT?

- Fourier Series: conceptually simple, but limited to periodic signals.
- Fourier Transforms: arbitrary signals, but continuous domain (Ω)
 - good for theory; not so good for numerical evaluation
- Discrete Fourier Transform: arbitrary DT signals, discrete domain (k)
 - good for computation → broadly used in “Digital Signal Processing”

Today: using the DFT to analyze frequency content of a signal.

Single Sinusoid

Create four signals

$$x_1[n] = \cos(8\pi n/100)$$

$$x_2[n] = \cos(8\pi n/100 - \pi/4)$$

$$x_3[n] = \cos(9\pi n/100)$$

$$x_4[n] = \cos(9\pi n/100 - \pi/2)$$

each with a duration of 1 second when the sample frequency is 44,100 Hz.

Compare the DFTs of the first 100 samples of each of these signals.

Python Code

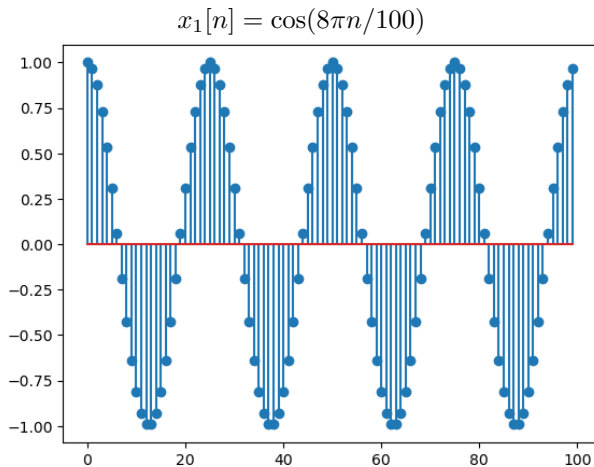
```
from math import cos, pi
from lib6003.audio import wav_write
from matplotlib.pyplot import stem, show

fs = 44100
x1 = [cos(8*pi*n/100) for n in range(fs)]
x2 = [cos(8*pi*n/100-pi/4) for n in range(fs)]
x3 = [cos(9*pi*n/100) for n in range(fs)]
x4 = [cos(9*pi*n/100-pi/2) for n in range(fs)]

wav_write(x1,fs,'x1.wav')
wav_write(x2,fs,'x2.wav')
wav_write(x3,fs,'x3.wav')
wav_write(x4,fs,'x4.wav')

stem(x1[0:100])
show()
```

Single Sinusoid



What is the frequency of this tone if the sample rate is 44,100 Hz?

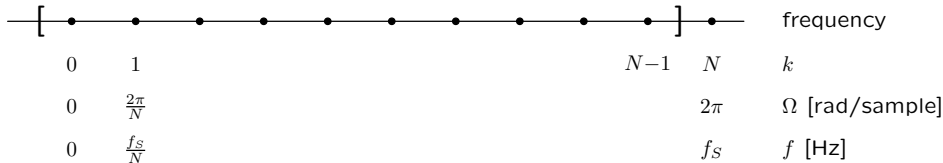
Single Sinusoid

What is the frequency of the tone generated by $x_1[n]$?

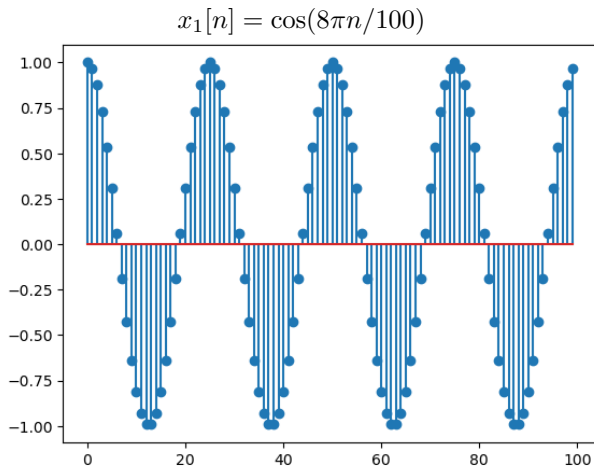
Since $x_1[n] = \cos(8\pi n/100)$, we know that the discrete frequency $\Omega_1 = \frac{8\pi}{100}$. Furthermore, the sample frequency $f = f_s$ corresponds to the maximum discrete frequency $\Omega = 2\pi$, and frequencies f in Hz are proportional to discrete frequencies Ω .

$$\frac{f}{f_s} = \frac{\Omega}{2\pi}$$

$$\text{So } f = \frac{\Omega f_s}{2\pi} = \frac{8\pi/100}{2\pi} \times 44,100 \text{ Hz} = 1764 \text{ Hz}$$



Single Sinusoid



Write a program to calculate the DFT of an input sequence.
Use that program to calculate $X_1[k]$, which is the DFT of the first 100 samples of $x_1[n]$.

Single Sinusoid

Write a program to calculate the DFT of an input sequence.

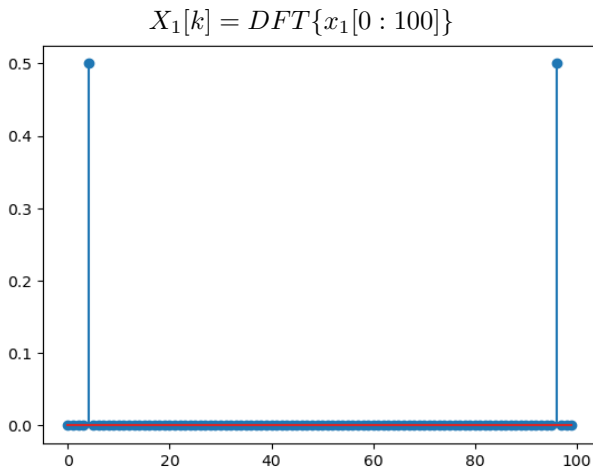
Use that program to calculate $X_1[k]$, which is the DFT of the first 100 samples of $x_1[n]$.

```
def dft(x):
    N = len(x)
    answer = [0 for k in range(N)]
    for k in range(N):
        for n in range(N):
            answer[k] += (1/N)*x[n]*e**(-2j*pi*k*n/N)
    return answer

X1 = dft(x1[0:100])
```

Single Sinusoid

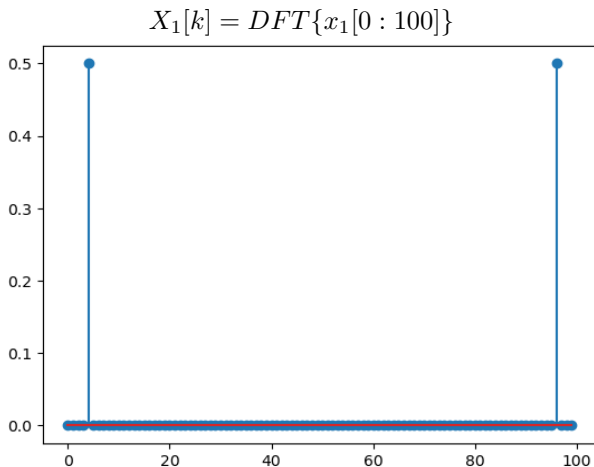
Plot the magnitude of $X_1[\cdot]$.



Which values of k are non-zero?

Single Sinusoid

Plot the magnitude of $X_1[\cdot]$.



Which values of k are non-zero?

$$k = \frac{\Omega N}{2\pi} = \frac{8\pi}{100} \times \frac{100}{2\pi} = 4$$

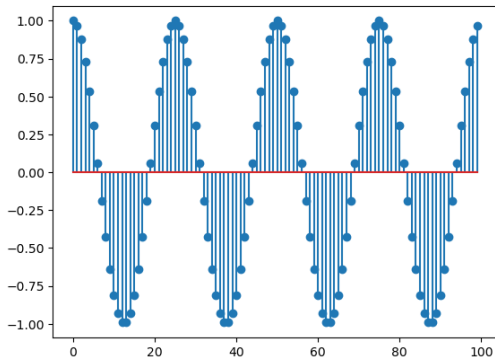
$k = -4$ is also non-zero (Euler's formula).

Also $k = 100 - 4 = 96$ is non-zero since $X[k]$ is periodic in N .

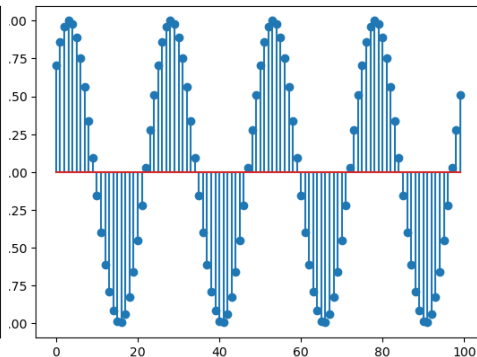
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

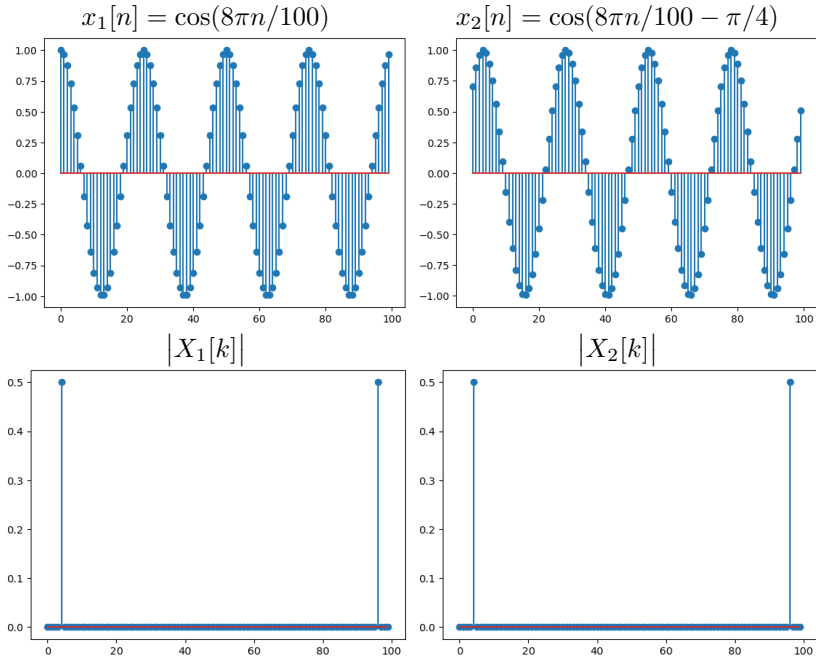
$$x_1[n] = \cos(8\pi n/100)$$



$$x_2[n] = \cos(8\pi n/100 - \pi/4)$$



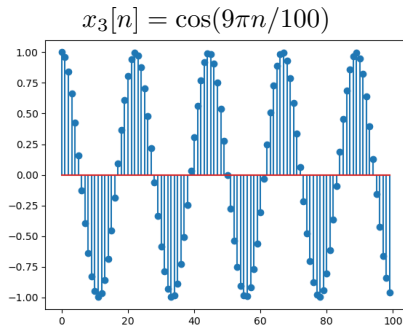
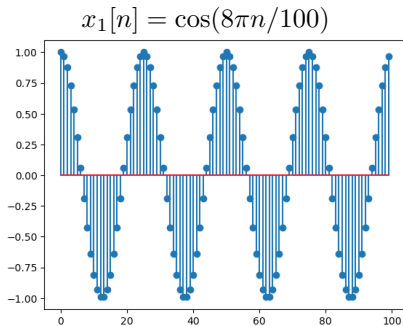
Compare Two Signals



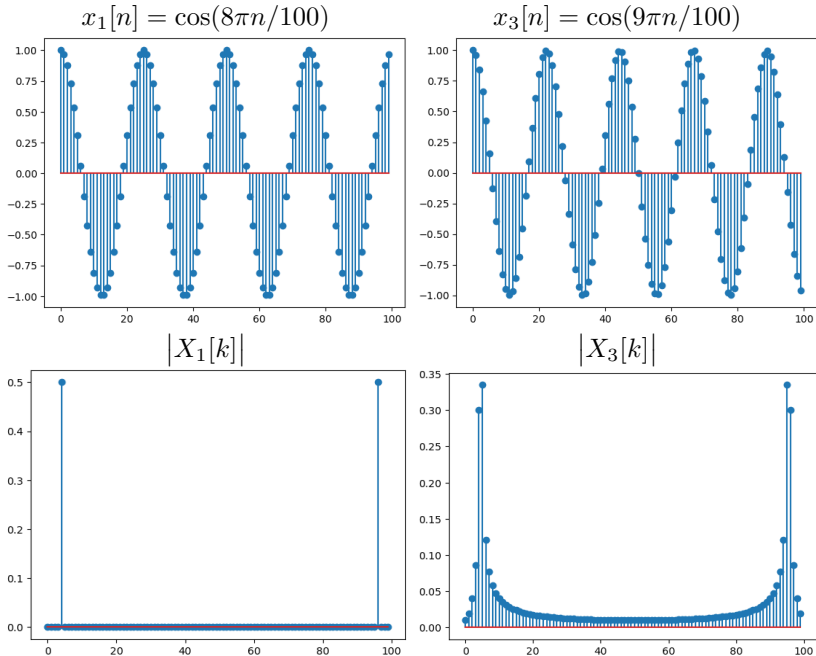
No difference in magnitudes (but the phases are different).

Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

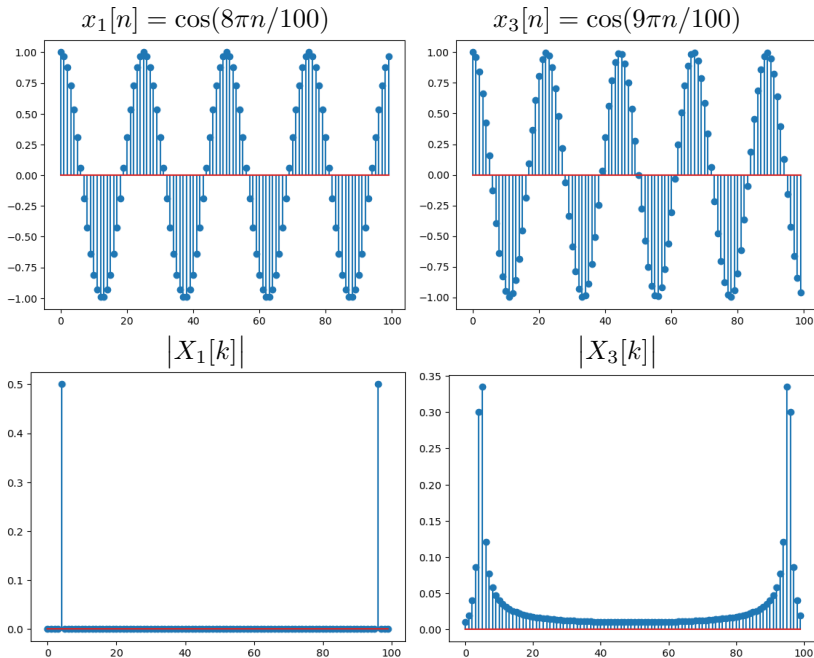


Compare Two Signals



Why are these DFTs so different?

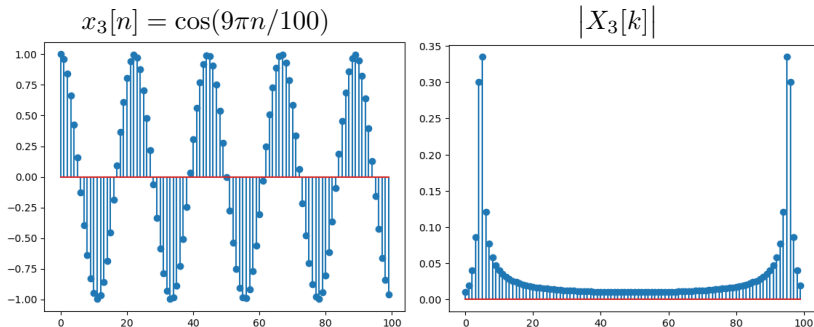
Compare Two Signals



$\Omega_1 \neq \Omega_3$. Even more importantly, $x_3[n]$ is not periodic in $N = 100$!

Single Sinusoid

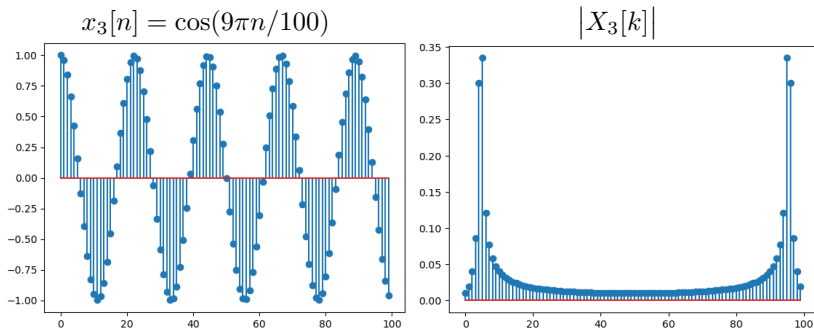
This blurring occurs because the signal is not periodic in the analysis window ($N = 100$).



What value of k corresponds to $\Omega = 9\pi/100$?

Single Sinusoid

This blurring occurs because the signal is not periodic in the analysis window ($N = 100$).



What value of k corresponds to $\Omega = 9\pi/100$?

$$\Omega = 9\pi/100 = 2\pi k/N$$

$$k = 4.5$$

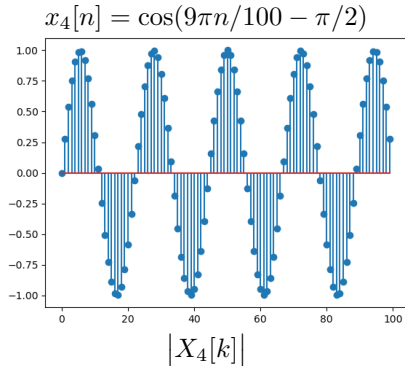
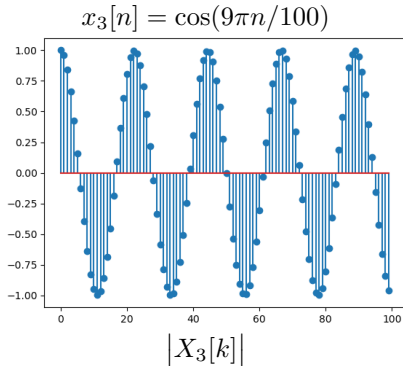
The signal frequency fell between the analysis frequencies.

Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

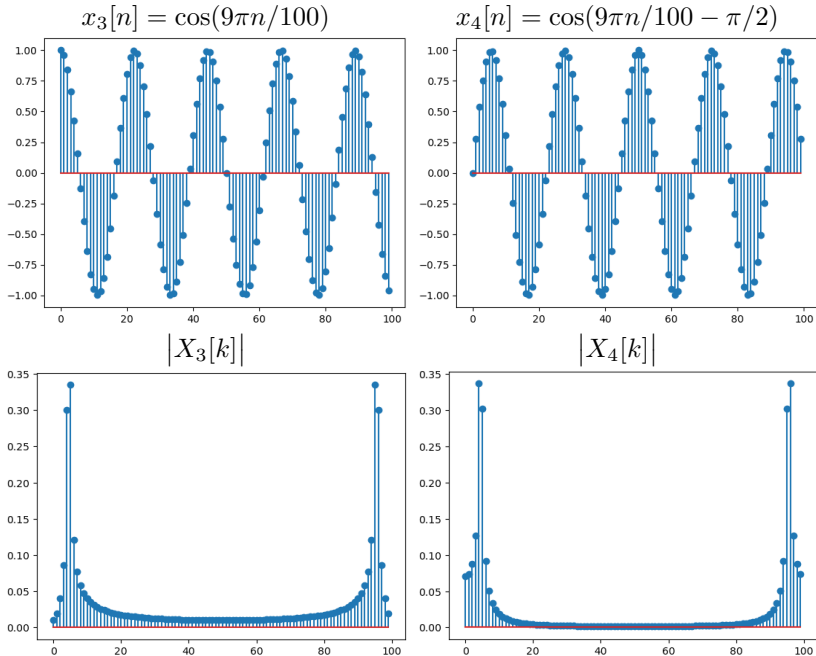
- $x_3[n] = \cos(9\pi n/100)$
- $x_4[n] = \cos(9\pi n/100 - \pi/2)$

Compare Two Signals



$\Omega_3 = \Omega_4$. But DC bigger: 5 positive half cycles versus 4 negative ones.

Compare Two Signals

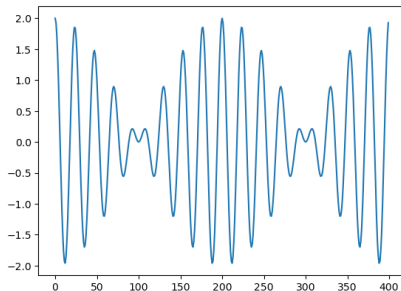


High freq. content of X_4 smaller than X_3 : $|x_4[99] - x_4[0]| < |x_3[99] - x_3[0]|$

Analyzing Signals with Multiple Frequencies

What is the minimum window size N needed to resolve $\Omega = 8\pi/100$ from $9\pi/100$?

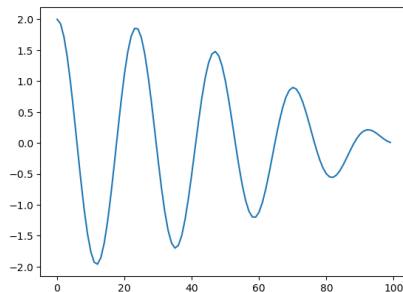
$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



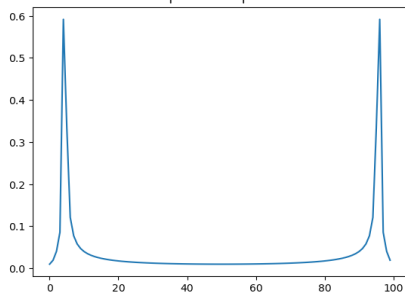
Analyzing Signals with Multiple Frequencies

If the analysis window is small (here $N=100$), the two frequencies $8\pi/100$ and $9\pi/100$ generate a single peak in the DFT at $k = 4$ (along with its partner at $k = 100-4 = 96$).

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



$$|X_5[k]|$$

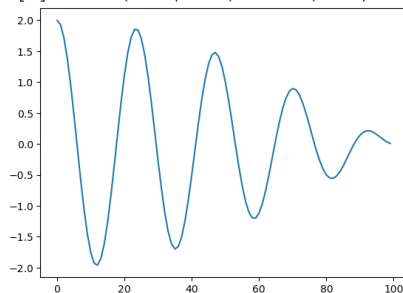


Analyzing Signals with Multiple Frequencies

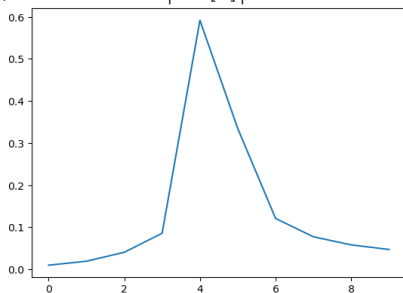
Two frequencies can look like one if analysis window is too small.

$N = 100$ zoomed

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



$$|X_5[k]|$$

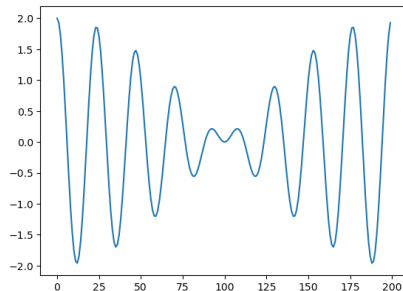


Analyzing Signals with Multiple Frequencies

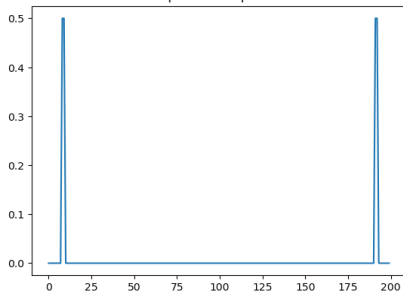
Two frequencies can look like one if analysis window is too small.

$N = 200$

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



$$|X_5[k]|$$

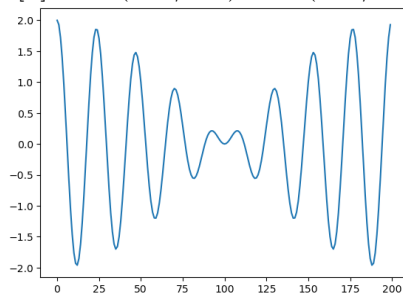


Analyzing Signals with Multiple Frequencies

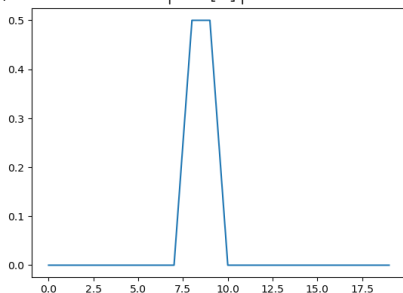
Two frequencies can look like one if analysis window is too small.

$N = 200$ zoomed

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



$$|X_5[k]|$$

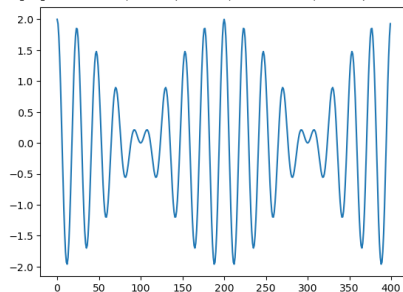


Analyzing Signals with Multiple Frequencies

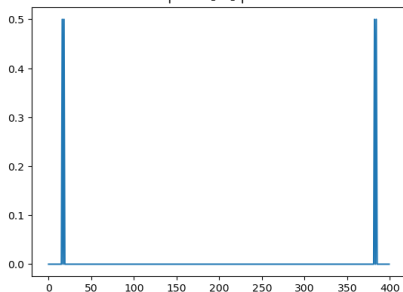
Two frequencies can look like one if analysis window is too small.

$N = 400$

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



$$|X_5[k]|$$

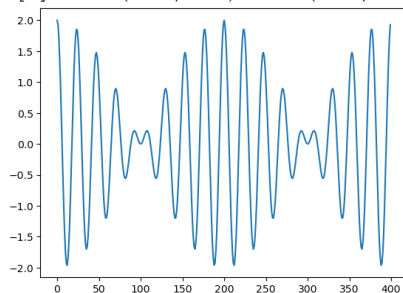


Analyzing Signals with Multiple Frequencies

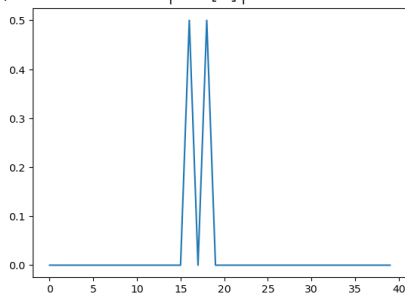
Two frequencies can look like one if analysis window is too small.

$N = 400$ zoomed

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$



$$|X_5[k]|$$

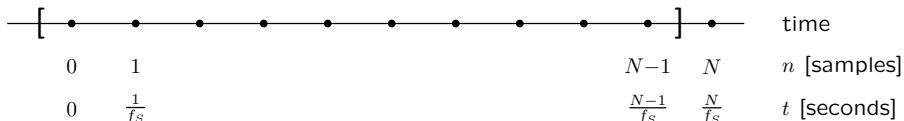


These frequencies are clearly resolved with $N = 400$.

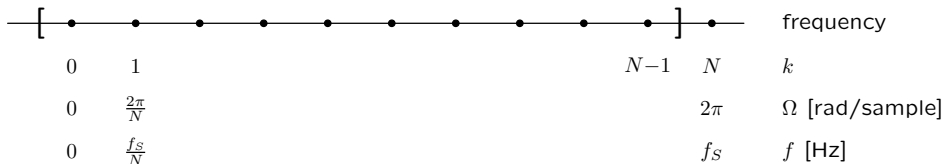
Frequency Scales

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians, which is equivalent to (f_S/N) Hz.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



Analyzing Signals with Multiple Frequencies

Two frequencies are resolved if they are separated by more than $\frac{2\pi}{N}$.

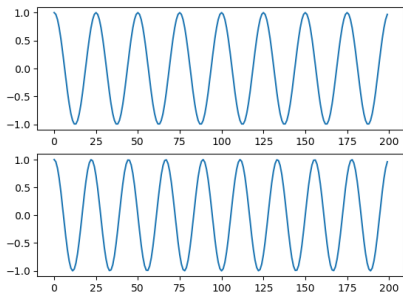
$\Omega_1 = \frac{8\pi}{100}$ and $\Omega_2 = \frac{9\pi}{100}$ will be resolved if

$$\Delta\Omega = \Omega_2 - \Omega_1 = \frac{9\pi}{100} - \frac{8\pi}{100} = \frac{\pi}{100} > \frac{2\pi}{N}$$

That is, if $N > 200$.

We can think of $\frac{2\pi}{N}$ as the frequency resolution of the DFT.

Notice 8 full cycles of Ω_1 and 9 full cycles of Ω_2 fit in $N = 200$.



Summary

Time and frequency resolution are important issues in all Fourier analyses.

Frequency resolution is determined by the number of samples N included in the analysis.

