

6.300 Quiz 1

Fall 2026

Name:

Solutions

MIT ID:

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use one hand-written 8.5×11 sheet of notes.

You may NOT use any electronic devices (such as calculators and phones).

Show your work in the space below each part, and enter your final answer in the box provided.

Work on the worksheet pages at the end of this booklet will be considered for partial credit if you point to it from the answer box. Do not remove any pages and do not hand in loose sheets: only this booklet is scanned.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\cos(a) + \cos(b) = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$\sin(a) + \sin(b) = 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2 \sin(a) \sin(b)$$

$$\sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b)$$

$$\sin(a+b) - \sin(a-b) = 2 \cos(a) \sin(b)$$

$$2 \cos(a) \cos(b) = \cos(a-b) + \cos(a+b)$$

$$2 \sin(a) \sin(b) = \cos(a-b) - \cos(a+b)$$

$$2 \sin(a) \cos(b) = \sin(a+b) + \sin(a-b)$$

$$2 \cos(a) \sin(b) = \sin(a+b) - \sin(a-b)$$

1. Simplify & Locate (20 points)

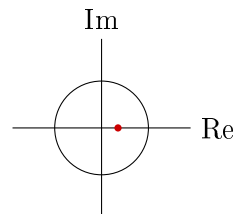
In each of the following, a complex number is given. Write the simplified form, either in polar or rectangular format, and mark its value on the complex plane with a dot. If an expression has more than one possible value, giving any one of them is fine.

(1)

$$\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^j$$

Simplified expression:

$$e^{-\pi/3} \approx 0.35$$



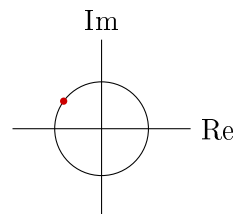
Solution. $\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\pi/3}$, so $(e^{j\pi/3})^j = e^{j \cdot j\pi/3} = e^{-\pi/3} \approx 0.35$: a positive real number, on the real axis inside the unit circle. (Writing the base as $e^{j(\pi/3+2\pi m)}$ gives $e^{-\pi/3-2\pi m}$; any one of these is accepted.)

(2)

$$\sum_{n=0}^8 e^{j\frac{\pi}{5}n}$$

Simplified expression:

$$e^{j4\pi/5}$$



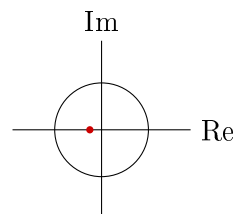
Solution. Geometric sum with $r = e^{j\pi/5}$: $\sum_{n=0}^8 r^n = \frac{1-r^9}{1-r}$, and $r^9 = e^{j9\pi/5} = e^{-j\pi/5}$, so the sum is $\frac{1-e^{-j\pi/5}}{1-e^{j\pi/5}} = \frac{e^{-j\pi/10}(e^{j\pi/10}-e^{-j\pi/10})}{e^{j\pi/10}(e^{-j\pi/10}-e^{j\pi/10})} = -e^{-j\pi/5} = e^{j4\pi/5}$. Shortcut: the ten 10th roots of unity $e^{j\pi n/5}$, $n = 0, \dots, 9$, sum to zero, so nine of them sum to minus the missing one, $-e^{j9\pi/5}$. Magnitude 1, angle 144° .

(3)

$$\left(\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{6}\right)\right)^4$$

Simplified expression:

$$-\frac{1}{4}$$



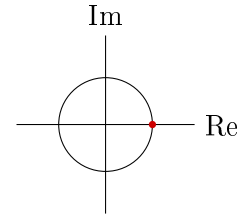
Solution. $\cos\frac{\pi}{3} = \sin\frac{\pi}{6} = \frac{1}{2}$, so the base is $\frac{1}{2}(1+j) = \frac{1}{\sqrt{2}}e^{j\pi/4}$. Fourth power: $\left(\frac{1}{\sqrt{2}}\right)^4 e^{j\pi} = \frac{1}{4} \cdot (-1) = -\frac{1}{4}$.

(4)

$$\sqrt[5]{\left(\frac{\sqrt{2}}{2}(1+j)\right)^8}$$

Simplified expression:

1

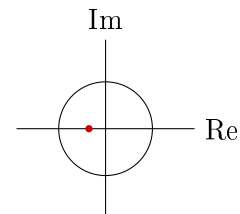


Solution. $\frac{\sqrt{2}}{2}(1+j) = e^{j\pi/4}$, so the eighth power is $e^{j2\pi} = 1$, and the fifth roots of 1 are $e^{j2\pi m/5}$, $m = 0, \dots, 4$. The principal root is 1; any of the five is accepted.

(5)

$$\text{Im} \left(\frac{1}{2e^{j(\tan^{-1}(2)+\tan^{-1}(3))}} \right)$$

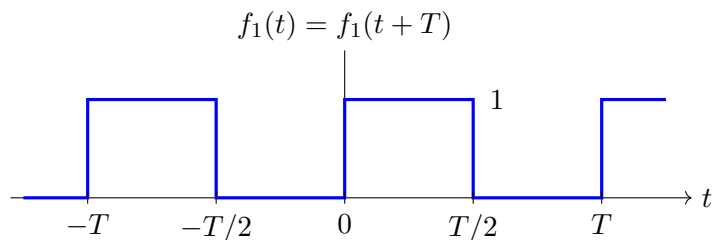
Simplified expression:

 $-\frac{\sqrt{2}}{4} \approx -0.35$ 

Solution. Let $a = \tan^{-1} 2$, $b = \tan^{-1} 3$. Then $\tan(a+b) = \frac{2+3}{1-2 \cdot 3} = -1$ with $a+b \in (\frac{\pi}{2}, \pi)$, so $a+b = \frac{3\pi}{4}$. The expression is $\frac{1}{2e^{j3\pi/4}} = \frac{1}{2}e^{-j3\pi/4}$, whose imaginary part is $-\frac{1}{2}\sin \frac{3\pi}{4} = -\frac{\sqrt{2}}{4}$. The result is a real number, so the dot goes on the real axis at -0.35 .

2. Signal Transformations (30 points)

A periodic signal $f_1(t)$ having fundamental period T is shown below. All of the Fourier series coefficients below are computed with period T .



- (1) Find the expression for its Fourier series coefficients $F_1[k]$, and enter the values of $F_1[k]$ for k from -3 to 3 below.

$$F_1[k] = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{-j}{\pi k} & k \text{ odd} \\ 0 & k \text{ even, } k \neq 0 \end{cases}$$

k	-3	-2	-1	0	1	2	3
$F_1[k]$	$\frac{j}{3\pi}$	0	$\frac{j}{\pi}$	$\frac{1}{2}$	$-\frac{j}{\pi}$	0	$-\frac{j}{3\pi}$

Solution. $f_1(t) = 1$ on $(0, T/2)$ and 0 on $(T/2, T)$, so $F_1[k] = \frac{1}{T} \int_0^{T/2} e^{-j2\pi kt/T} dt = \frac{1}{T} \cdot \frac{e^{-j2\pi kt/T}}{-j2\pi k/T} \Big|_0^{T/2} = \frac{1 - e^{-j\pi k}}{j2\pi k}$. Since $e^{-j\pi k} = (-1)^k$, the numerator is 2 for odd k and 0 for even $k \neq 0$, giving $\frac{2}{j2\pi k} = \frac{-j}{\pi k}$ for odd k . The $k = 0$ term is the average, $\frac{1}{2}$. (Equivalent form: $F_1[k] = e^{-j\pi k/2} \frac{\sin(\pi k/2)}{\pi k}$.)

- (2) A symmetric signal $f_2(t)$ can be obtained by shifting $f_1(t)$ so that $f_2(t) = f_1(t - t_0)$. Find t_0 in terms of T , and express its Fourier series coefficients $F_2[k]$ in terms of $F_1[k]$.

$$t_0 = -\frac{T}{4} \quad (\text{or } \frac{3T}{4})$$

$$F_2[k] = e^{j\pi k/2} F_1[k]$$

Solution. The pulse of f_1 is centred at $t = T/4$; $f_2(t) = f_1(t - t_0)$ moves the centre to $T/4 + t_0$. An even signal needs the pulse centred on 0 (or, by periodicity, on $T/2$), so $t_0 = -T/4$, equivalently $3T/4$. Time-shift property: $F_2[k] = e^{-j2\pi kt_0/T} F_1[k] = e^{j\pi k/2} F_1[k]$. Explicitly, $F_2[k] = \frac{\sin(\pi k/2)}{\pi k}$ for $k \neq 0$ and $F_2[0] = \frac{1}{2}$: real and even in k , as the coefficients of a real even signal must be.

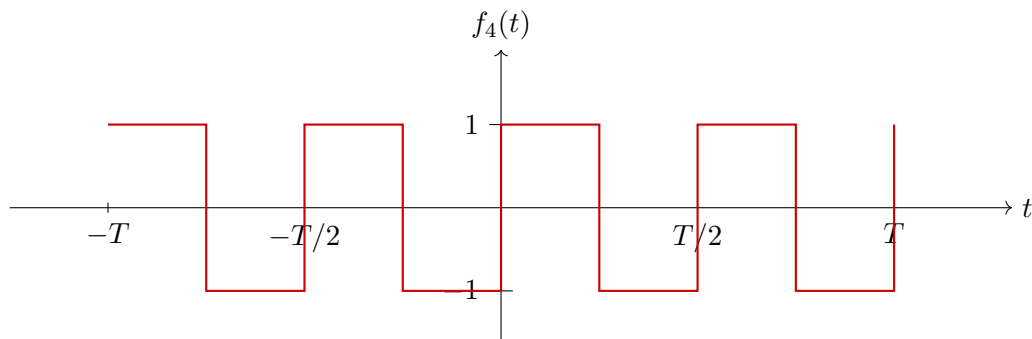
- (3) An anti-symmetric signal $f_3(t)$ can be obtained by adding a constant to $f_1(t)$ so that $f_3(t) = f_1(t) + y$. Find the constant y , and express its Fourier series coefficients $F_3[k]$ in terms of $F_1[k]$.

$$y = \boxed{-\frac{1}{2}}$$

$$F_3[k] = \boxed{F_1[k] \text{ for } k \neq 0, \quad F_3[0] = 0}$$

Solution. An odd signal has zero average, and the average of f_1 is $\frac{1}{2}$, so $y = -\frac{1}{2}$. Then $f_3 = +\frac{1}{2}$ on $(0, T/2)$ and $-\frac{1}{2}$ on $(T/2, T)$, which satisfies $f_3(-t) = -f_3(t)$. Adding a constant changes only the DC coefficient: $F_3[0] = F_1[0] + y = 0$, and $F_3[k] = F_1[k] = \frac{-j}{\pi k}$ (odd k) otherwise, purely imaginary as for an odd signal.

- (4) Signal $f_4(t) = 1 - 2f_1(-2t)$. Plot $f_4(t)$ for t from $-T$ to T below, express its Fourier series coefficients $F_4[k]$ (computed with period T) in terms of $F_1[k]$, and enter the values of $F_4[k]$ for k from -3 to 3 below.

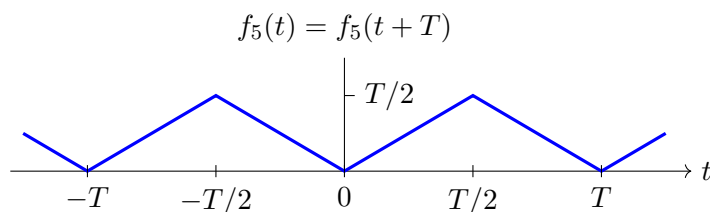


$$F_4[k] = \boxed{\delta[k] - 2F_1[-k/2] \text{ for } k \text{ even, } 0 \text{ for } k \text{ odd}}$$

k	-3	-2	-1	0	1	2	3
$F_4[k]$	0	$\frac{2j}{\pi}$	0	0	0	$-\frac{2j}{\pi}$	0

Solution. Plot: $f_1(-2t) = 1$ when $-2t \bmod T \in (0, T/2)$, i.e. when $t \bmod \frac{T}{2} \in (\frac{T}{4}, \frac{T}{2})$, so $f_4 = 1 - 2f_1(-2t)$ is $+1$ on $(0, T/4)$ and -1 on $(T/4, T/2)$, repeating every $T/2$: four cycles of an odd ± 1 square wave between $-T$ and T (drawn above). Coefficients, with period T : $g(t) = f_1(-2t) = \sum_m F_1[m] e^{j2\pi m(-2t)/T} = \sum_m F_1[-m] e^{j2\pi(2m)t/T}$, so $G[k] = F_1[-k/2]$ for even k and 0 for odd k (scaling by 2 moves index m to $2m$; the reversal sends $m \rightarrow -m$). Then $F_4[k] = \delta[k] - 2G[k]$: $F_4[0] = 1 - 2 \cdot \frac{1}{2} = 0$, $F_4[\pm 2] = -2F_1[\mp 1] = \mp \frac{2j}{\pi}$, and $F_4[\pm 1] = F_4[\pm 3] = 0$.

- (5) For the following signal $f_5(t)$, find its relationship with $f_1(t)$, express its Fourier series coefficients $F_5[k]$ in terms of $F_1[k]$, and enter the values of $F_5[k]$ for k from -3 to 3 below.



Expression describing the relationship of $f_5(t)$ with $f_1(t)$:

$$\frac{df_5}{dt} = 2f_1(t) - 1 \quad \left(\text{i.e. } f_5(t) = \int_0^t (2f_1(\tau) - 1) d\tau \right)$$

$$F_5[k] = \frac{T}{j\pi k} F_1[k] \quad (k \neq 0), \quad F_5[0] = \frac{T}{4}$$

k	-3	-2	-1	0	1	2	3
$F_5[k]$	$-\frac{T}{9\pi^2}$	0	$-\frac{T}{\pi^2}$	$\frac{T}{4}$	$-\frac{T}{\pi^2}$	0	$-\frac{T}{9\pi^2}$

Solution. f_5 has slope $+1$ on $(0, T/2)$ and -1 on $(T/2, T)$, which is exactly $2f_1(t) - 1$ (equivalently $2f_3(t)$). Derivative property: if $f_5 \leftrightarrow F_5[k]$ then $\frac{df_5}{dt} \leftrightarrow j \frac{2\pi k}{T} F_5[k]$. The coefficients of $2f_1 - 1$ are $2F_1[k]$ for $k \neq 0$, so $j \frac{2\pi k}{T} F_5[k] = 2F_1[k]$, i.e. $F_5[k] = \frac{T}{j\pi k} F_1[k] = \frac{T}{j\pi k} \cdot \frac{-j}{\pi k} = -\frac{T}{\pi^2 k^2}$ for odd k , and 0 for even $k \neq 0$. The derivative says nothing about $k = 0$: $F_5[0]$ is the average of a triangle of height $T/2$, which is $T/4$. Check: $f_5(0) = \frac{T}{4} - \frac{2T}{\pi^2} \left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) = \frac{T}{4} - \frac{2T}{\pi^2} \cdot \frac{\pi^2}{8} = 0$.

3. Sampling and Aliasing (20 points)

A signal with three frequency components f_1 , f_2 and f_3 of 300 Hz, 400 Hz, and 500 Hz is sampled with the different sampling rates below. Please include units in your answers.

- (1) When the signal is sampled with f_{s1} of 1000 Hz, find the corresponding base-band frequencies Ω_1 , Ω_2 , and Ω_3 of the discrete-time signal.

$$\Omega_1 = \boxed{0.6\pi \text{ rad/sample}}$$

$$\Omega_2 = \boxed{0.8\pi \text{ rad/sample}}$$

$$\Omega_3 = \boxed{\pi \text{ rad/sample}}$$

Solution. A tone at f sampled at f_s becomes $\cos(2\pi f n / f_s)$, so $\Omega = 2\pi f / f_s$: $2\pi \cdot \frac{300}{1000} = 0.6\pi$, $2\pi \cdot \frac{400}{1000} = 0.8\pi$, $2\pi \cdot \frac{500}{1000} = \pi$ rad/sample. All three are within $[0, \pi]$ (500 Hz sits exactly at $f_s/2$), so nothing folds.

- (2) When the signal is sampled with another sampling rate f_{s2} to obtain a discrete-time signal, then played back using the same sampling rate f_{s2} , it was found that only two of the original frequencies were played out. Identify a possible f_{s2} and the corresponding two frequencies being played out.

$$f_{s2} = \boxed{900 \text{ Hz (one possibility)}}$$

$$f_{out1} = \boxed{300 \text{ Hz}}$$

$$f_{out2} = \boxed{400 \text{ Hz}}$$

Solution. After sampling at f_s and playing back at f_s , a component at f comes out at the frequency in $[0, f_s/2]$ that f folds onto, $\min_m |f - m f_s|$. Two outputs from three inputs means two components land on the same frequency. With $f_{s2} = 900$ Hz: 300 and 400 Hz are below $f_s/2 = 450$ and come out unchanged, while $500 \rightarrow |500 - 900| = 400$ Hz, coinciding with f_2 . Outputs: 300 Hz and 400 Hz. (Another valid choice: $f_{s2} = 700$ Hz gives $400 \rightarrow 300$, $500 \rightarrow 200$, outputs 200 and 300 Hz.)

- (3) When the signal is sampled with a third sampling rate f_{s3} to obtain a discrete-time signal, then played back using the same sampling rate f_{s3} , it was found that the lowest frequency being played out, f_{out1} , is only 100 Hz. Find the sampling rate f_{s3} and the other frequencies that are being played out.

$$f_{s3} = \boxed{600 \text{ Hz}}$$

$$f_{out2} = \boxed{200 \text{ Hz}}$$

$$f_{out3} = \boxed{300 \text{ Hz}}$$

Solution. A lowest output of 100Hz means some component folds to 100Hz: $|f - mf_s| = 100$ for one of $f \in \{300, 400, 500\}$. The candidates are $f_s = 600$ ($500 \rightarrow 100$), 500 ($400 \rightarrow 100$), 400 and 200 ($300 \rightarrow 100$ and $500 \rightarrow 100$). Checking the other two components: at $f_s = 600$ Hz, $400 \rightarrow |400 - 600| = 200$ and 300 stays at 300 (it sits at $f_s/2$), giving three distinct outputs 100, 200, 300 Hz; at 500 Hz the 500 Hz tone goes to DC; at 400 and 200 Hz two tones land on 100 Hz and one on DC. So $f_{s3} = 600$ Hz, $f_{out2} = 200$ Hz, $f_{out3} = 300$ Hz.

4. Discrete Varieties (30 points)

- (1) Signal $x[n] = 1 + \sin\left(\frac{2\pi n}{5}\right) + 2\cos\left(\frac{2\pi n}{5}\right) + 3\cos\left(\frac{4\pi n}{5} + \frac{\pi}{2}\right)$. Find its DT Fourier series coefficients when computed with period $N = 5$, and fill in the $X[k]$ values for the corresponding k in the table below.

k	0	1	2	3	4	5	6	7	8	9
$X[k]$	1	$1 - \frac{j}{2}$	$\frac{3j}{2}$	$-\frac{3j}{2}$	$1 + \frac{j}{2}$	1	$1 - \frac{j}{2}$	$\frac{3j}{2}$	$-\frac{3j}{2}$	$1 + \frac{j}{2}$

Solution. $x[n] = 1 + \frac{1}{2j} (e^{j2\pi n/5} - e^{-j2\pi n/5}) + (e^{j2\pi n/5} + e^{-j2\pi n/5}) + \frac{3}{2} (e^{j\pi/2} e^{j4\pi n/5} + e^{-j\pi/2} e^{-j4\pi n/5})$, so $X[0] = 1$, $X[\pm 1] = 1 \mp \frac{j}{2}$, $X[\pm 2] = \pm \frac{3j}{2}$, with $X[-1] = X[4]$ and $X[-2] = X[3]$. Convention: $x[n] = \sum_{k=\langle N \rangle} X[k] e^{j2\pi kn/N}$; $X[k]$ is 5-periodic, shown for $k = 0, \dots, 9$.

- (2) If the same signal $x[n]$ above is computed with period $N = 10$, find the corresponding Fourier series coefficients $X_{10}[k]$ and fill in the table below.

k	0	1	2	3	4	5	6	7	8	9
$X_{10}[k]$	1	0	$1 - \frac{j}{2}$	0	$\frac{3j}{2}$	0	$-\frac{3j}{2}$	0	$1 + \frac{j}{2}$	0

Solution. Same signal, so the same exponentials: $e^{j2\pi kn/5} = e^{j2\pi(2k)n/10}$. Each harmonic k of the $N = 5$ series is harmonic $2k$ of the $N = 10$ series, so $X_{10}[2k] = X[k]$ and the odd-indexed coefficients are zero: $X_{10}[0] = 1$, $X_{10}[2] = 1 - \frac{j}{2}$, $X_{10}[4] = \frac{3j}{2}$, $X_{10}[6] = X_{10}[-4] = -\frac{3j}{2}$, $X_{10}[8] = X_{10}[-2] = 1 + \frac{j}{2}$.

- (3) Consider a time-stretched version of the signal, $y[n] = \begin{cases} x\left[\frac{n}{2}\right], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$, computed with period $N = 10$. Find the corresponding Fourier series coefficients $Y_{10}[k]$ and fill in the table below.

k	0	1	2	3	4	5	6	7	8	9
$Y_{10}[k]$	$\frac{1}{2}$	$\frac{1}{2} - \frac{j}{4}$	$\frac{3j}{4}$	$-\frac{3j}{4}$	$\frac{1}{2} + \frac{j}{4}$	$\frac{1}{2}$	$\frac{1}{2} - \frac{j}{4}$	$\frac{3j}{4}$	$-\frac{3j}{4}$	$\frac{1}{2} + \frac{j}{4}$

Solution. Only the even samples $n = 2m$ are nonzero, and $y[2m] = x[m]$: $Y_{10}[k] = \frac{1}{10} \sum_{n=0}^9 y[n]e^{-j2\pi kn/10} = \frac{1}{10} \sum_{m=0}^4 x[m]e^{-j2\pi k(2m)/10} = \frac{1}{2} \cdot \frac{1}{5} \sum_{m=0}^4 x[m]e^{-j2\pi km/5} = \frac{1}{2}X[k]$, with $X[k]$ taken 5-periodic. The spectrum of x is repeated twice and halved; the table is half of the table in (1).

- (4) Signal $z[n] = (-1)^n \cdot x[n]$ is computed with period $N = 10$. Find the corresponding Fourier series coefficients $Z_{10}[k]$ and fill in the table below.

k	0	1	2	3	4	5	6	7	8	9
$Z_{10}[k]$	0	$-\frac{3j}{2}$	0	$1 + \frac{j}{2}$	0	1	0	$1 - \frac{j}{2}$	0	$\frac{3j}{2}$

Solution. $(-1)^n = e^{j\pi n} = e^{j2\pi \cdot 5n/10}$, so multiplying by $(-1)^n$ shifts every harmonic of the $N = 10$ series by 5: $z[n] = \sum_k X_{10}[k] e^{j2\pi(k+5)n/10}$, i.e. $Z_{10}[k] = X_{10}[k-5]$. The table is the table of (2) rotated by five places: the nonzero entries move from $k = 0, 2, 4, 6, 8$ to $k = 5, 7, 9, 1, 3$.

- (5) Signal $x'[n]$ has the same fundamental period as signal $x[n]$, and its Fourier series coefficients $X'[k]$ (computed with $N = 5$) have the following relationship with $X[k]$:

$$X'[k] = \operatorname{Re}(X[k])$$

Write the expression for signal $x'[n]$.

$$x'[n] = \boxed{1 + 2 \cos\left(\frac{2\pi n}{5}\right)}$$

Solution. $\operatorname{Re}(X[k]) = \frac{1}{2}(X[k] + X^*[k])$. Conjugating the coefficients of a real signal reverses time, $X^*[k] \leftrightarrow x^*[-n] = x[-n]$, so $x'[n] = \frac{1}{2}(x[n] + x[-n])$: the even part of x . The $\sin(2\pi n/5)$ term and $3 \cos(4\pi n/5 + \pi/2) = -3 \sin(4\pi n/5)$ are odd and drop out, leaving $x'[n] = 1 + 2 \cos(2\pi n/5)$. Directly from the table: $\operatorname{Re} X[k]$ is 1, 1, 0, 0, 1 for $k = 0, \dots, 4$, which synthesises $1 + e^{j2\pi n/5} + e^{-j2\pi n/5}$.

Worksheet (intentionally blank)

Worksheet (intentionally blank)