

6.3000: Signal Processing

DFT and Circular Convolution

October 29, 2026

Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.



Consider three ways to calculate the convolution of $x[n]$ with itself.

1. direct convolution:

$$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

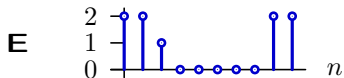
2. using DTFTs:

$$y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega) e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

3. using DFTs of length $N=16$:

$$y_3[n] = 16 \sum_{k=0}^{15} X^2[k] e^{j\frac{2\pi k}{16}n} \quad \text{where} \quad X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n] e^{-j\frac{2\pi k}{16}n}$$

The plots on the right show the **first ten samples** of five signals. Match the signals on the left with the corresponding plots on the right.



Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

$$\begin{aligned} f[n] &= \sum_{k=0}^{N-1} F[k] e^{j \frac{2\pi k}{N} n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j \frac{2\pi k}{N} n} \\ &= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j \frac{2\pi k}{N} m} \right) e^{j \frac{2\pi k}{N} n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j \frac{2\pi k}{N} (n-m)} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m] \end{aligned}$$

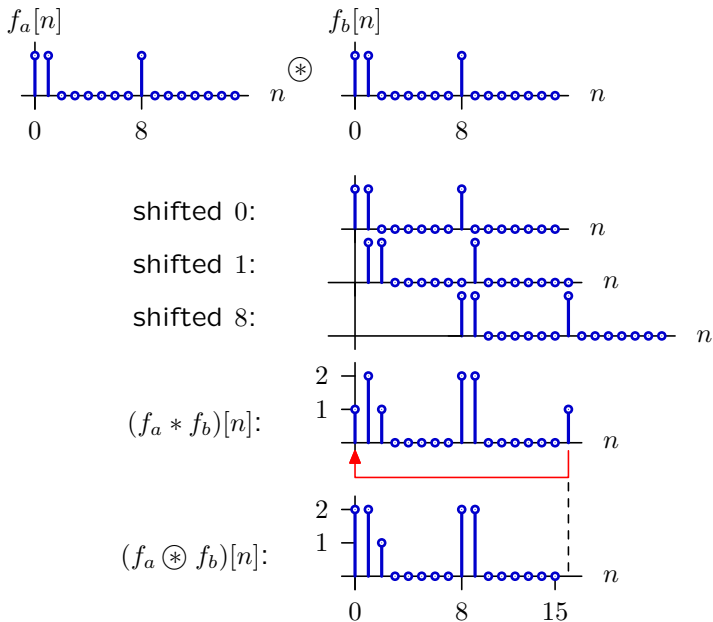
where $f_{ap}[n] = f_a[n \bmod N]$ is a periodically extended version of $f_a[n]$.

We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N} (f_a \circledast f_b)[n] \quad \xrightarrow{\text{DFT}} \quad F_a[k] \times F_b[k]$$

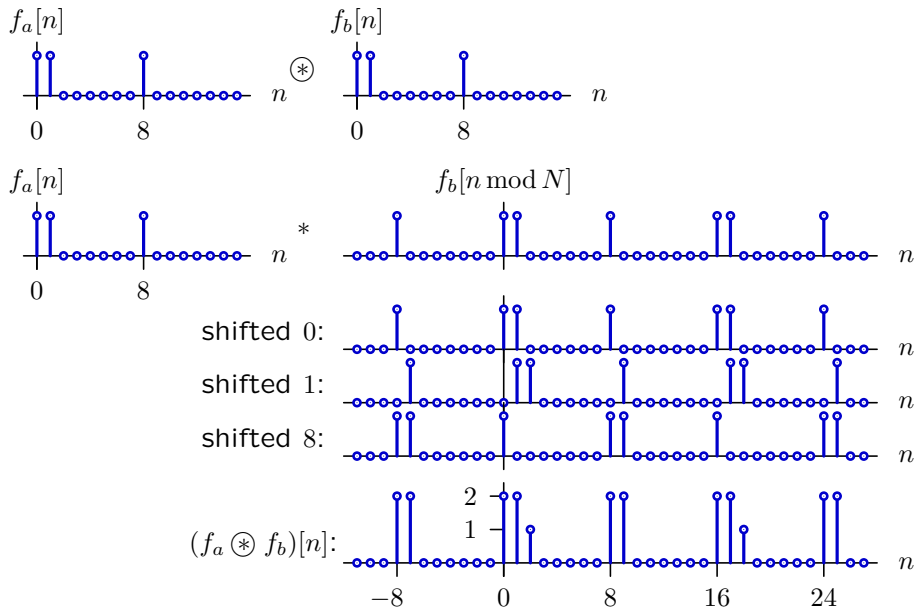
Circular Convolution

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.

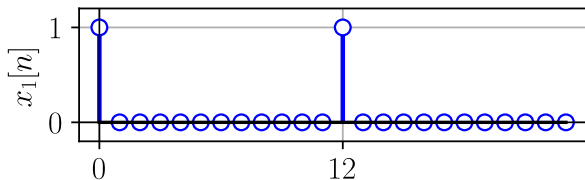


Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.

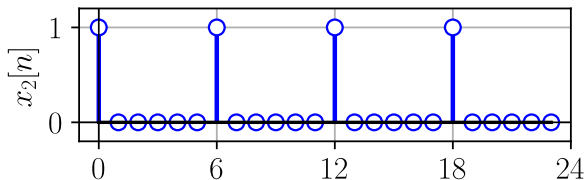


Impulse Trains: DFT



Let $N = 24$. Determine $X_1[k]$, the DFT of $x_1[n]$.

Impulse Trains: DFT



Let $N = 24$. Determine $X_2[k]$, the DFT of $x_2[n]$.

Impulse Trains: DFT

Suppose that N is an integer multiple of L . Using an analysis window of length N , determine the DFT of

$$f[n] = \sum_m \delta[n - mL].$$

Hint: Generalize the work we've done so far.

Impulse Trains: CTFT

Let $T > 0$. Determine $F(\omega)$, the Fourier transform of

$$f(t) = \sum_m \delta(t - mT).$$

Hint: $f(t)$ is periodic.

Impulse Trains: DTFT

Let $N > 0$. Determine $F(\Omega)$, the Fourier transform of

$$f[n] = \sum_m \delta[n - mN].$$

Hint: $f[n]$ is periodic.

Lessons Learned

Multiplication of DFTs corresponds to circular convolution in the time domain: $F[k]G[k] \iff \frac{1}{N}(f \circledast g)[n]$.

Circular convolution: $\frac{1}{N}(f \circledast g)[n] \iff F[k]G[k]$

- Compute the usual convolution $(f * g)[n]$ to start.
- Wrap $(f * g)[n]$ into $[0, N - 1]$ and scale by $1/N$.

Impulse trains: The Fourier transform of an impulse train is another impulse train!

$$f[n] = \sum_m \delta[n - mL] \iff F[k] = \frac{1}{L} \sum_m \delta[k - m\frac{N}{L}]$$

$$f(t) = \sum_m \delta(t - mT) \iff F(\omega) = \frac{2\pi}{T} \sum_m \delta(\omega - m\frac{2\pi}{T})$$