6.3000 Quiz 2 Fall 2026

Name: **Solutions**

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two 8.5 × 11 sheets of notes (four sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all answers in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

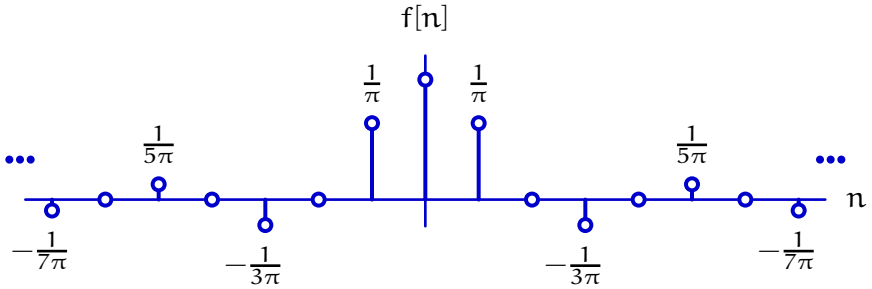
We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Mystery Filter

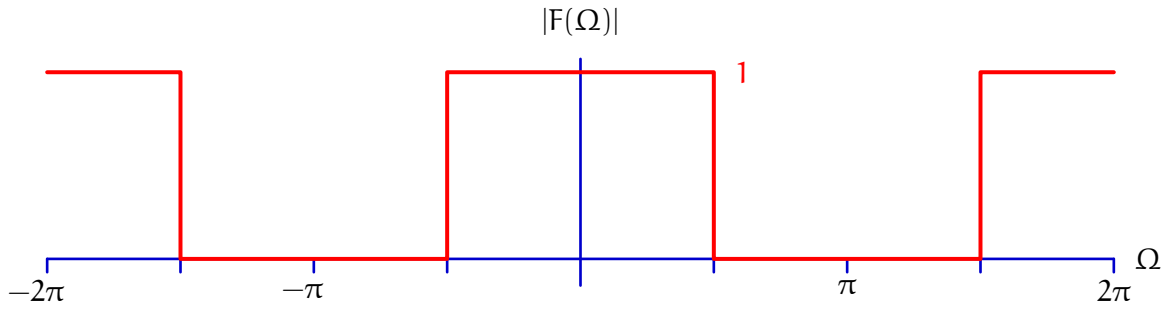
Let $f[n]$ represent the unit sample response of an ideal lowpass filter. The figure below shows the portion of $f[n]$ that falls in the range $-7 \leqslant n \leqslant 7$:



Part a. Determine the cutoff frequency Ω_c of this filter:

$\Omega_c =$ $\frac{\pi}{2}$

and sketch the magnitude of the frequency response $|F(\Omega)|$ for frequencies $-2\pi \leqslant \Omega \leqslant 2\pi$ on the axes below. Label the important parameters of your sketch.



Find a closed-form expression for $f[n]$ and enter it in the following box:

$f[n] =$ $\frac{1}{\pi n}\sin(\pi n/2)$

Determine the numerical value of $f[0]$ and enter it in the box below:

$f[0] =$ $\frac{1}{2}$

The unit sample response of an ideal lowpass filter can be determined from the DTFT of the frequency response as follows:

$$f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}d\Omega = \frac{1}{\pi n}\sin(\Omega_c n)$$

The zeros of the sin function are at integer multiples of π. The first zero of $f[n]$ (shown above) is at $n = 2$. Therefore $2\Omega_c$ must be π, so $\Omega_c = \pi/2$.

We can find $f[0]$ by applying L'Hopital's rule for $n = 0$ in the general expression, with the result that $f[0] = 1/2$.

Part b. Let $p[n]$ represent a discrete-time signal whose discrete-time Fourier Transform (DTFT) is given by

$$P(\Omega) = F(\Omega) - 1$$

where $F(\Omega)$ represents the low-pass frequency response found in part a. Determine an expression for $p[n]$ in terms of $f[n]$ and enter this expression in the box below.

$p[n] =$ $f[n] - \delta[n]$

The DTFT of the unit sample signal $\delta[n]$ is

$$\sum_n \delta[n] e^{-j\Omega n} = 1$$

Therefore $p[n] = f[n] - \delta[n]$.

Part c. Let $g[n]$ represent a discrete-time signal whose discrete-time Fourier Transform (DTFT) is given by

$$G(\Omega) = F(\Omega - \pi)$$

where $F(\Omega)$ represents the low-pass frequency response found in part a. Determine an expression for $g[n]$ in terms of $f[n]$ and enter this expression in the box below.

$g[n] =$ $e^{j\pi n} f[n]$

$$g[n] = \frac{1}{2\pi}\int_{2\pi} G(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} F(\Omega - \pi) e^{j\Omega n} d\Omega$$

Let $\lambda = \Omega - \pi$ then

$$g[n] = \frac{1}{2\pi}\int_{2\pi} F(\lambda) e^{j(\lambda+\pi)n} d\lambda = e^{j\pi n} f[n]$$

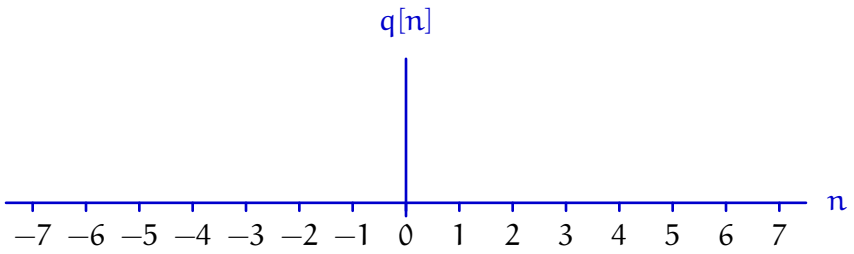
Part d. Let $q[n]$ represent a discrete-time signal whose discrete-time Fourier Transform (DTFT) is given by

$$Q(\Omega) = F(3\Omega) \times (1 + 2\cos(\Omega))$$

where $F(\Omega)$ represents the low-pass frequency response found in part a. Determine a closed-form expression for $q[n]$ and it in the box below:

$q[n] =$ TBD

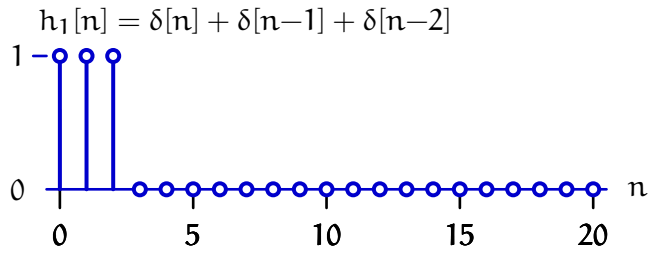
Sketch $q[n]$ for $-7 \leqslant n \leqslant 7$ on the axes below.



$$\begin{aligned}
q[n] &= \frac{1}{2\pi}\int_{2\pi} Q(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} F(3\Omega) \times (1 + 2\cos(\Omega))e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{6\pi} F(3\Omega) \times (e^{-j\Omega} + 1 + e^{j\Omega})e^{j\Omega n}d\Omega \\
&= \frac{1}{6\pi}\int_{6\pi} F(\lambda) \times (e^{-j\lambda/3} + 1 + e^{j\lambda/3})e^{j\lambda n/3}d\lambda \\
&= \frac{1}{6\pi}\int_{6\pi} F(\lambda)e^{j\lambda(n-1)/3}d\lambda + \frac{1}{6\pi}\int_{6\pi} F(\lambda)e^{j\lambda(n-1)/3}d\lambda + \frac{1}{6\pi}\int_{6\pi} F(\lambda)e^{j\lambda(n-1)/3}d\lambda
\end{aligned}$$

2 Discrete-Time Signals and Systems

Let S_1 represent a linear, time-invariant system with unit-sample response $h_1[n]$.



Part a. The input/output relation for system S_1 can be represented by the following linear difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

where $x[n]$ represents the input to system S_1 and $y[n]$ represents the resulting output of that system.

Determine the numerical values of a_1, a_2, b_0, b_1, b_2 and enter those values in the boxes below:

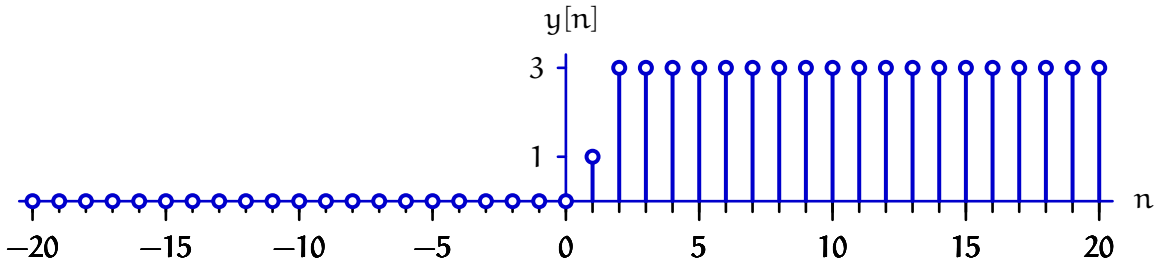
$a_1 =$	0	$a_2 =$	0		
$b_0 =$	1	$b_1 =$	1	$b_2 =$	1

$$y[n] = (x * h_1)[n] = \sum_m h_1[m]x[n-m] = h_1[0]x[n] + h_1[1]x[n-1] + h_1[2]x[n-2] = x[n] + x[n-1] + x[n-2]$$

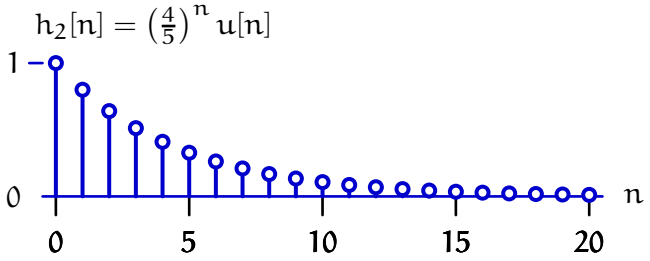
Part b. Determine the output $y[n]$ of system S_1 when the input to that system is

$$x[n] = \begin{cases} \delta[n \bmod 3] & n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Sketch the output $y[n]$ on the axes below.



Let S_2 represent a linear, time-invariant system with unit-sample response $h_2[n]$.

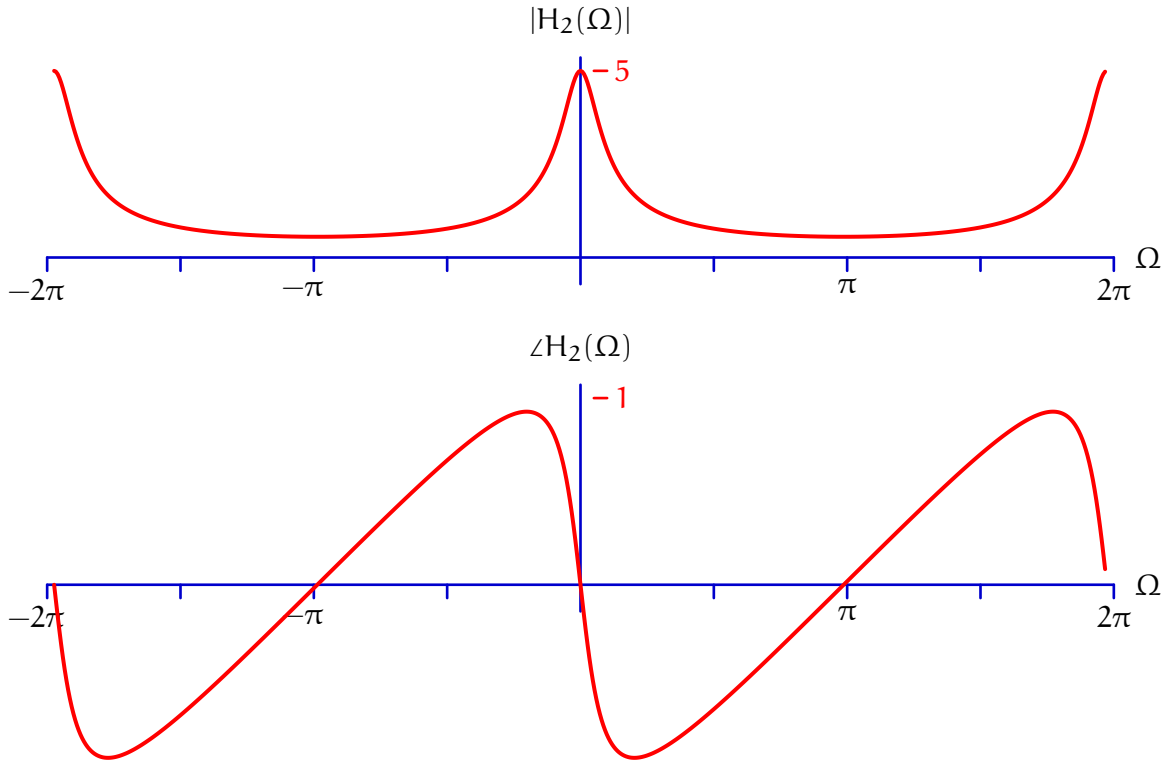


Notice that $h_2[n] = 0$ for all $n < 0$.

Part c. Find the frequency response $H_2(\Omega)$ of system S_2 and enter an expression for $H_2(\Omega)$ in the box below:

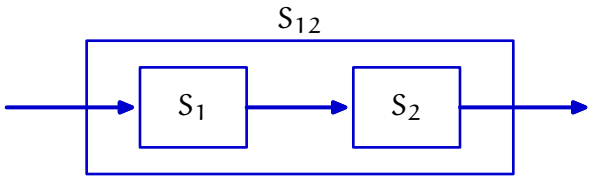
$H_2(\Omega) =$ $\frac{1}{1-\frac{4}{5}e^{-j\Omega}}$

Sketch the magnitude and phase of $H_2(\Omega)$ in the range $-2\pi \leqslant \Omega \leqslant 2\pi$ on the axes below. Label all of the important parameters of your sketch.



$$H_2(\Omega) = \sum_n h_2[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}e^{-j\Omega}\right)^n = \frac{1}{1-\frac{4}{5}e^{-j\Omega}}$$

Part d. Let S_{12} represent a new system in which S_1 and S_2 are connected in cascade as shown below:



Determine the frequency response H_{12} of the system S_{12} and enter an expression for H_{12} in the box below.

$H_{12}(\Omega) =$ $\dfrac{1 + e^{-j\Omega} + e^{-2j\Omega}}{1 - \frac{4}{5}e^{-j\Omega}}$

Let $h_{12}[n]$ represent the unit-sample response of the system S_{12}. Enter expressions for the first four values of $h_{12}[n]$ in the boxes below.

$h_{12}[0] =$ 1 $h_{12}[1] =$ 9/5

$h_{12}[2] =$ 61/125 $h_{12}[3] =$ 305/625

Part e. If the input to system S_{12} is

$$x[n] = (-1)^n$$

then the output can be expressed as follows:

$$y[n] = \lambda(-1)^n$$

Determine λ and enter its numerical value in the box below:

$\lambda =$ TBD

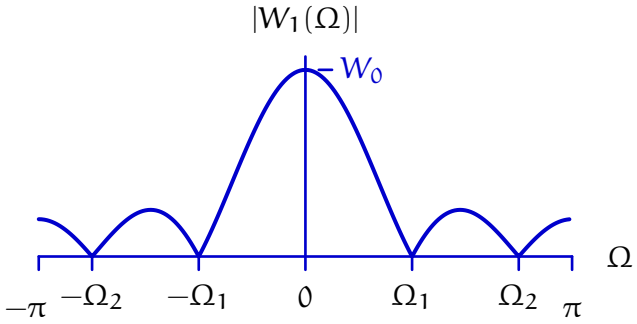
3 From DTFT to DFT

An aperiodic discrete-time signal $w_1[n]$ is described as follows:

$$w_1[n] = \begin{cases} 1 & 0 \leqslant n < M \\ 0 & \text{otherwise} \end{cases}$$

where M is a positive integer.

Part a. The magnitude of the discrete-time Fourier transform (DTFT) of $w_1[n]$ is shown below for the frequency interval $-\pi \leqslant \Omega \leqslant \pi$:



Determine the value of M and enter that value in the box below:

$M =$ 5

Determine the value of W_0 and enter that value in the box below:

$W_0 =$ 5

Determine the value of Ω_1 and enter it in the box below:

$\Omega_1 =$ $2\pi/5$

Determine the value of Ω_2 and enter it in the box below:

$\Omega_2 =$ $4\pi/5$

$$W_1(\Omega) = \sum_{n=-\infty}^{\infty} w_1[n]e^{-j\Omega n} = \sum_{n=0}^{M-1} e^{-j\Omega n} = \frac{1 - e^{-j\Omega M}}{1 - e^{-j\Omega}}$$

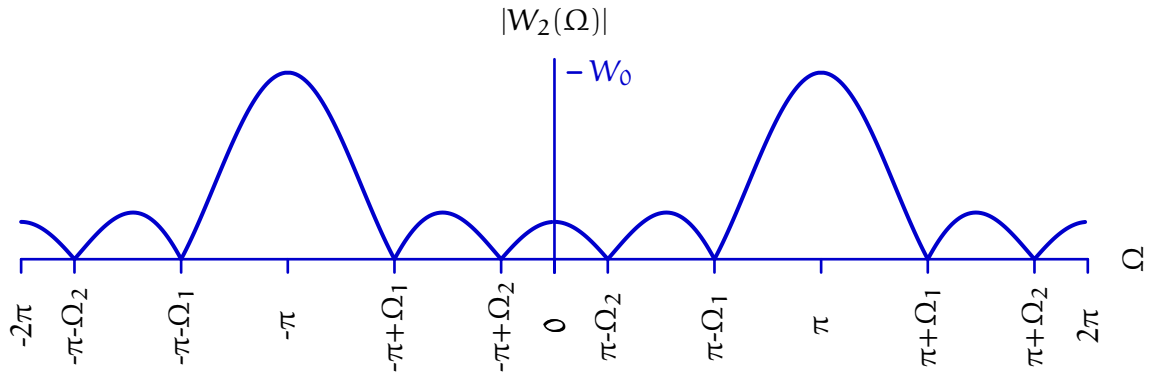
$W_1\Omega = 0$ when ΩM is a multiple of 2π, which is when Ω is a multiple of $2\pi/M$. It follows that the plot of $|W_1(\Omega)|$ shown above results when $M = 5$.

Part b. A new signal $w_2[n]$ is derived from $w_1[n]$ as follows:

$$w_2[n] = \begin{cases} w_1[n] & \text{if } n \text{ is even} \\ -w_1[n] & \text{if } n \text{ is odd} \end{cases}$$

Find an expression for $W_2(\Omega)$ in terms of $W_1(\Omega)$ and plot the magnitude of $W_2(\Omega)$ over the range $-2\pi \leqslant \Omega \leqslant 2\pi$ on the axes below. b

$W_2(\Omega) =$ $W_1(\Omega - \pi)$



$w_2[n] = w_1[n] \times (-1)^n = w_1[n] \times e^{j\pi n}$

$W_2(\Omega) = \sum_n w_2[n] \times e^{j\Omega n} = \sum_n w_1[n] e^{j\pi n} e^{-j\Omega n} = \sum_n w_1[n] e^{-j(\Omega-\pi)n} = W_1(\Omega - \pi)$

Part c. Let $F_8[k]$ denote the DFT of $w_1[n]$ computed with an analysis window length $N = 8$. Determine the values of $F_8[k]$ for $k = 0, 2, 4$ and enter those values in the boxes below:

$F_8[0] =$ 5/8 $F_8[2] =$ 1/8 $F_8[4] =$ 1/8

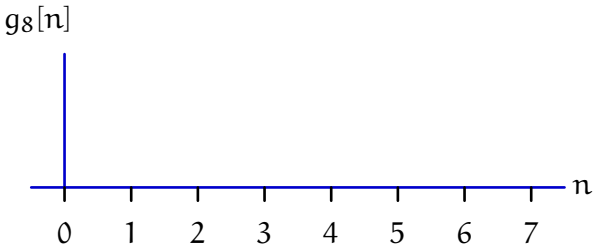
$$F_8[k] = \frac{1}{8}\sum_{n=0}^{7} w_1[n]e^{-j2\pi nk/8} = \frac{1}{8}\sum_{n=0}^{4} e^{-j2\pi nk/8}$$

$$F_8[0] = (1+1+1+1+1)/8 = 5/8$$

$$F_8[2] = (1+j-1-j+1)/8 = 1/8$$

$$F_8[4] = (1-1+1-1+1)/8 = 1/8$$

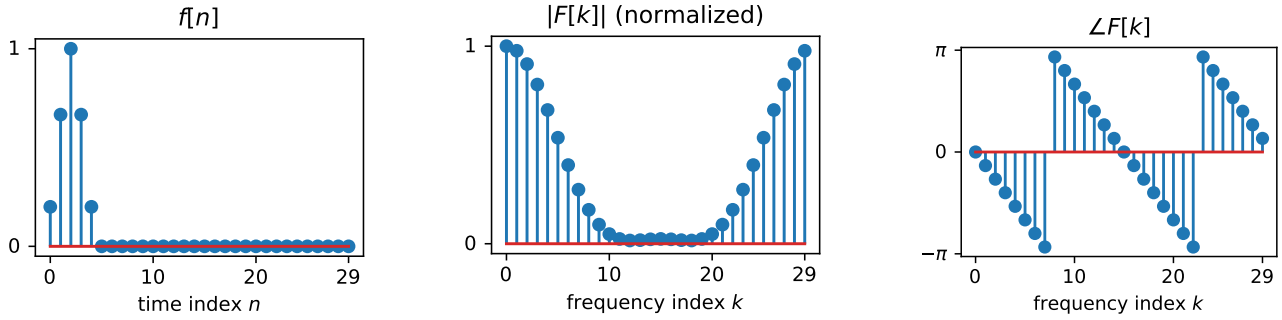
Part d. Let $g_8[n]$ denote the signal with DFT $G_8[k] = F_8[k] \times F_8[k]$. The analysis window is of length $N = 8$. Sketch $g_8[n]$ on the axes provided below. Label the important features of your sketch.



Multiplication in frequency is equivalent to convolution in time. Therefore, the function $g_8[n]$ is the circular convolution of $w_1[n]$ with itself.

4 Related DFTs

Let $f[n]$ represent a periodic, discrete-time signal with period N=30 so that $f[n] = f[n+30]$ for all n. The following plots show one period of $f[n]$ along with the magnitude and angle of its Discrete Fourier Transform (DFT) computed with an analysis window of length N=30. The magnitude plot has been scaled so that the maximum magnitude is 1.



Ten signals ($g_1[n]$ through $g_{10}[n]$) are derived from $f[n]$ as shown below.

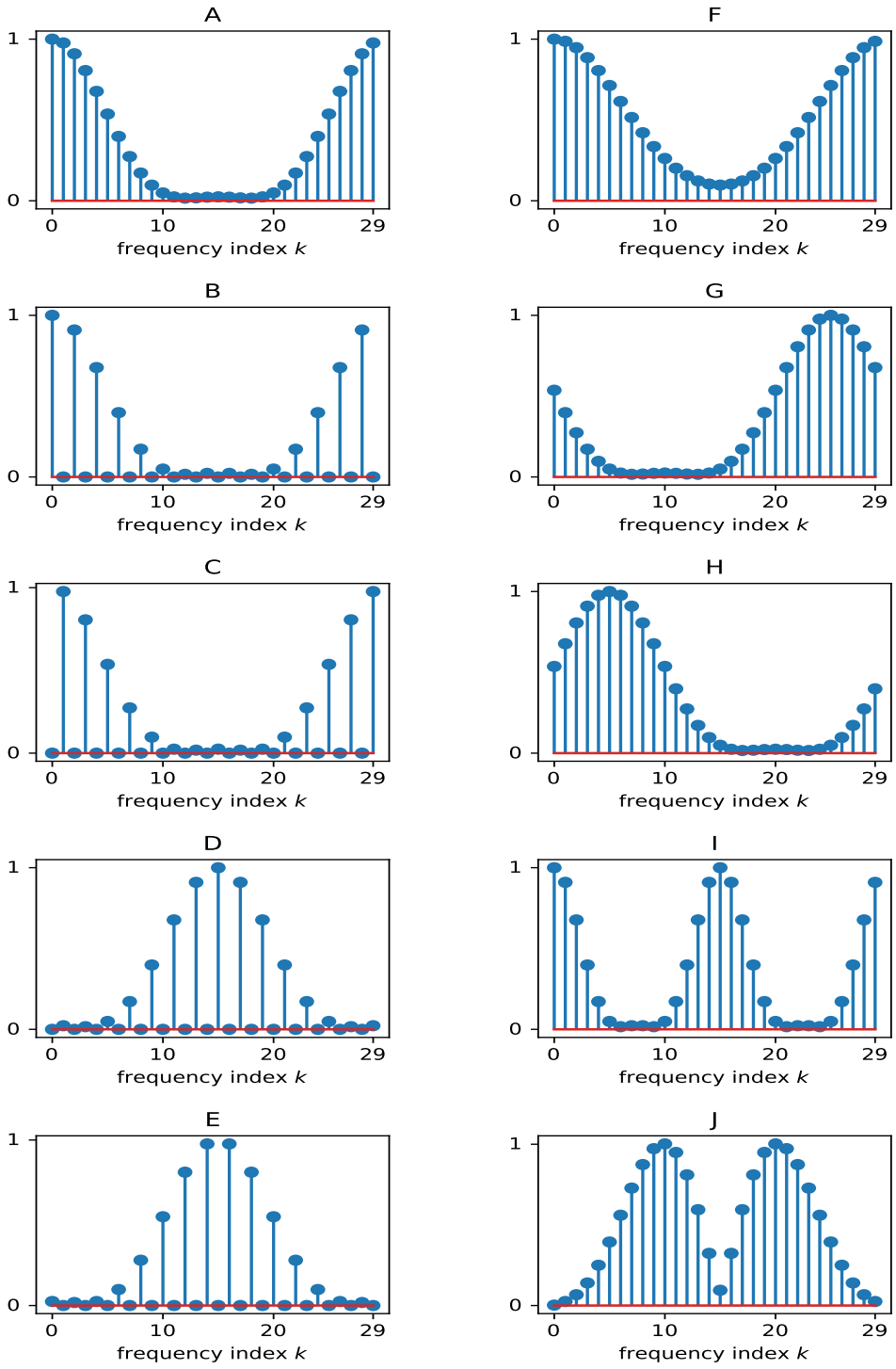
	normalized magnitude	angle
$g_1[n] = f[-n]$	A	j
$g_2[n] = f[n-1]$	A	h
$g_3[n] = f[n+1]$	A	b
$g_4[n] = f^2[n]$	F	c
$g_5[n] = e^{-j\pi n/3} f[n]$	G	e
$g_6[n] = e^{j\pi n/3} f[n]$	H	d
$g_7[n] = \Big(f[n] + f[(n-15)]\Big)/2$	B	f
$g_8[n] = \Big(f[n] + f[(n-15)]\Big)/2$	C	g
$g_9[n] = \begin{cases} f[n/2] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$	I	i
$g_{10}[n] = f[n+2]$	A	a

Determine which of the panels (A-J) on the following page shows the magnitude of the DFT of each of the derived signals and write the letter for that panel in the corresponding box in the magnitude column above.

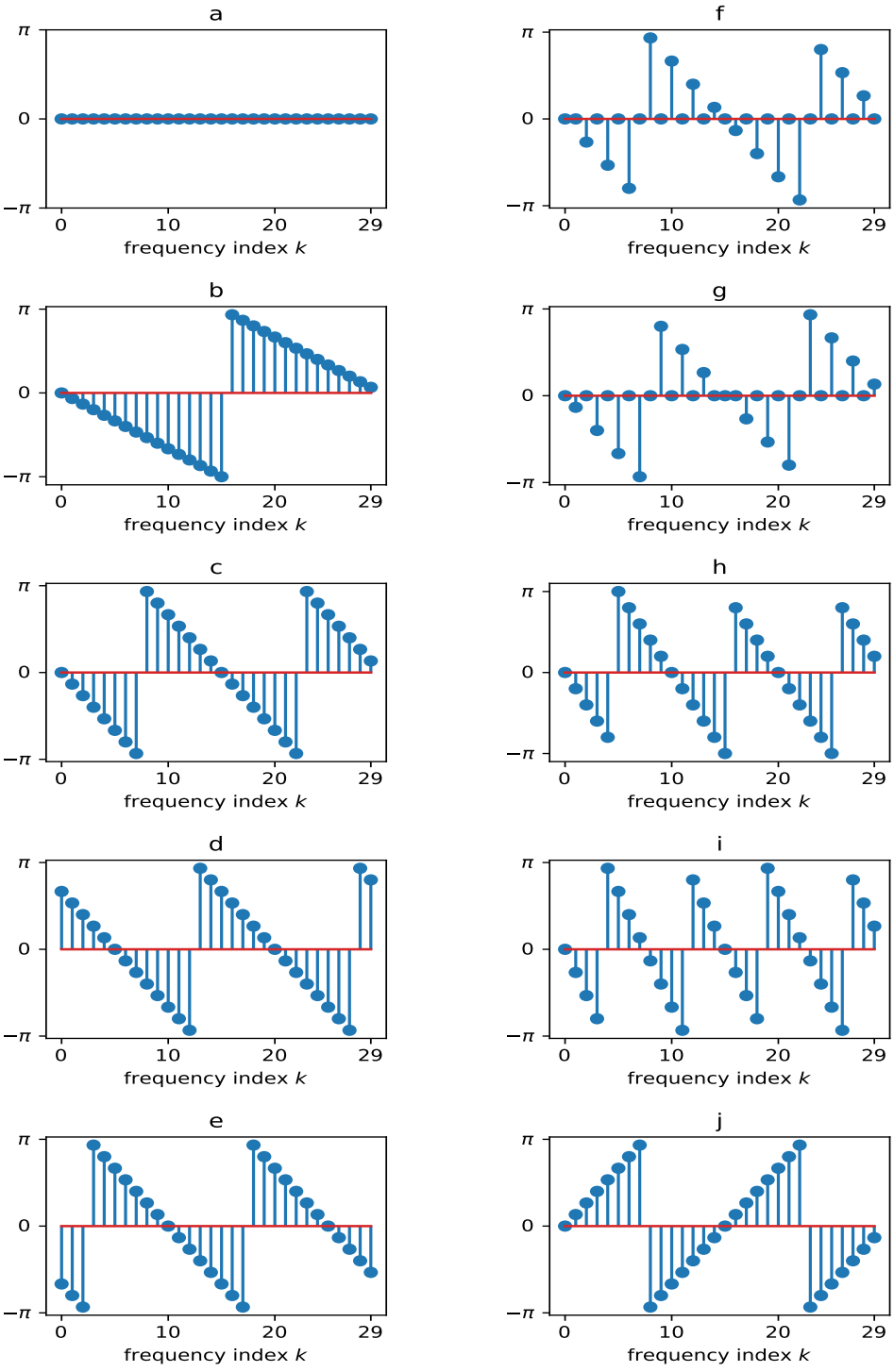
Similarly, determine which of the panels (a-j) on the subsequent page shows the angle of the DFT of each of the derived signals and write the letter for that panel in the corresponding box in the angle column above.

The same panel may be used more than once.

Magnitude Plots for Problem 4: the magnitudes in each plot have been scaled so that the largest magnitude in each plot is 1.



Angle Plots for Problem 4.



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