6.300 Signal Processing

Week 3, Recitation B:
Discrete-Time Fourier Series

- DTFS
- Fourier Series properties

September 24, 2026

Discrete-Time Fourier Series

Synthesis:

$$x[n] = x[n+N] = \sum_{k=<N>} X[k]e^{j\frac{2\pi}{N}kn}$$

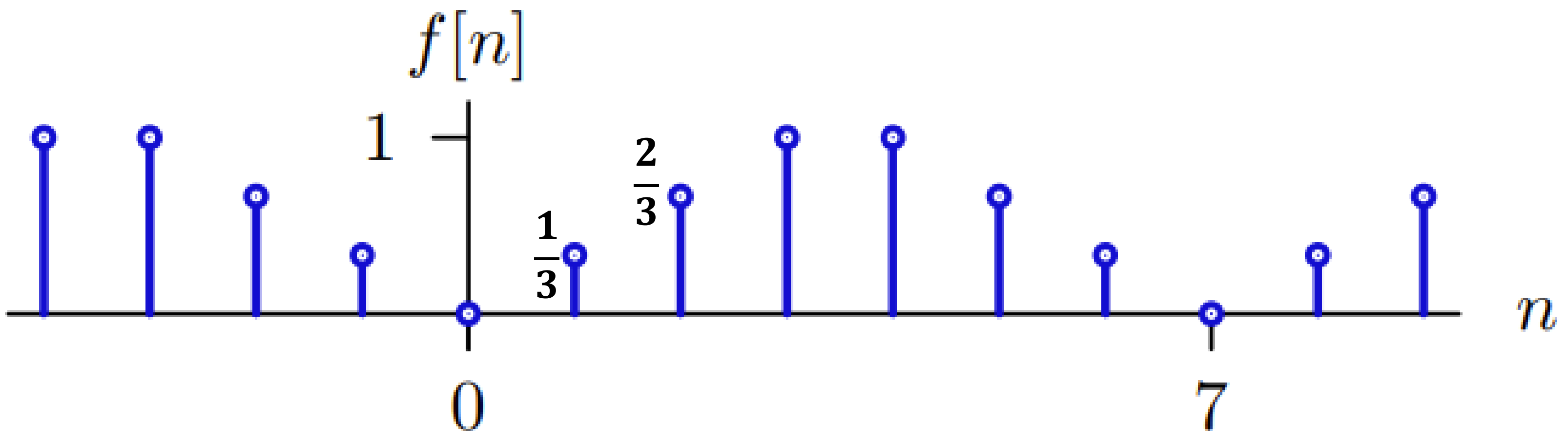
Analysis:

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\Omega_0 kn}$$

discrete in time => periodic in frequency

Find the DT Fourier Series Coefficients

Let $f[n]$ represent a periodic DT signal with period $N = 7$:



Determine the Fourier series coefficients $F[k]$ for $f[n]$.

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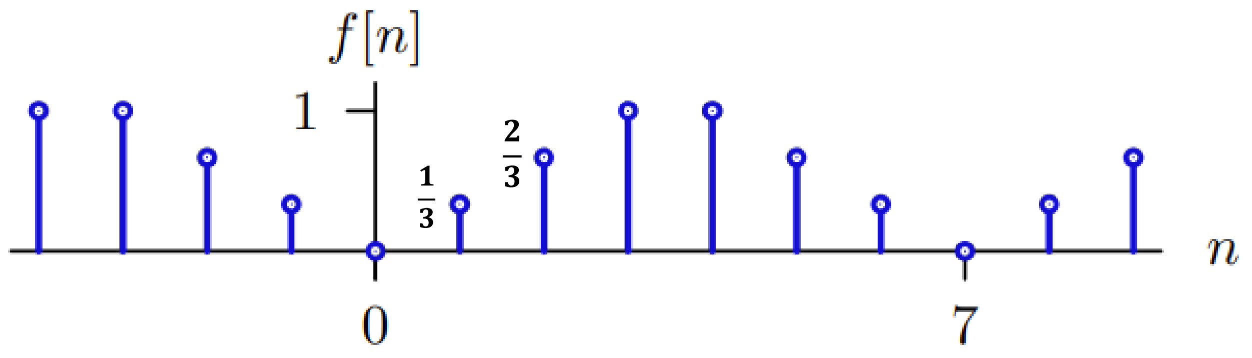


Determine the Fourier series coefficients $F[k]$ for $f[n]$.

$$F[k] = \frac{1}{7}\sum_{n=0}^{6} f[n] e^{-j\frac{2\pi}{7}kn}$$

$$= \frac{1}{7}\left(\frac{1}{3}e^{-j\frac{2\pi}{7}k} + \frac{2}{3}e^{-j\frac{2\pi}{7}2k} + e^{-j\frac{2\pi}{7}3k} + e^{-j\frac{2\pi}{7}4k} + \frac{2}{3}e^{-j\frac{2\pi}{7}5k} + \frac{1}{3}e^{-j\frac{2\pi}{7}6k}\right)$$

This is a completely well-formed answer – but we can simplify.

Find the DT Fourier Series Coefficients

Simplifying ...

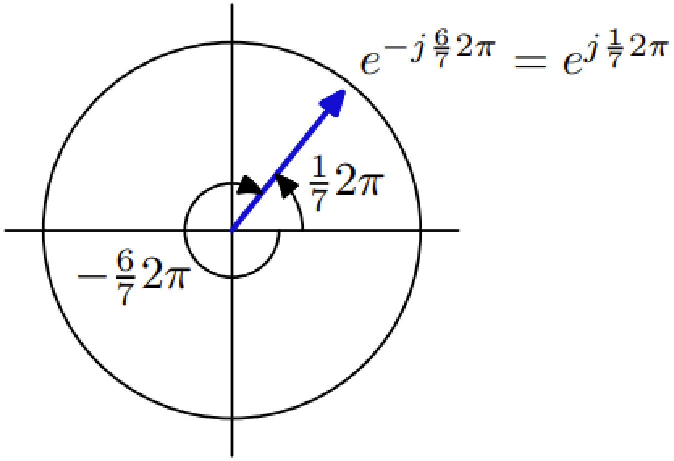
$$F[k] = \frac{1}{7}\left(\frac{1}{3}e^{-j\frac{2\pi}{7}k} + \frac{2}{3}e^{-j\frac{2\pi}{7}2k} + e^{-j\frac{2\pi}{7}3k} + e^{-j\frac{2\pi}{7}4k} + \frac{2}{3}e^{-j\frac{2\pi}{7}5k} + \frac{1}{3}e^{-j\frac{2\pi}{7}6k}\right)$$

The last exponential term can be rewritten with a positive exponent:

$$e^{-j\frac{2\pi}{7}6k} = e^{j\frac{2\pi}{7}7k}e^{-j\frac{2\pi}{7}6k} = e^{j\frac{2\pi}{7}k}$$

where we have used the fact that $e^{j\frac{2\pi}{7}7k} = 1$.

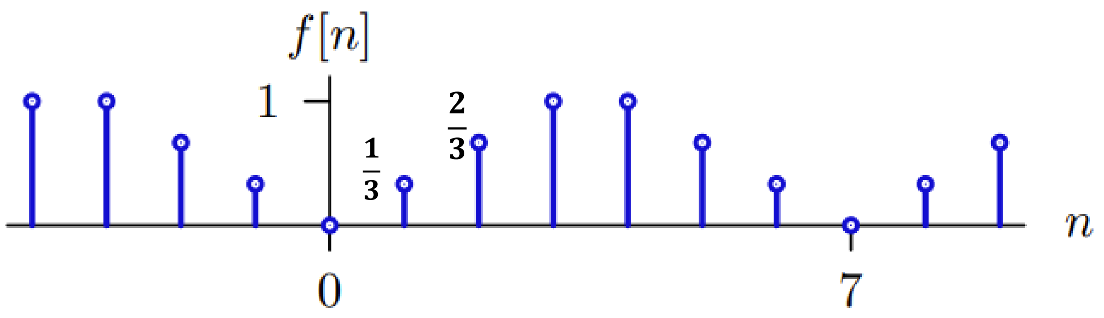
This identity is also apparent in the complex plane.



Given that $e^{-j\frac{6}{7}2\pi} = e^{j\frac{1}{7}2\pi}$ it follows that $\left(e^{-j\frac{6}{7}2\pi}\right)^k = \left(e^{j\frac{1}{7}2\pi}\right)^k$

Find the DT Fourier Series Coefficients

We could get the same answer by summing a different set of time indices.



Sum $n = -3$ to 3 instead of 0 to 6:

$$F[k] = \frac{1}{7}\sum_{n=-3}^{3} f[n]e^{-j\frac{2\pi}{7}kn}$$

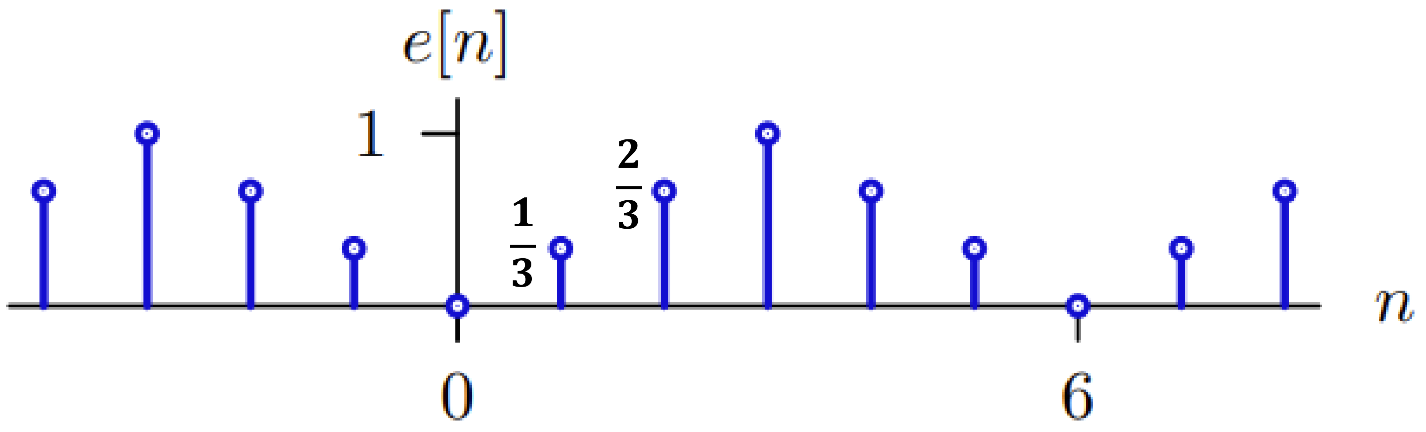
$$= \frac{1}{7}\left(e^{j\frac{2\pi}{7}3k} + \frac{2}{3}e^{j\frac{2\pi}{7}2k} + \frac{1}{3}e^{j\frac{2\pi}{7}1k} + \frac{1}{3}e^{-j\frac{2\pi}{7}k} + \frac{2}{3}e^{-j\frac{2\pi}{7}2k} + e^{-j\frac{2\pi}{7}3k}\right)$$

$$= \frac{2}{21}\cos\left(\frac{2\pi k}{7}\right) + \frac{4}{21}\cos\left(\frac{4\pi k}{7}\right) + \frac{6}{21}\cos\left(\frac{6\pi k}{7}\right)$$

Whichever way we do the math, the answer reduces to the sum of three cosine terms.

Find the DT Fourier Series Coefficients

How would the answer change if the period were $N = 6$?



Determine the Fourier series coefficients $E[k]$ for $e[n]$.

Find the DT Fourier Series Coefficients

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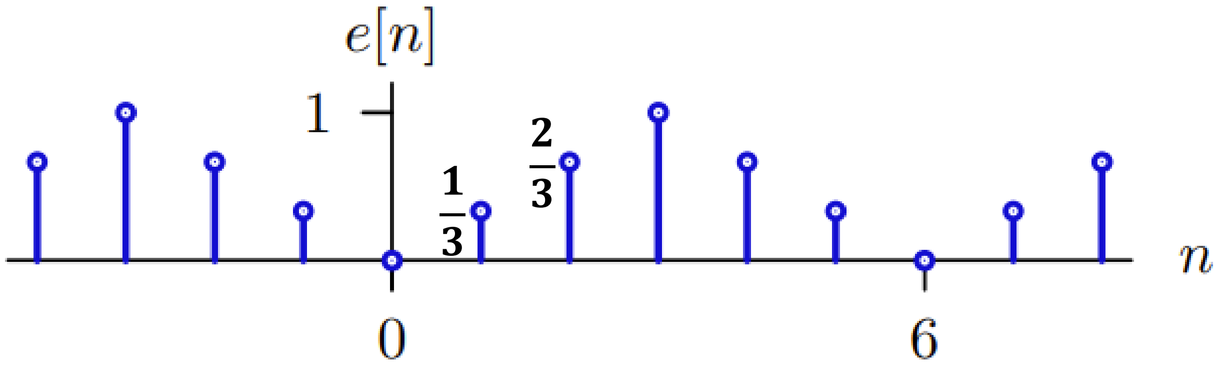


Determine the Fourier series coefficients $E[k]$ for $e[n]$.

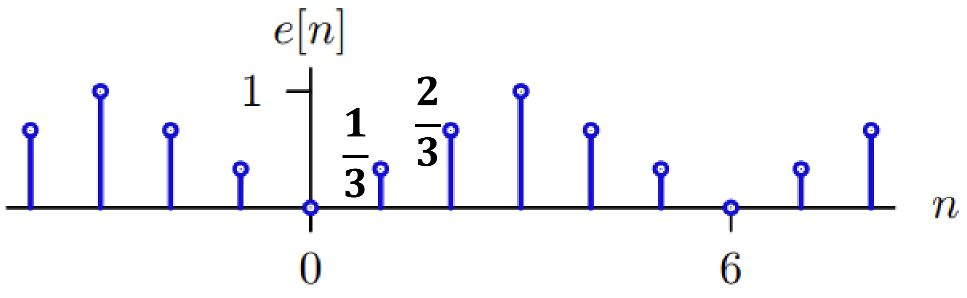
$$E[k] = \frac{1}{6}\sum_{n=0}^{5} e[n]e^{-j\frac{2\pi}{6}kn}$$

$$= \frac{1}{6}\left(\frac{1}{3}e^{-j\frac{2\pi}{6}k} + \frac{2}{3}e^{-j\frac{2\pi}{6}2k} + \frac{3}{3}e^{-j\frac{2\pi}{6}3k} + \frac{2}{3}e^{-j\frac{2\pi}{6}4k} + \frac{1}{3}e^{-j\frac{2\pi}{6}5k}\right)$$

Can we simplify the answer by summing over indices centered on 0?

Find the DT Fourier Series Coefficients

How would the answer change if the period were $N = 6$?



Can we simplify the answer by summing over indices centered on 0?

Yes. But we must be careful at the edges.

Include $n = -3$ or $n = 3$ but not both.

$$E[k] = \frac{1}{6}\sum_{n=-3}^{2} e[n]e^{-j\frac{2\pi}{6}kn}$$

$$= \frac{1}{6}\left(e^{j\frac{2\pi}{6}3k} + \frac{2}{3}e^{j\frac{2\pi}{6}2k} + \frac{1}{3}e^{j\frac{2\pi}{6}k} + \frac{1}{3}e^{-j\frac{2\pi}{6}k} + \frac{2}{3}e^{-j\frac{2\pi}{6}2k}\right)$$

Notice that the $n = -3$ and $n = 3$ terms are equal.

$$e^{j\frac{2\pi}{6}3k} = e^{-j\frac{2\pi}{6}3k} = (e^{\pm j\pi})^k = (-1)^k$$

Find the DT Fourier Series Coefficients

- What are the Fourier Series coefficients of the following signal?

$$x[n] = 1 + \cos(\frac{2\pi}{5} n)$$

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First, fundamental frequency $\Omega_0 = \frac{2\pi}{5}$, signal $x[n]$ is periodic with $N = \frac{2\pi}{2\pi/5} = 5$

$$x[n] = 1 + \cos(\frac{2\pi}{5}n) = 1 + \frac{1}{2}(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n})$$

$$\boxed{x[n] = \sum_{k=<N>} X[k]e^{j\Omega_0 kn}}$$

We have three non-zero $X[k]$'s: $k = 0, k = \pm 1$

$X[0] = 1$ $\quad$ $X[1] = \frac{1}{2}$ $\quad$ $X[-1] = \frac{1}{2}$

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First, fundamental frequency $\Omega_0 = \frac{\pi}{4}$, signal $x[n]$ is periodic with $N = \frac{2\pi}{\pi/4} = 8$

$$x[n] = 1 + \sin(\frac{\pi}{4}n) \ = 1 - \frac{j}{2}(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n})$$

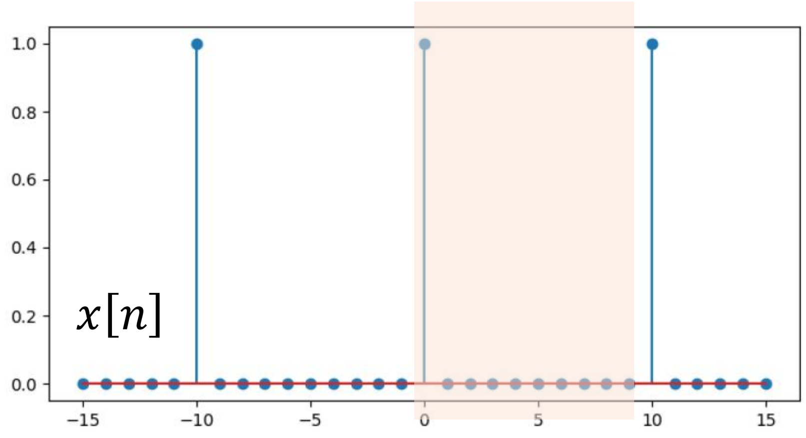
We have three non-zero $X[k]$'s: $k = 0, k = \pm 1$

$X[0] = 1$ $\qquad$ $X[1] = -\frac{j}{2}$ $\qquad$ $X[-1] = \frac{j}{2}$

Find the DT Fourier Series Coefficients

- What are the Fourier Series coefficients of the following signal?

$$x[n] = \begin{cases} 1 & \text{if } n \bmod 10 \equiv 0 \\ 0 & \text{otherwise} \end{cases}$$



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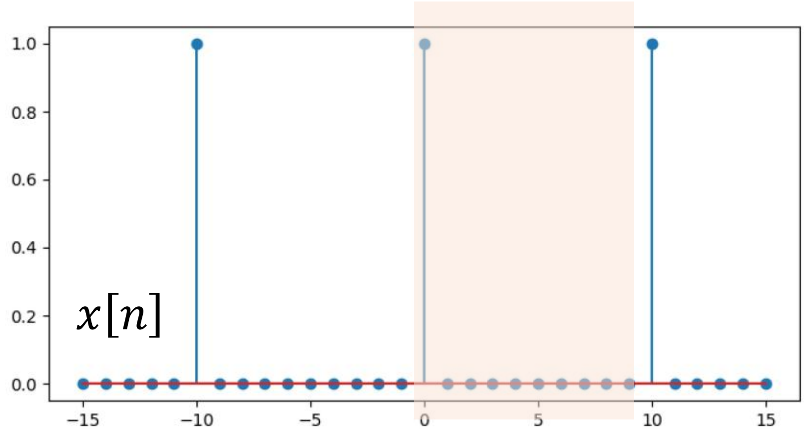
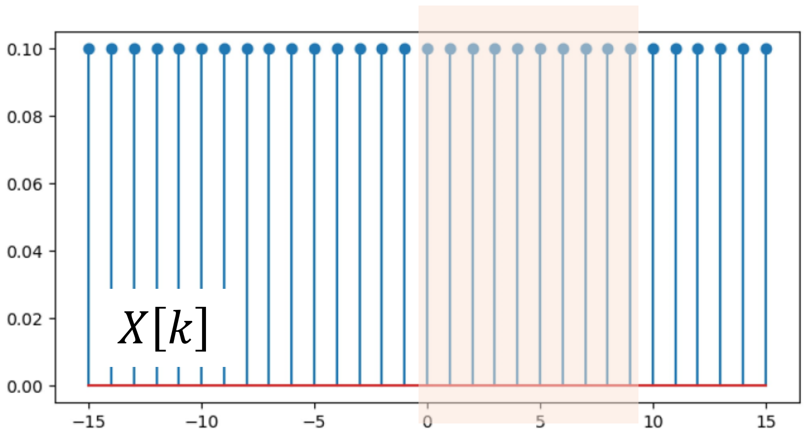


First, $x[n]$ is periodic in $N=10$

$$x[n] = x[n+10] = \sum_{k=0}^{9} X[k] e^{j\frac{2\pi}{10}kn}$$

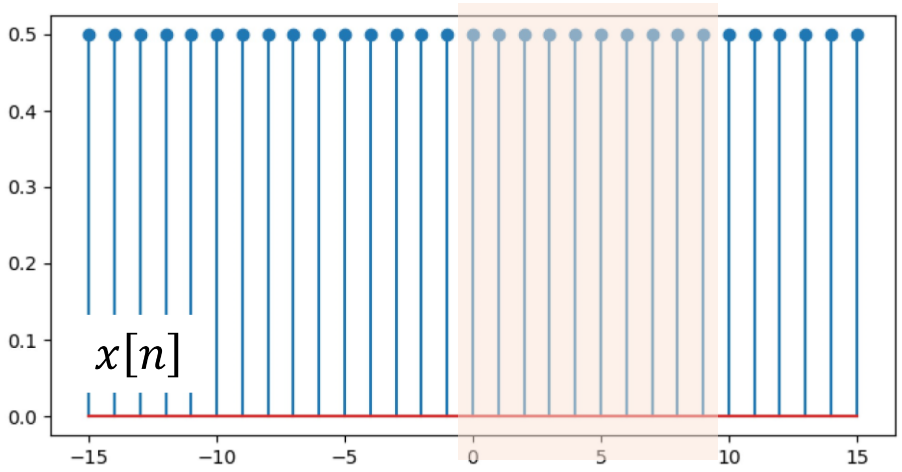


$$X[k] = \frac{1}{10}\sum_{n=0}^{9} x[n] e^{-j\frac{2\pi}{10}kn} = \frac{1}{10}$$ for every k

Find the DT Fourier Series Coefficients

- What are the Fourier Series coefficients of the following signal with a period of N=10?

$$x[n] = 0.5$$



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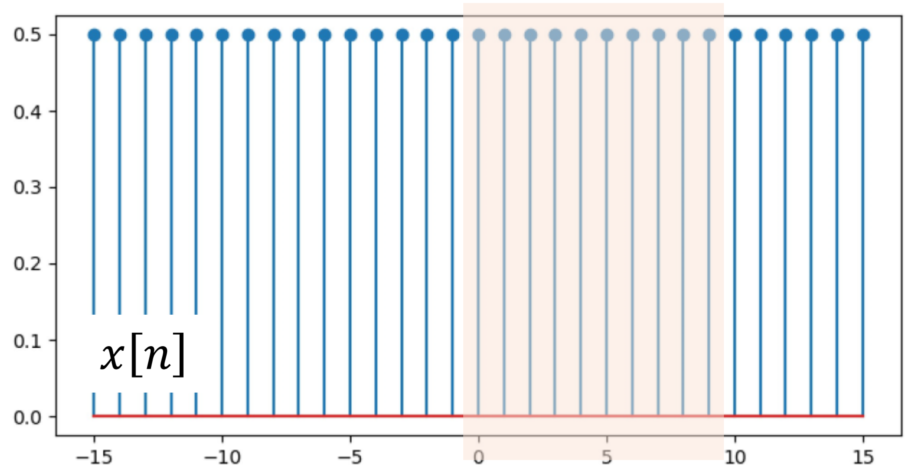
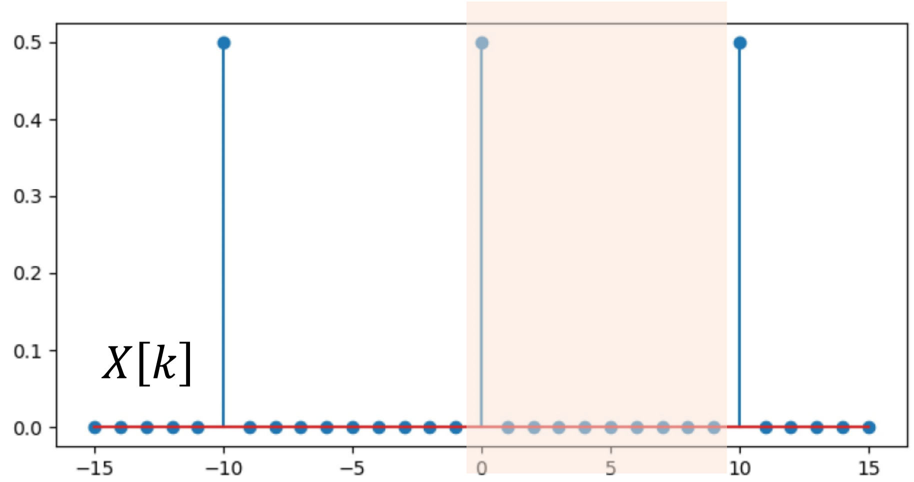


$$X[k] = \frac{1}{10}\sum_{n=0}^{9} x[n]e^{-j\frac{2\pi}{10}kn}$$

$$= \frac{1}{10}\cdot 0.5 \cdot \sum_{n=0}^{9} e^{-j\frac{2\pi}{10}kn}$$

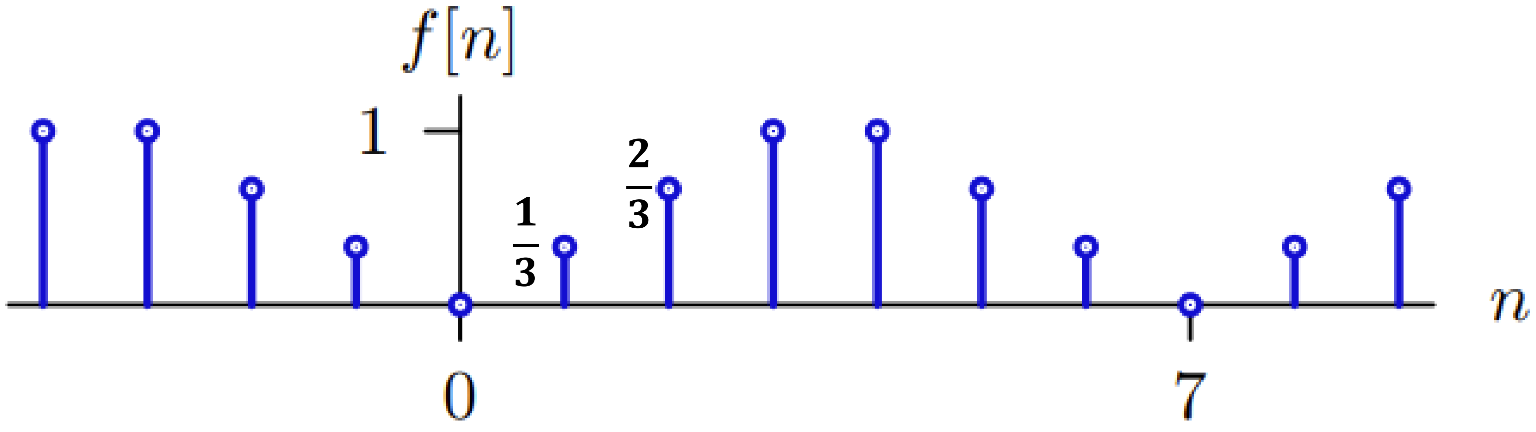
$$= \begin{cases} 0.5 & if\ k = 0 \\ 0 & if\ k \neq 0 \end{cases}$$



Find the DT Fourier Series Coefficients

Consider a new signal $g[n]$ derived from $f[n]$ as follows:

$$g[n] = 9 - 3f[n-1]$$



Find the DTFS coefficients of $g[n]$.

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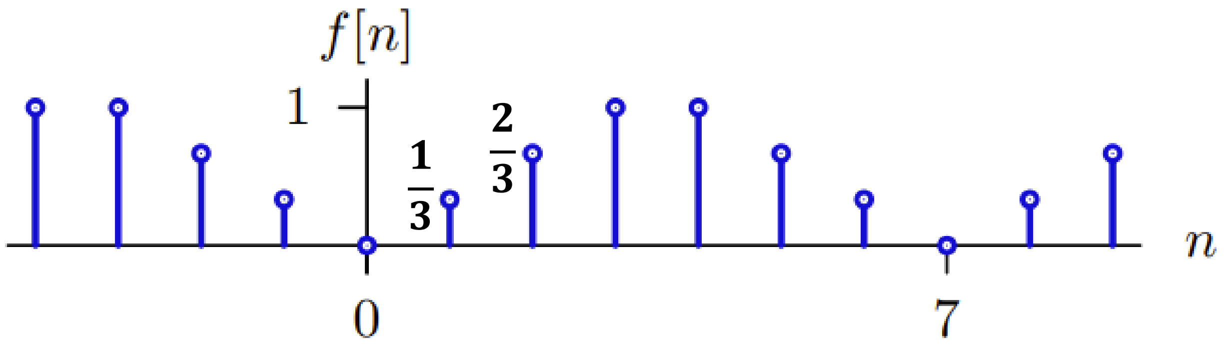


Find the DTFS coefficients of $g[n]$.

The straightforward approach is to calculate $g[n]$ for all n.

An easier approach is to use properties of the Fourier series.
We can use linearity to break the problem into two easier pieces:

$$g[n] = g_1[n] - g_2[n]$$

where $g_1[n] = 9$ and $g_2[n] = 3f[n-1]$.

Find the DT Fourier Series Coefficients

We can use linearity to break the problem into two easier pieces.

$$g[n] = g_1[n] - g_2[n]$$

where $g_1[n] = 9$ and $g_2[n] = 3f[n-1]$.

$$G_1[k] = \frac{1}{7}\sum_{n=0}^{6} 9e^{-j\frac{2\pi}{7}kn} = 9\delta[k]$$

Notice that we must use the same period $N = 7$ for $G_1[k]$, $G_2[k]$, and $G[k]$ in order to (later) apply linearity.

$g_2[n]$ combines a delay of 1 sample with multiplying by a scale factor 3.
The delay of 1 simply multiplies the Fourier coefficients (of $f[n]$) by $e^{-j\frac{2\pi}{7}k}$.
Scaling by 3 similarly multiplies the Fourier coefficients (of $f[n-1]$) by 3.
The net result is

$$G_2[k] = 3e^{-j\frac{2\pi}{7}k}F[k]$$

and

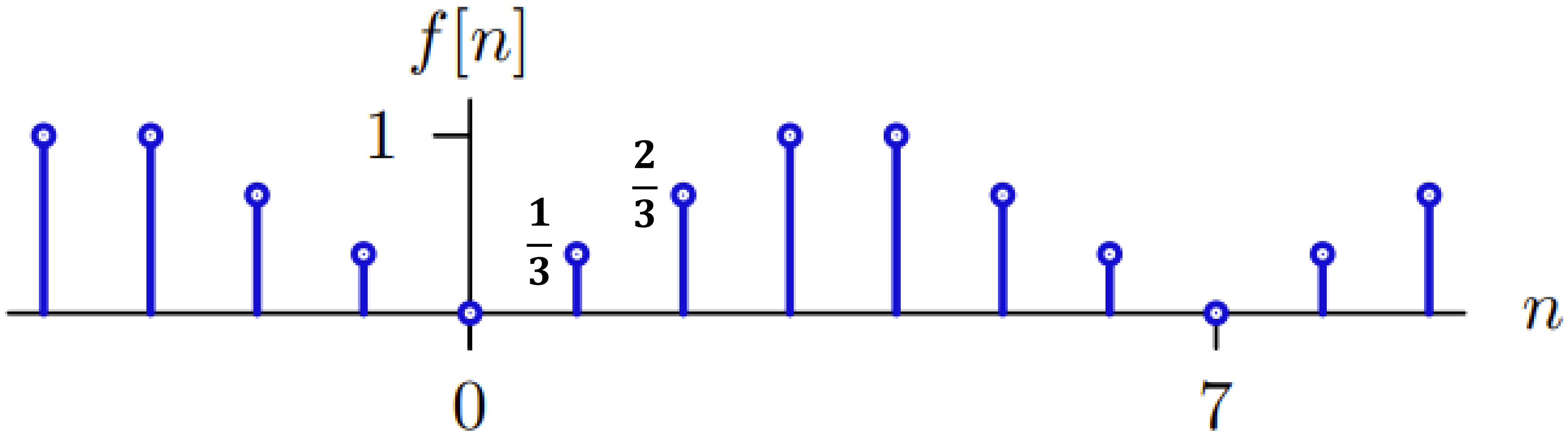
$$G[k] = 9\delta[k] - 3e^{-j\frac{2\pi}{7}k}F[k]$$

Find the DT Fourier Series Coefficients

Consider another new signal

$$h[n] = (-1)^n f[n]$$

where



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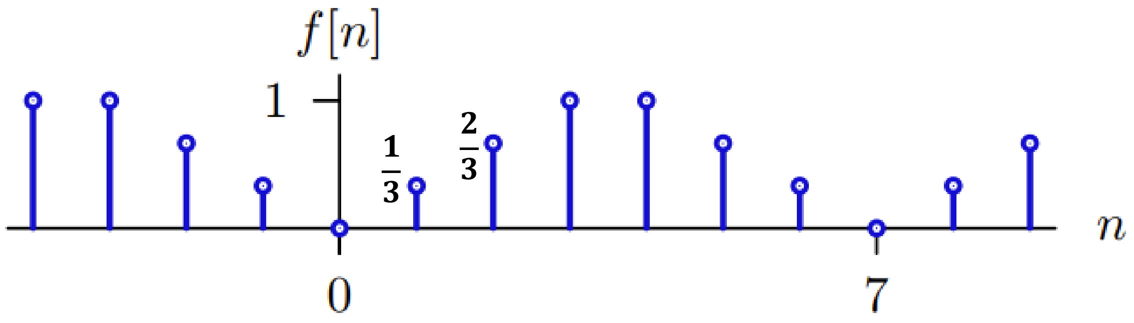


Find the DTFS coefficients of $h[n]$.

Let $f_1[n] = (-1)^n f[n]$.
Notice that $f_1[n]$ is not periodic in $N = 7$.
We will have to analyze $f_1[n]$ with $N = 14$!

Find the DT Fourier Series Coefficients

How does changing $N = 7$ to $N = 14$ affect the Fourier series coefficients?

If the period is $N = 7$ then

$$F_7[k] = \frac{1}{7}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{7}kn}$$

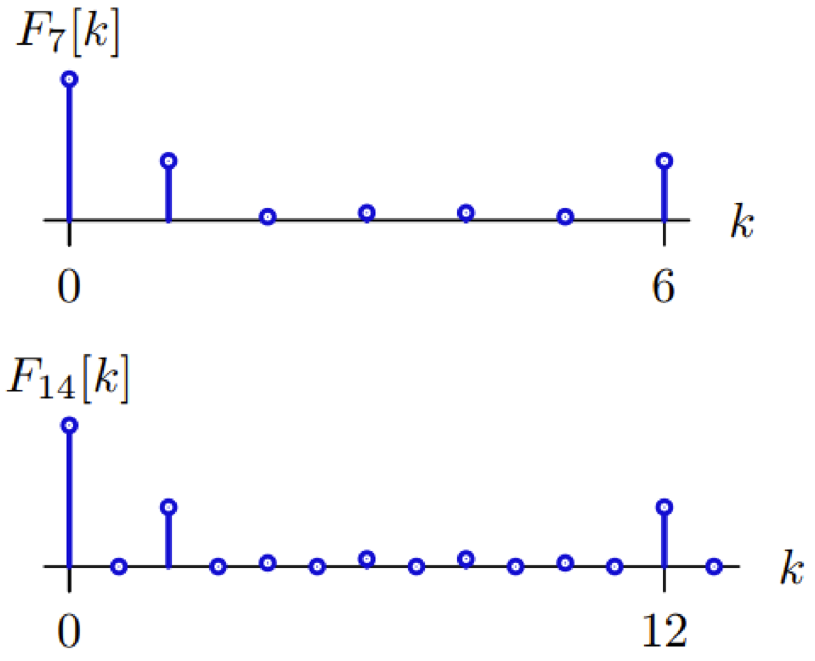
If the period is $N = 14$ then

$$\begin{aligned}
F_{14}[k] &= \frac{1}{14}\sum_{n=0}^{13} f[n]e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{14}kn} + \frac{1}{14}\sum_{n=7}^{13} f[n]e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{14}kn} + \frac{1}{14}\sum_{m=0}^{6} \underbrace{f[m+7]}_{f[m]}\ \underbrace{e^{-j\frac{2\pi}{14}k(m+7)}}_{e^{-j\frac{2\pi}{14}km}e^{-j\frac{2\pi}{14}7k}} \\
&= \frac{1}{14}\sum_{n=0}^{6} f[n]\left(1+(-1)^k\right)e^{-j\frac{2\pi}{14}kn} = \begin{cases} F_7[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Find the DT Fourier Series Coefficients

How does changing $N = 7$ to $N = 14$ affect the Fourier series coefficients?

$$F_{14}[k] = \begin{cases} F_7[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$



The components of F_7 are **stretched** in F_{14}.

There is no fundamental in F_{14}, the harmonics are 0, 2, 4, … 12.

Find the DT Fourier Series Coefficients

Now find the DTFS coefficients for $h[n]$:

$$h[n] = (-1)^n f[n]$$

$$\begin{aligned}
H[k] &= \frac{1}{14}\sum_{n=0}^{13}(-1)^n f[n] e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{13} e^{j\pi n} f[n] e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{13} f[n] e^{-j\frac{2\pi}{14}(k-7)n} \\
&= F_{14}[k-7] \\
&= \begin{cases} F_7[(k-7)/2] & \text{if } k-7 \text{ is even} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} F_7[(k-7)/2] & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$