6.300 Signal Processing

Week 2, Lecture A:
Continuous-Time Fourier Series (Trig Form)

- Fourier Series
- Convergence of Fourier Series
- Symmetry of Fourier Series

Lecture/Recitation Notes, office hour info are available on CATSOOP:

https://sigproc.mit.edu/fall26

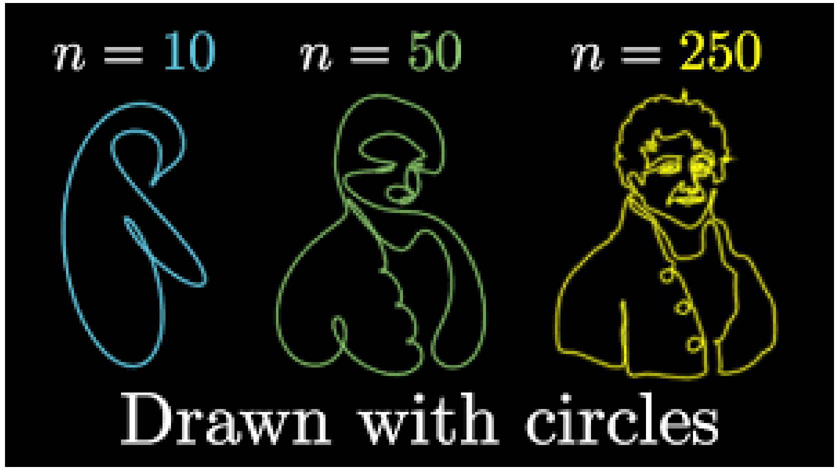
Fourier Series

Series: representing a signal as a sum of simpler signals.

- Taylor or Maclaurin's series

Function	Maclaurin Series
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
$\sin x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
$\cos x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad (\text{if } -1 < x < 1)$
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (\text{if } -1 < x \leq 1)$

- Draw only with circles

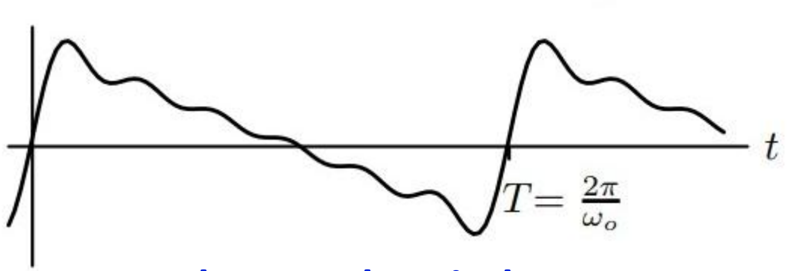


- Fourier series are sums of harmonically related sinusoids:

$$f(t) = f(t+T) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)\right)$$

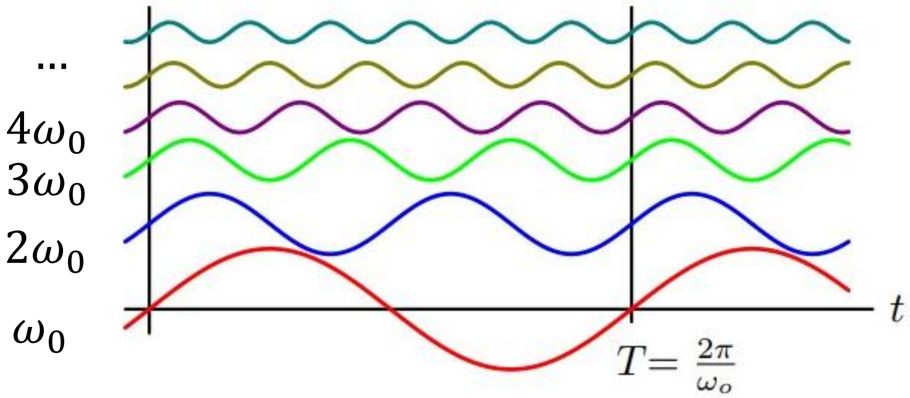
Fourier Series: Express periodic signals as a sum of sinusoids

Periodic signal: $f(t) = f(t+T)$



- **Fundamental period:** T
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T}$

Basis function $\cos(k\omega_0 t), sin(k\omega_0 t)$



Harmonically related: $\omega = k\omega_0$

Decomposition:

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

$$c_0 = \frac{1}{T} \int_T f(t)\, dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_o t)\, dt; \;\; k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_o t)\, dt; \;\; k = 1, 2, 3, \ldots$$

$$\boxed{\text{CTFS: } f(t) \to c_k, d_k}$$

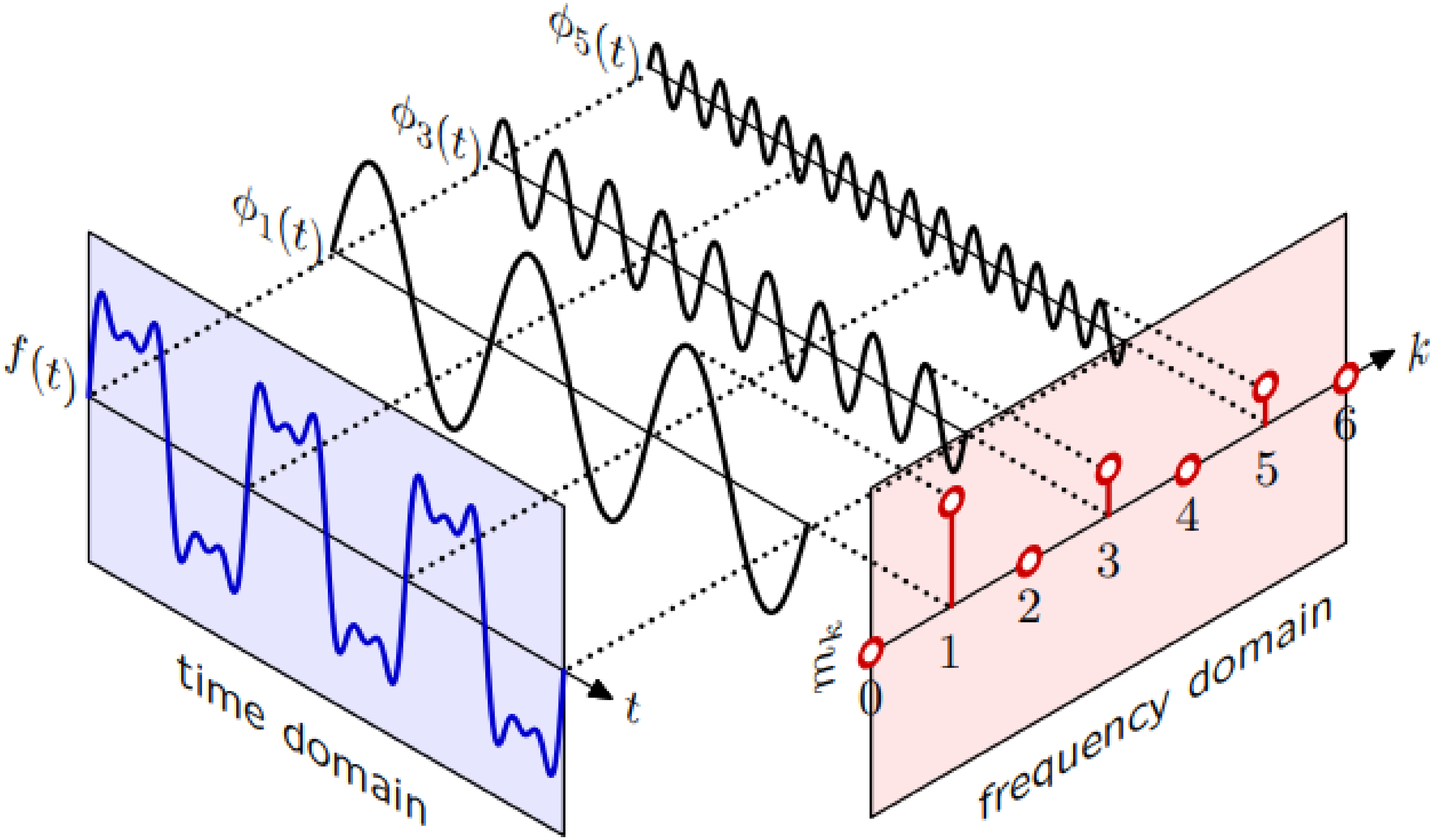
Two different ways of looking at a signal

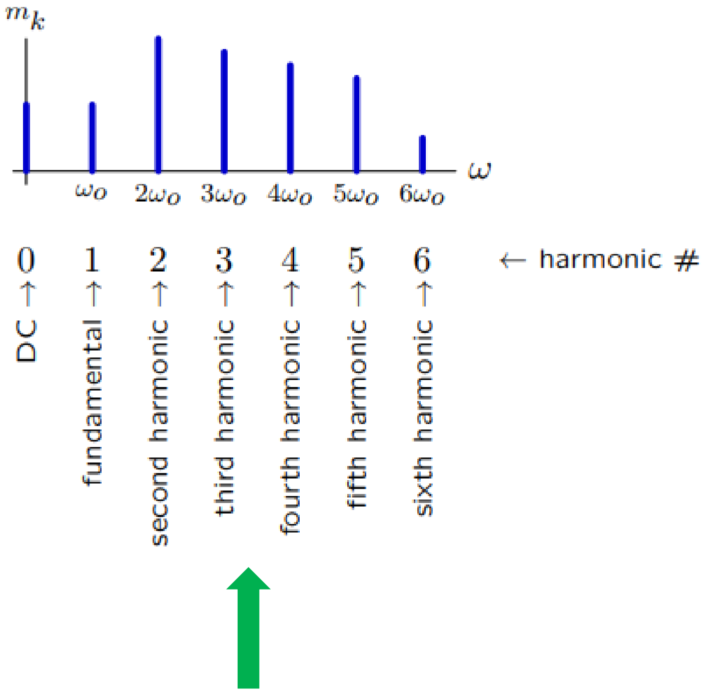
Representing a function $f(t)$ as a sum of harmonics

$$f(t) = \sum_{k=0}^{\infty} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) = \sum_{k=0}^{\infty} m_k \cos\left(k\omega_o t + \phi_k\right)$$

$$m_k^2 = c_k^2 + d_k^2,$$
$$tan\phi_k = d_k / c_k$$

provides an alternative view of a signal (based on frequency).

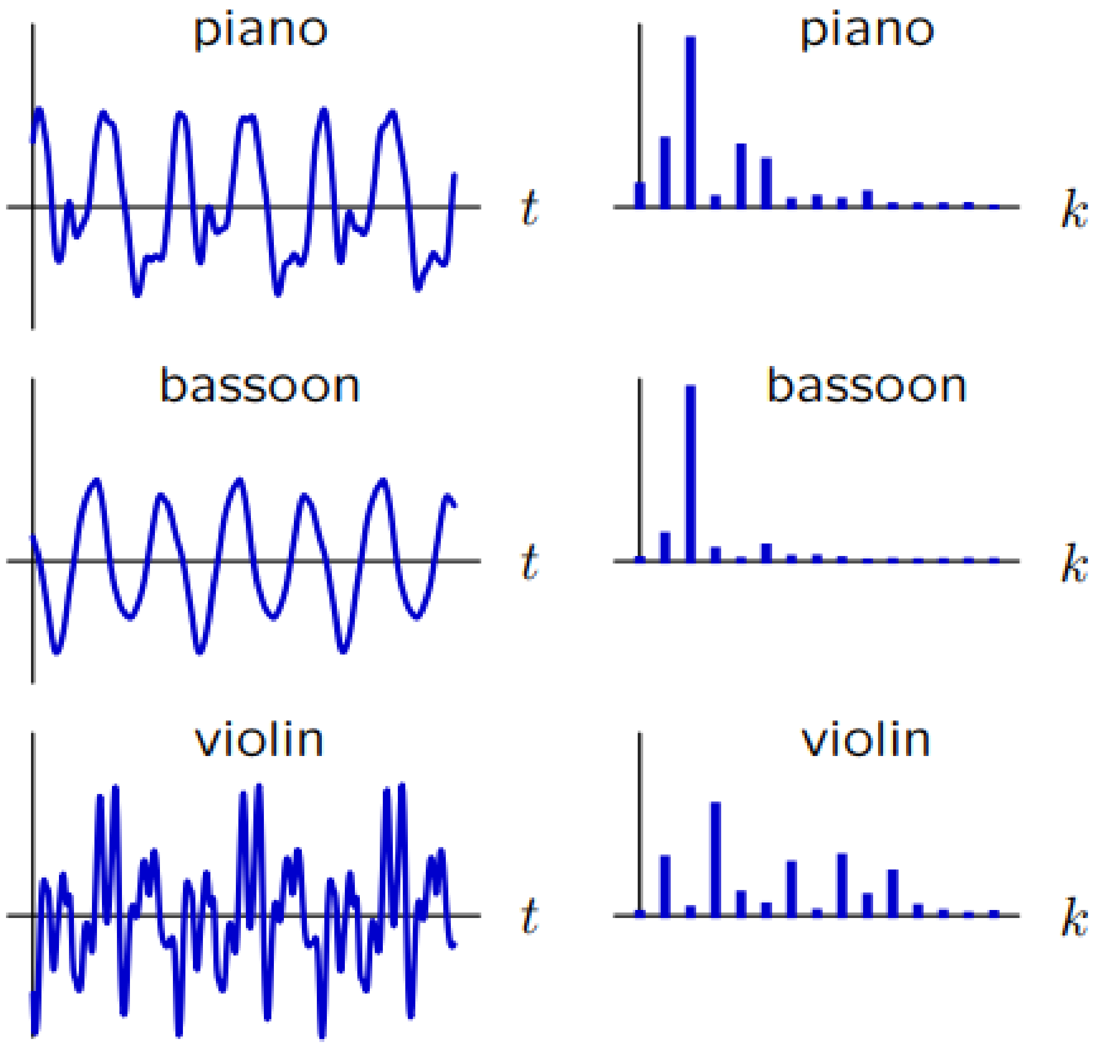




Harmonic Structure

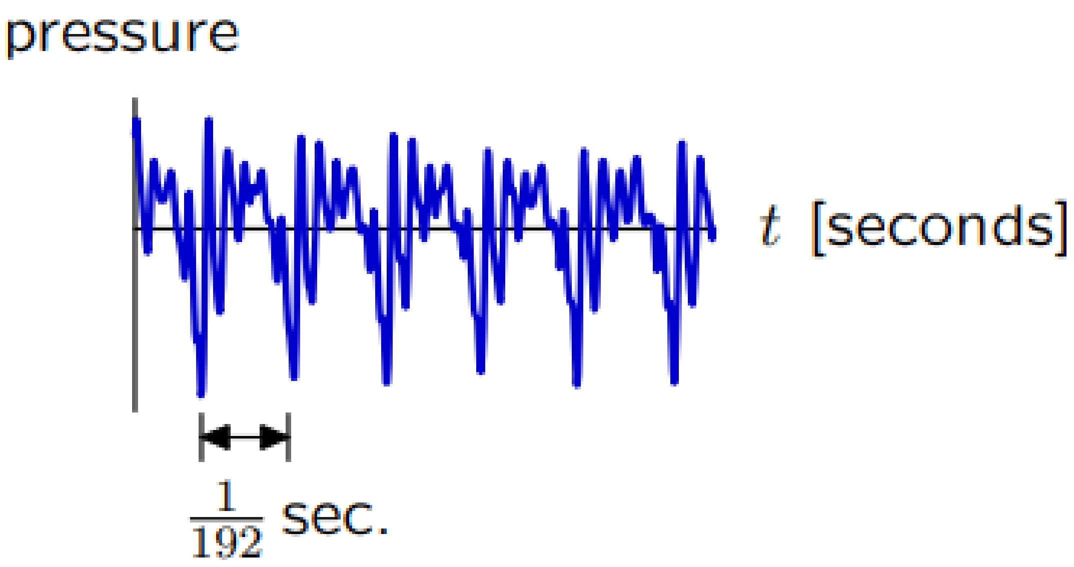
Harmonic Structure

Time- and frequency-based views highlight different aspects of a signal.



Speech Sounds as Signals

Vowel sounds can also be represented as signals.

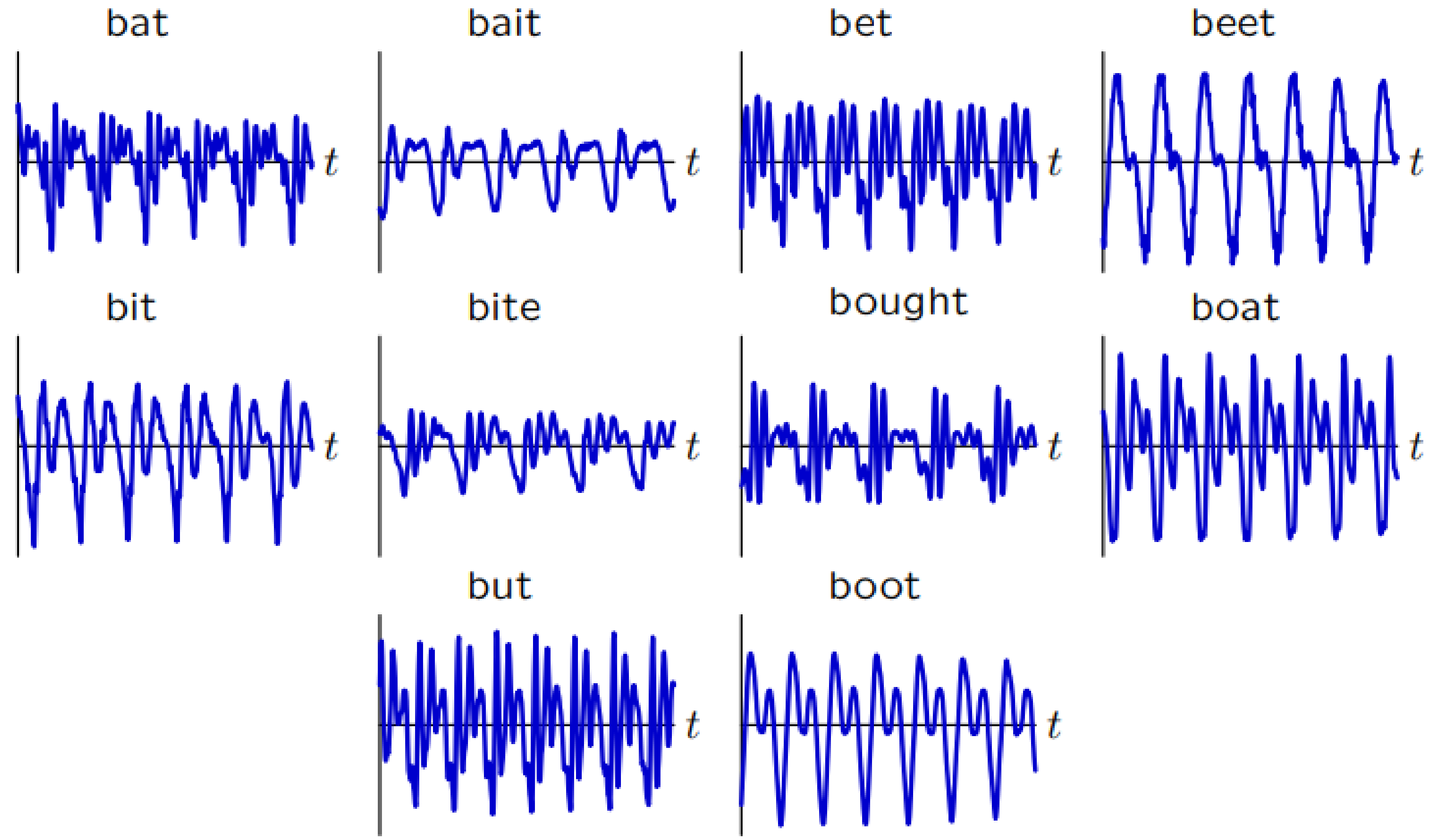


The vowel part of this signal (the "ah" in "bat") is nearly periodic with a frequency of 192 Hz.

Other information (such as which vowel is uttered) is less obvious.

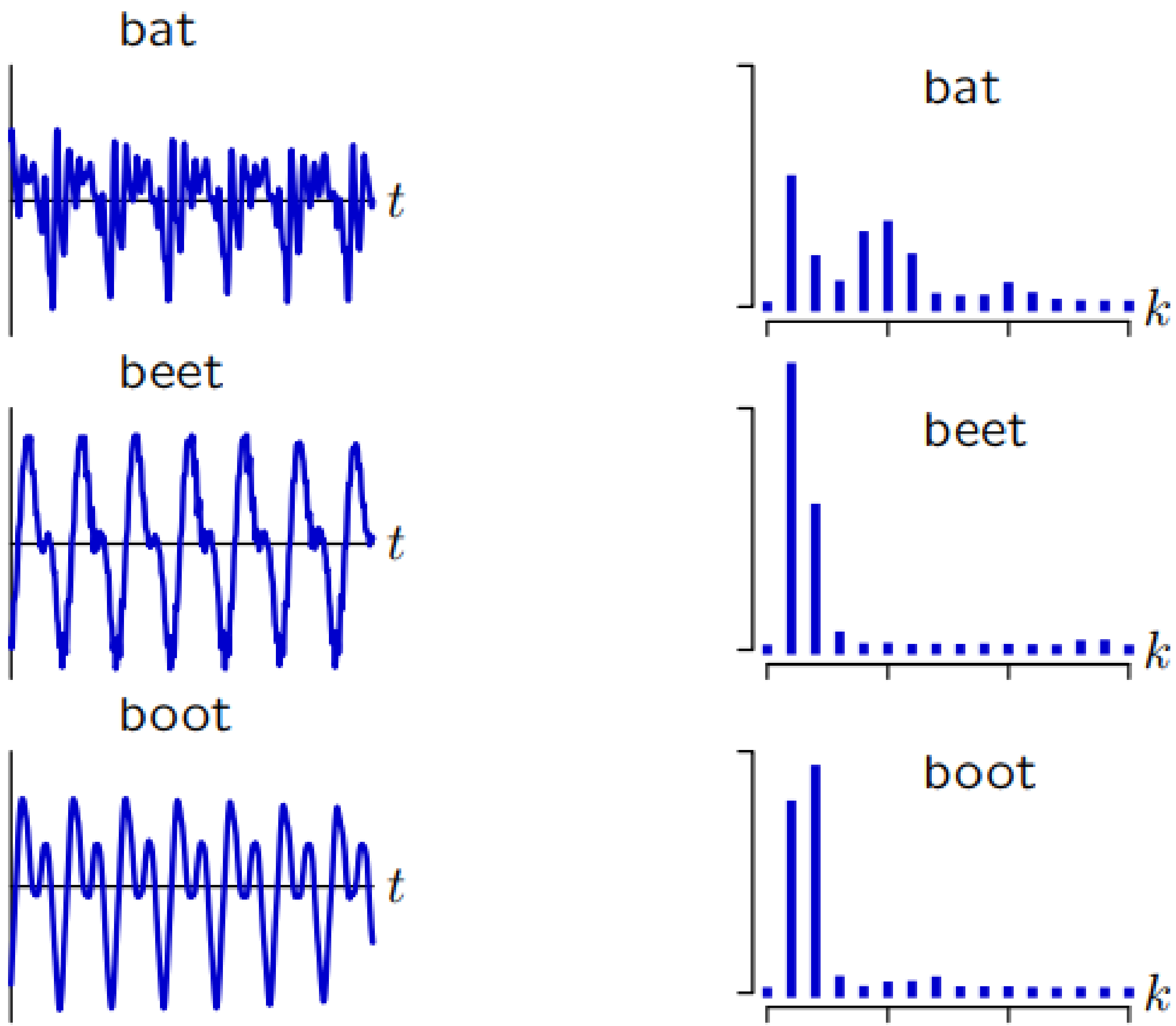
Speech Sounds as Signals

All of the following sounds have the same pitch, but they sound different.



Harmonic Structure

Harmonic content is natural way to describe vowel sounds.



Why focus on Fourier Series

- What's so special about sines and cosines?

➢ Sinusoidal functions have interesting mathematical properties.

  ➢ Harmonically related sinusoids are orthogonal to each other over [0, T)

- Average over a period:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) dt = 0 \qquad \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right) dt = \begin{cases} T & if\ k = 0 \\ 0 & otherwise \end{cases}$$

- Orthogonality of the basis functions:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) dt = 0$$

k and m are positive integers

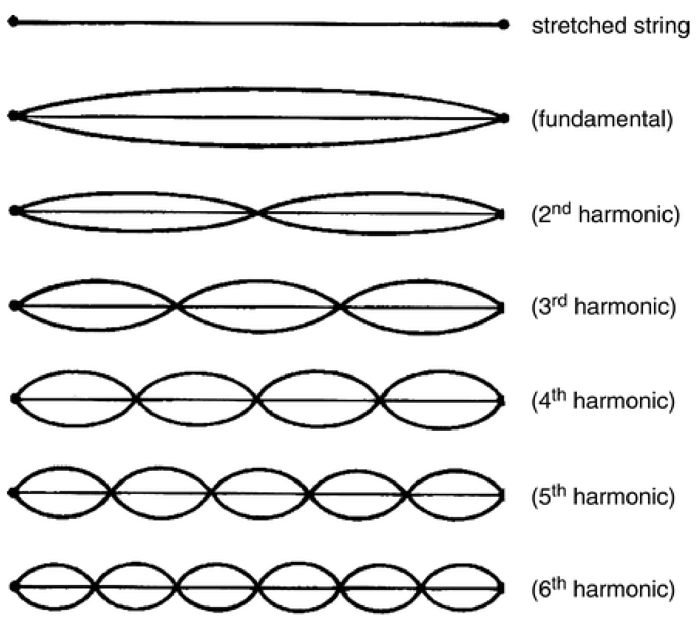
$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) dt = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) \sin\left(\frac{2\pi m}{T}t\right) dt = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

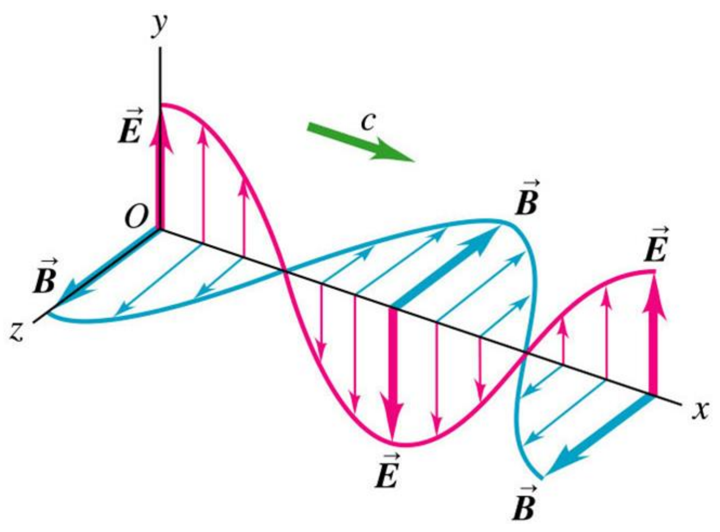
Why focus on Fourier Series

- Sines and cosines have interesting mathematical properties – orthogonality.
- Sines and cosines also play important roles in physics – especially the physics of waves.

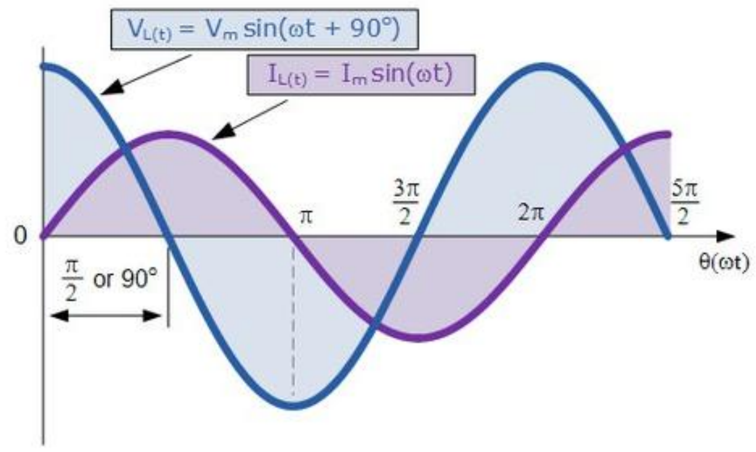
Vibrating string



Light waves



Electrical waves



Check yourself!

- What are the Fourier series coefficients associated with the following signal?

$$f(t) = 0.8\sin(6\pi t) - 0.3\cos(6\pi t) + 0.75\cos(12\pi t)$$

$$\omega_o = \ ?$$

$$c_k = ?$$

$$d_k = ?$$

Check yourself!

- What are the Fourier series coefficients associated with the following signal?

$$f(t) = 0.8\sin(6\pi t) - 0.3\cos(6\pi t) + 0.75\cos(12\pi t)$$

$$\omega_o = 6\pi$$

$$c_0 = 0$$

$$c_1 = -0.3$$

$$c_2 = 0.75$$

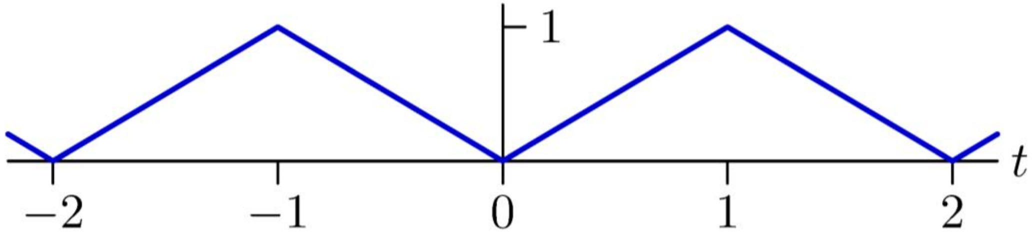
$$d_1 = 0.8$$

All the other c_k 's and d_k 's are zero.

Example of synthesis

Find the Fourier series coefficients for the following triangle wave:

$$f(t) = f(t+2)$$



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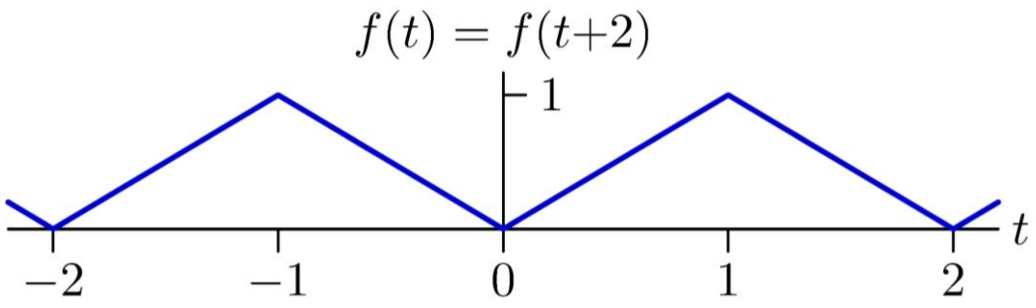


$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$\boxed{\int_0^1 t\cos(k\pi t)dt = \frac{\pi kt \cdot \sin(\pi kt) + \cos(\pi kt)}{\pi^2 k^2}\Bigg|_0^1}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt = 2\int_0^1 t\cos(\pi kt)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$d_k = 0$ (by symmetry)

$$\boxed{\int_0^1 t\sin(k\pi t)dt = \frac{\sin(\pi kt) - \pi kt \cdot \cos(\pi kt)}{\pi^2 k^2}\Bigg|_0^1}$$

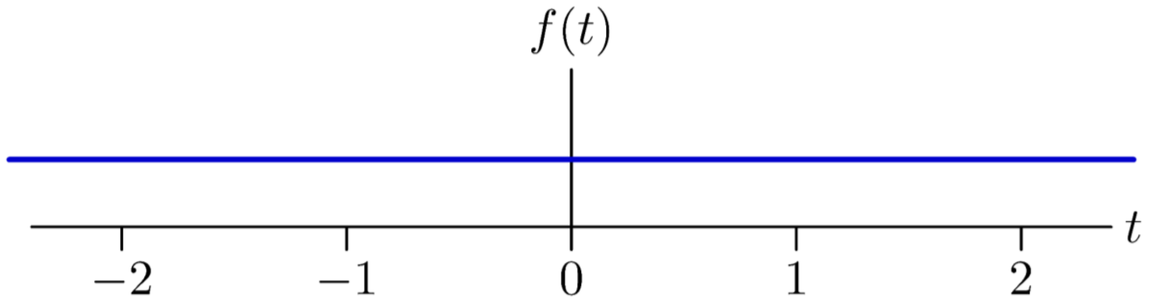
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{0} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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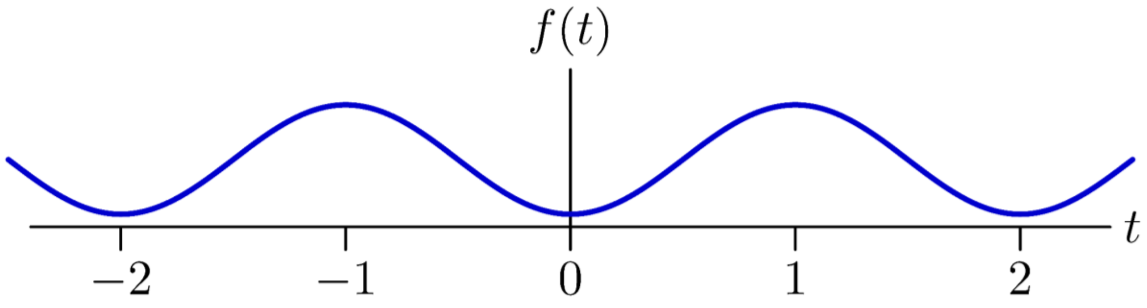
$$f(t) = \frac{1}{2} - \sum_{\substack{k = 1 \\ k \text{ odd}}}^{1} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



Example of synthesis

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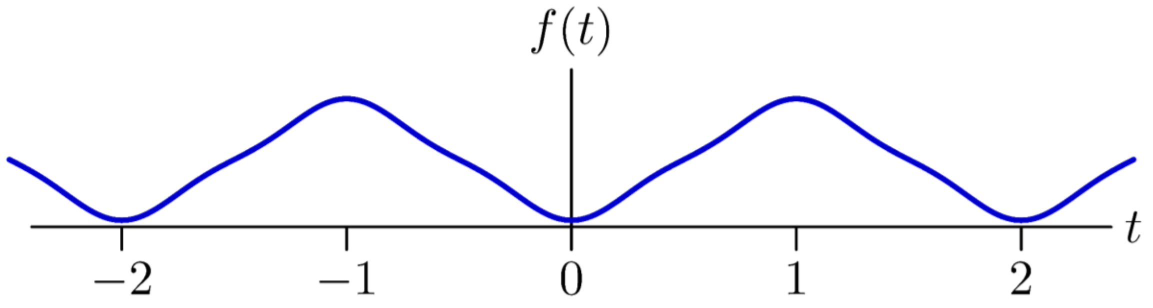
Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{3} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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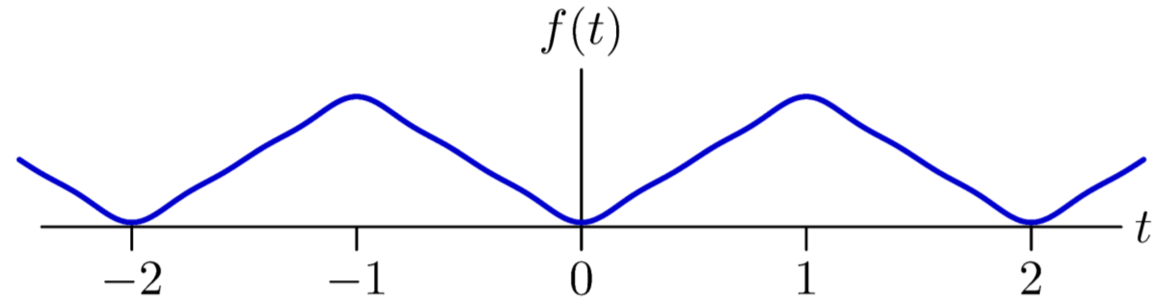
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{5} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



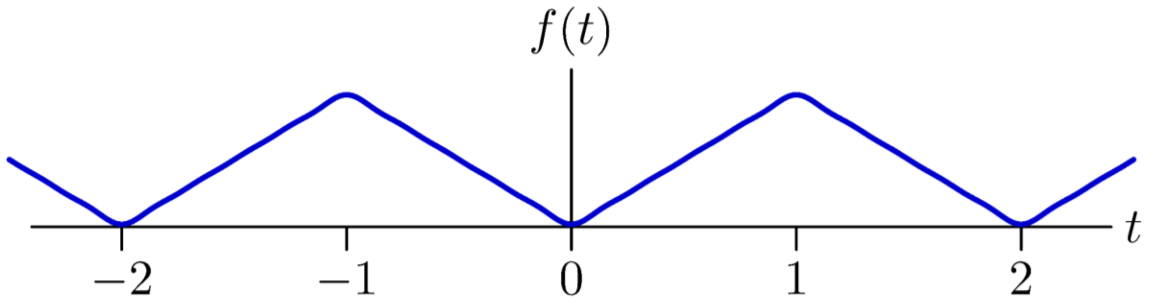
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



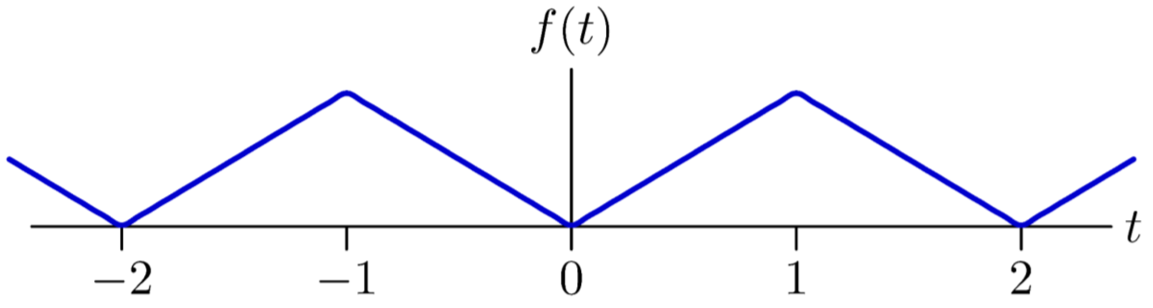
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



Example of synthesis

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Start with the Fourier coefficients

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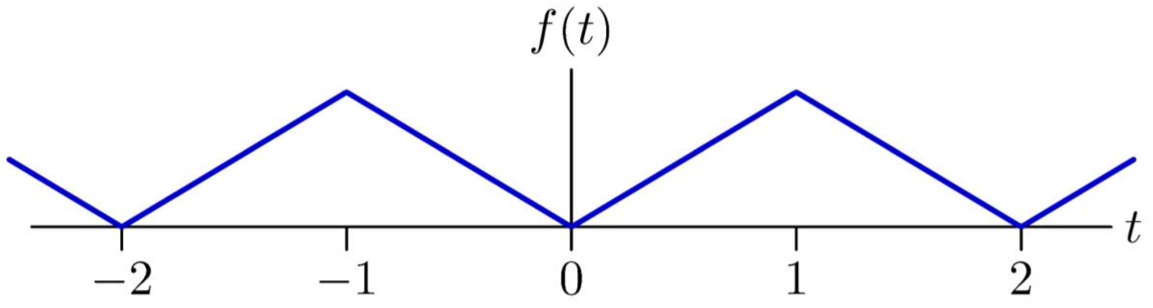
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

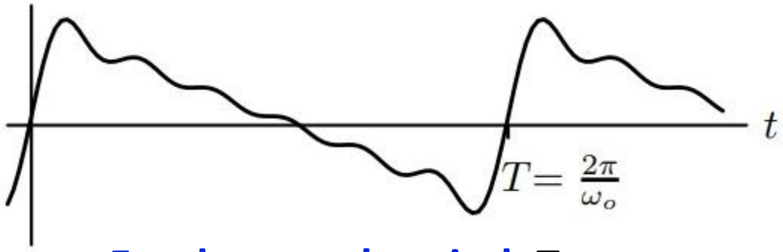


The synthesized function approaches original as number of terms increases.

Can Fourier Series approximate any periodic signals?

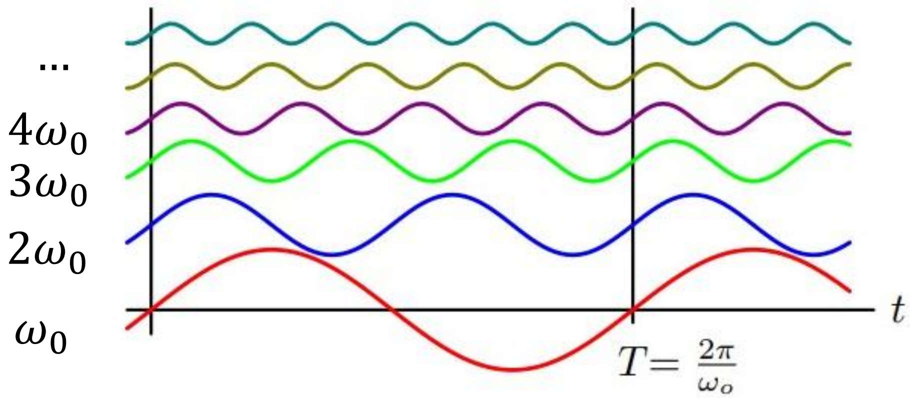
$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

Periodic signal: $f(t) = f(t + T)$



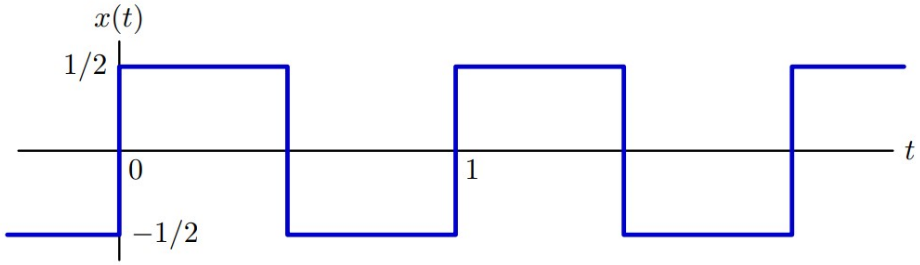
- **Fundamental period:** T
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T}$

Basis function $\cos(k\omega_0 t)$



Harmonically related: $\omega = k\omega_0$

What about discontinuous functions?

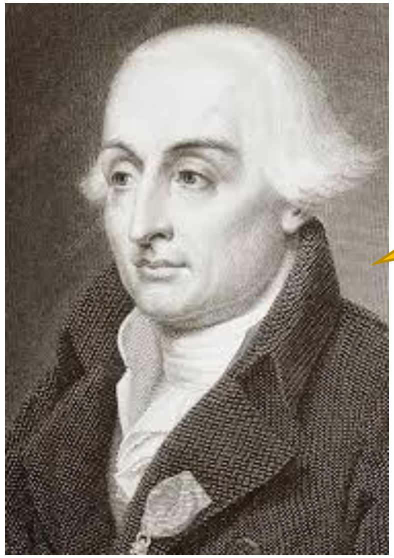


A debate two hundred years ago…

Fourier defended the idea that such a series is meaningful.
Lagrange ridiculed the idea that discontinuities could be generated from a sum of continuous signals.

Not a problem

Jean-Baptiste Joseph Fourier



Joseph-Louis Lagrange

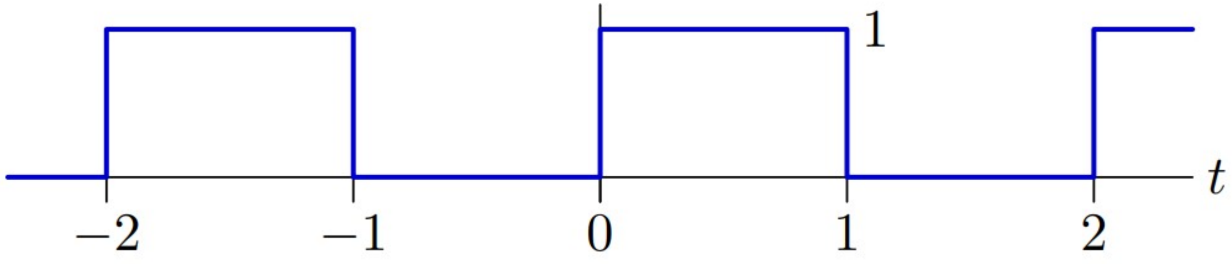
No way

Q: What do you think?

We can test this idea empirically – using computation

- Find the Fourier series coefficients for the following square wave:

$$f(t) = f(t+2)$$



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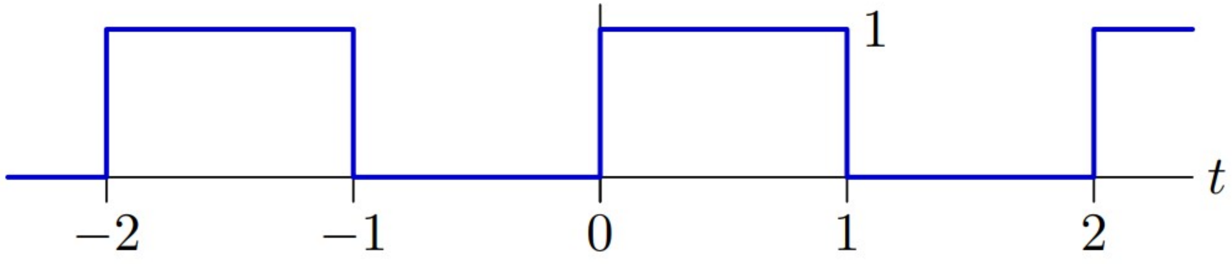

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$$\omega_o = \frac{2\pi}{T} = \pi$$

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$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, .. \\ 0 & \text{otherwise} \end{cases}$$

Fourier Synthesis of a Square Wave

- Generate f(t) from the Fourier coefficients in the previous slide:

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

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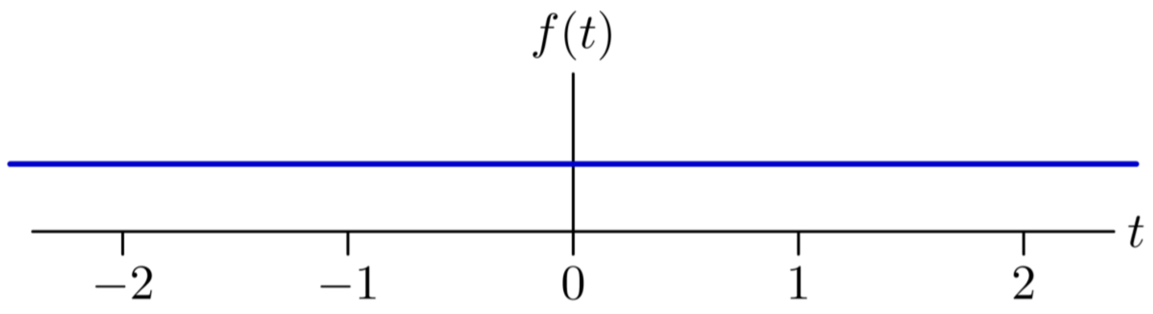

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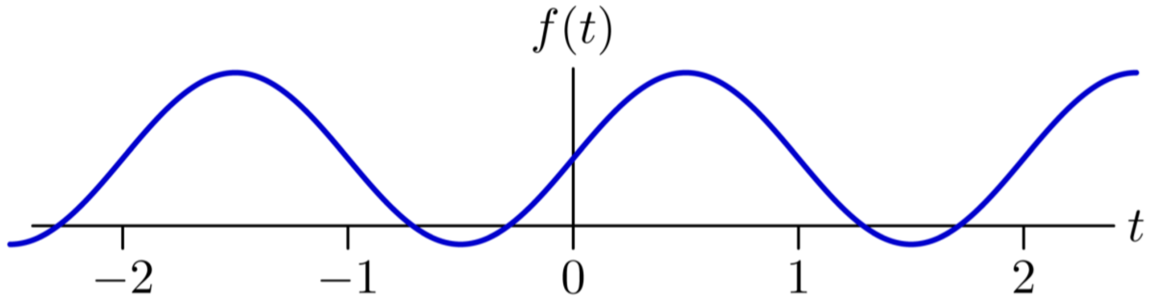

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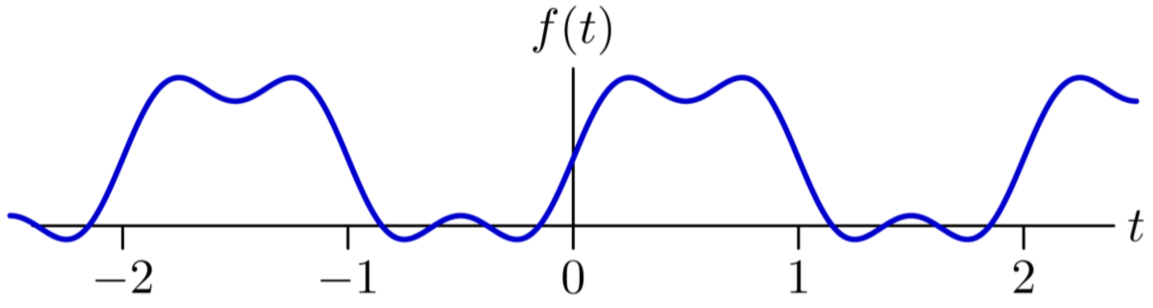

Fourier Synthesis of a Square Wave

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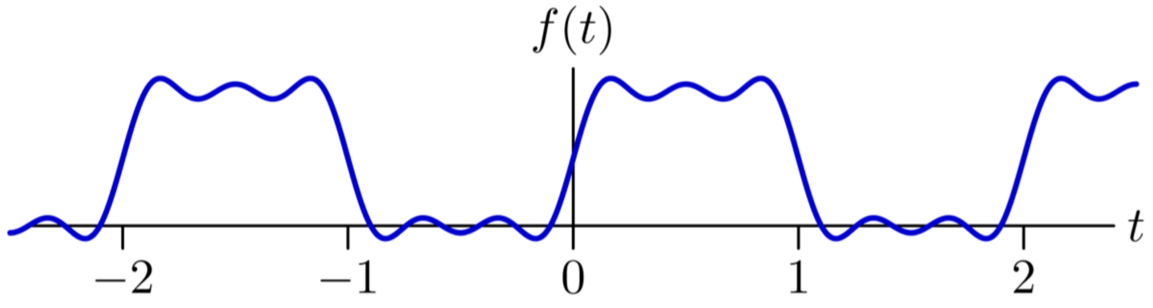

Fourier Synthesis of a Square Wave

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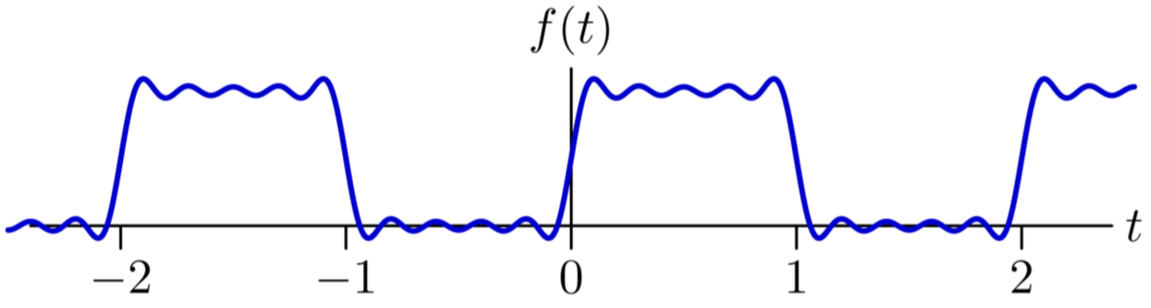
$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{2}{k\pi} \sin(k\pi t)$$

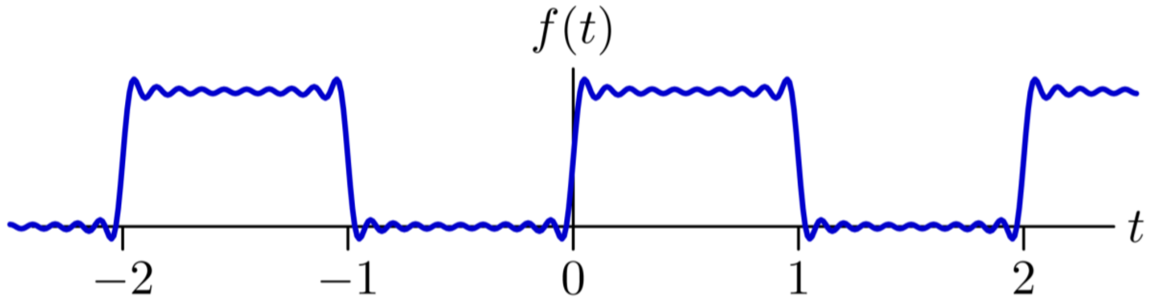


Fourier Synthesis of a Square Wave

- Generate f(t) from the Fourier coefficients in the previous slide:

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{2}{k\pi} \sin(k\pi t)$$

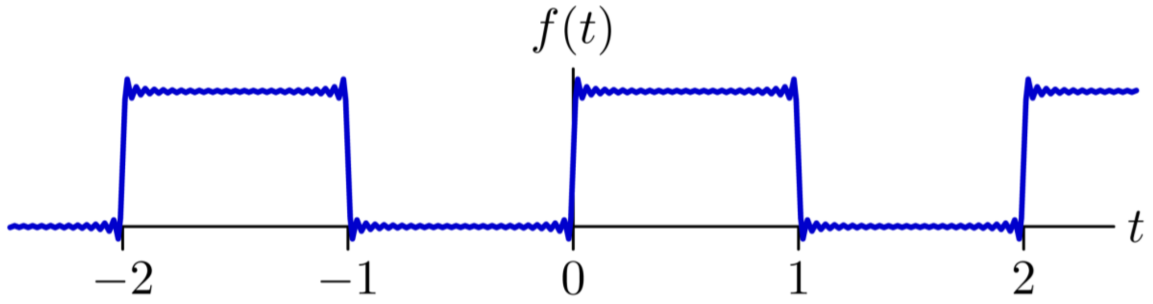


Fourier Synthesis of a Square Wave

- Generate f(t) from the Fourier coefficients in the previous slide:

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} + \sum_{\substack{k\,=\,1 \\ k\ \text{odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k\,=\,1 \\ k\ \text{odd}}}^{49} \frac{2}{k\pi} \sin(k\pi t)$$

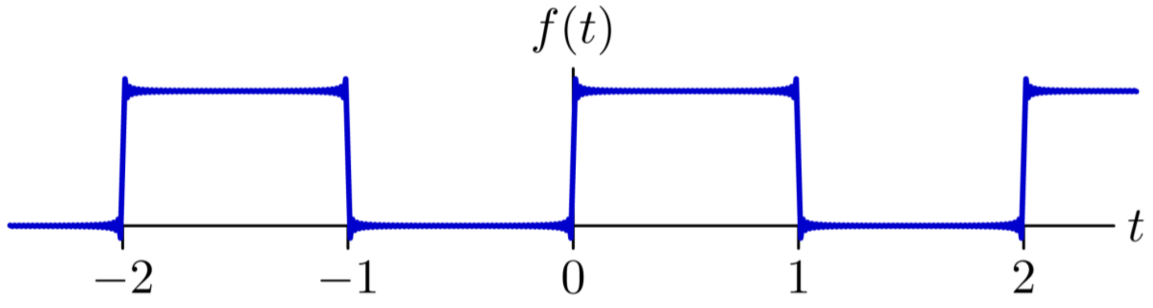


Fourier Synthesis of a Square Wave

- Generate f(t) from the Fourier coefficients in the previous slide:

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{2}{k\pi} \sin(k\pi t)$$

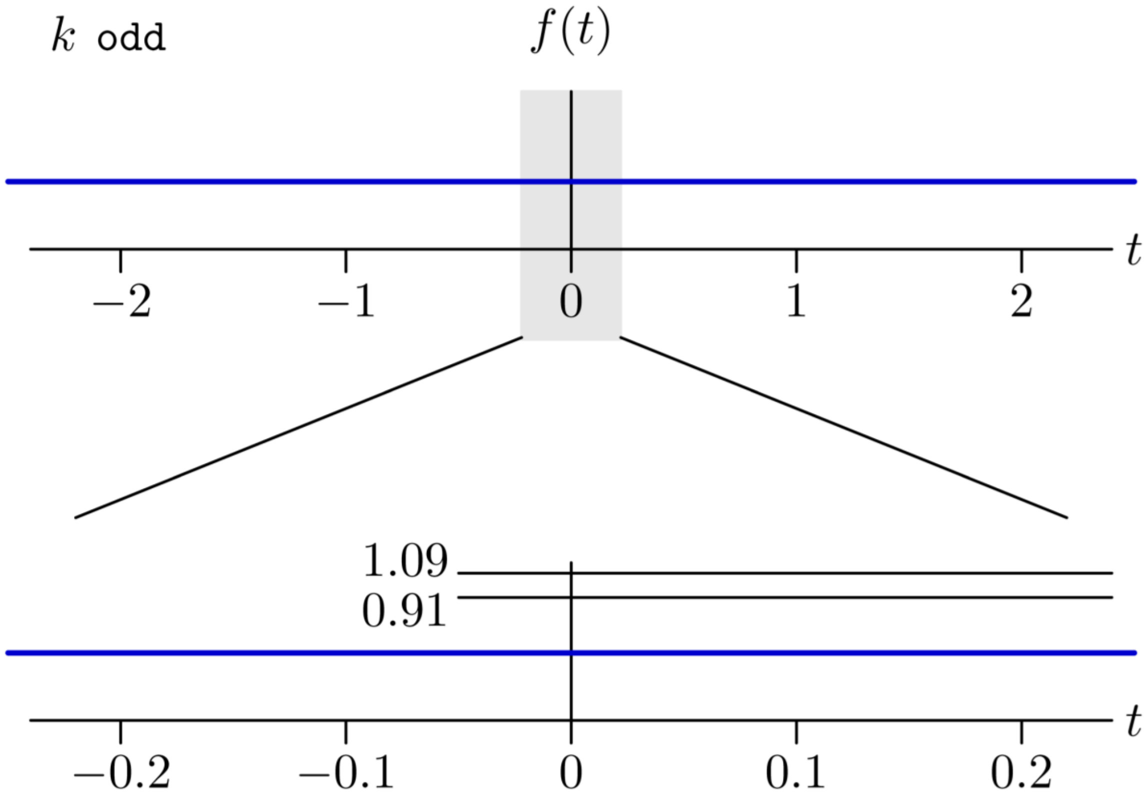


The synthesized function approaches original as number of terms increases.

Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

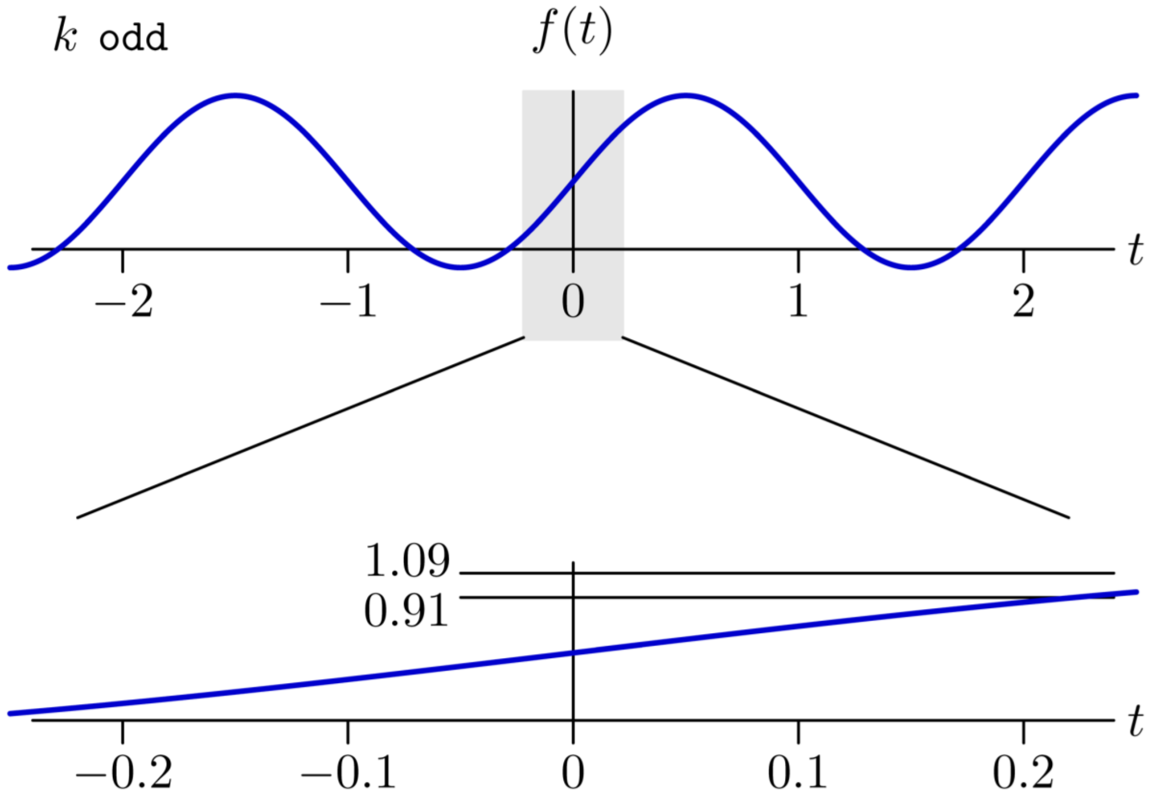
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{0} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

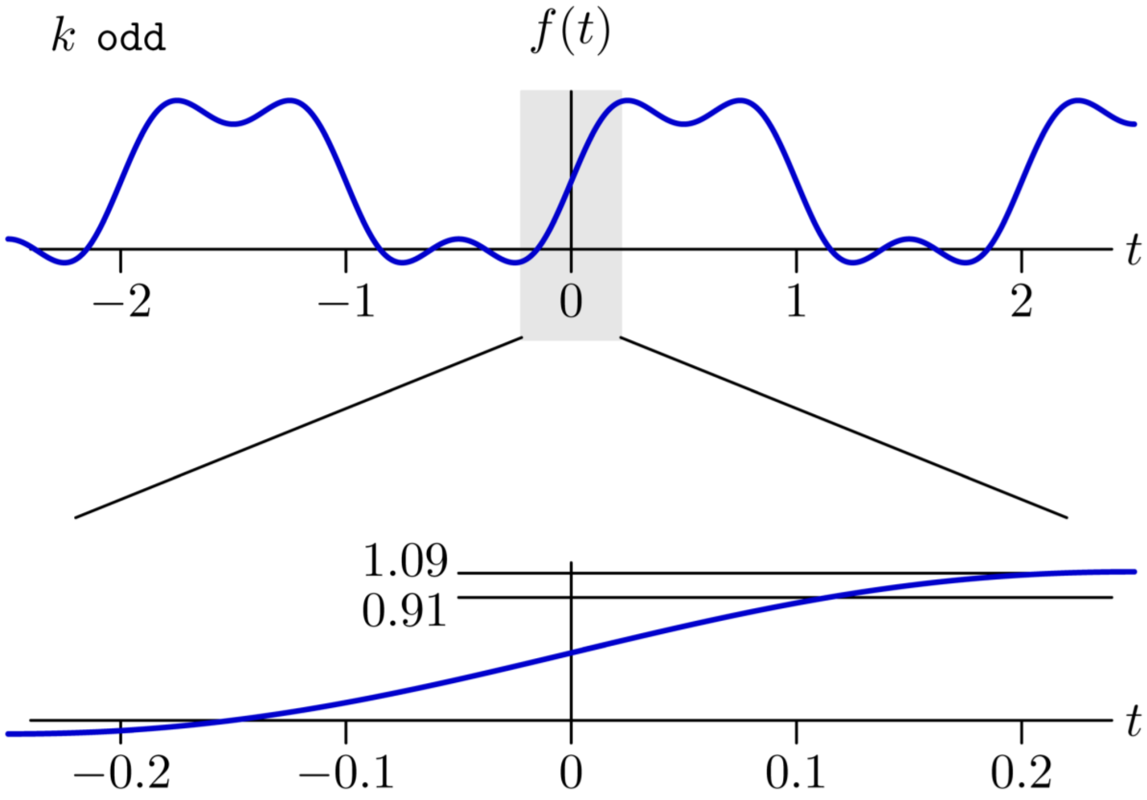
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{1} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

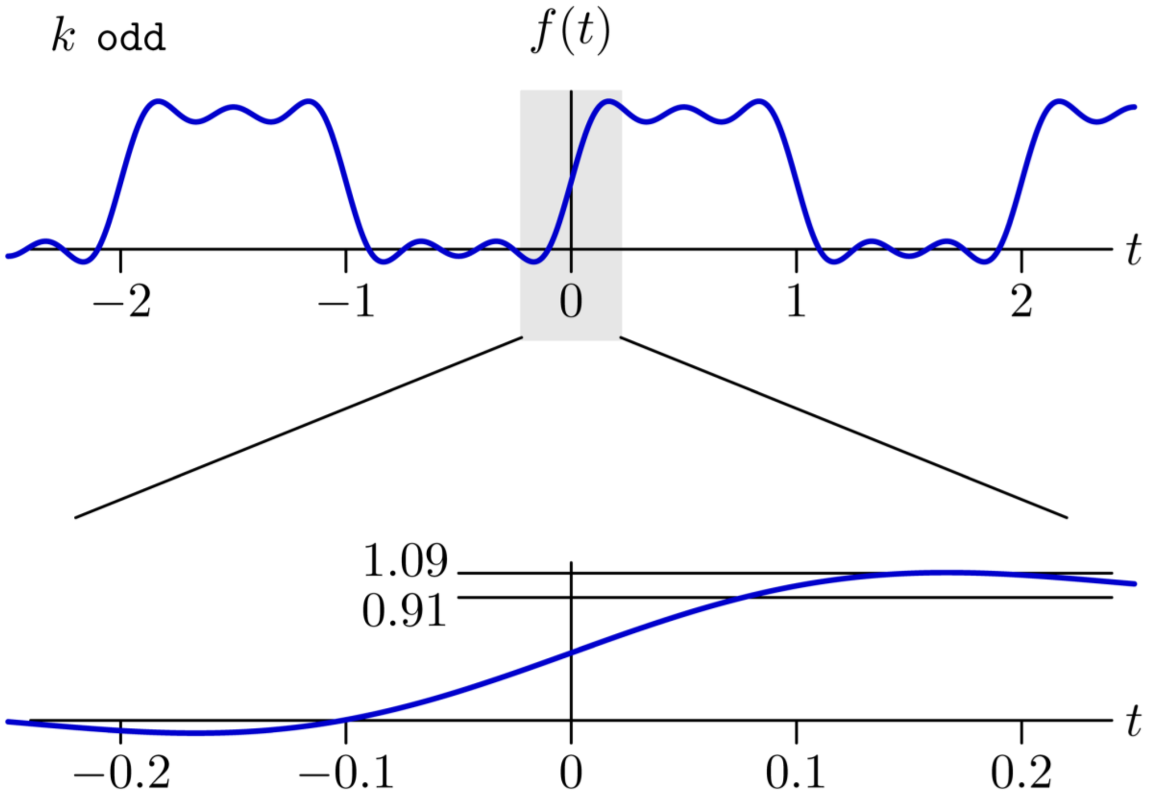
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{3} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

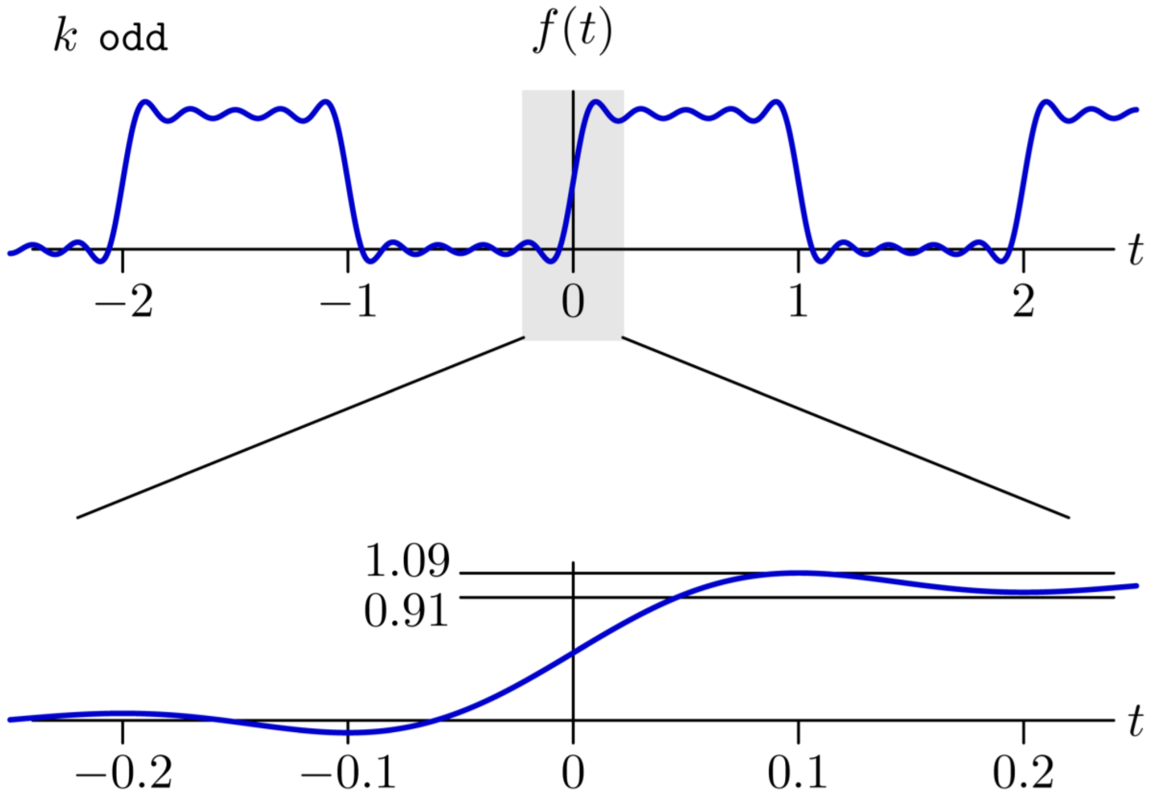
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{5} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

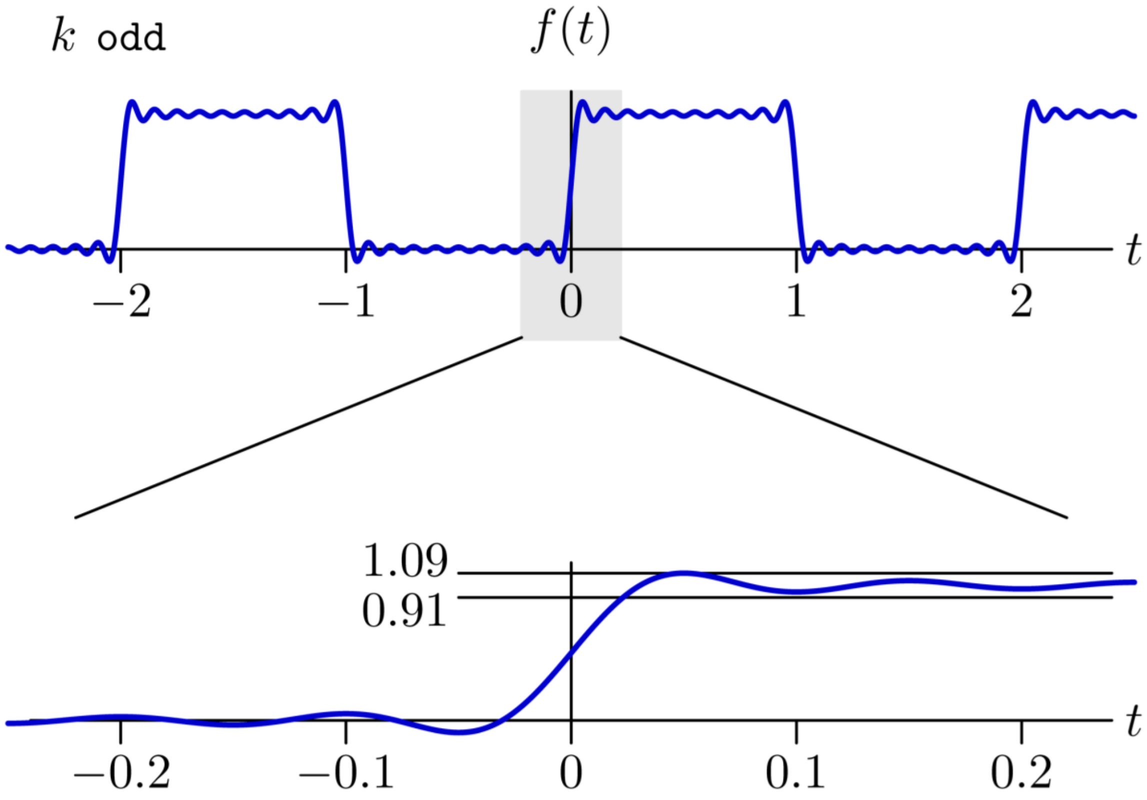
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

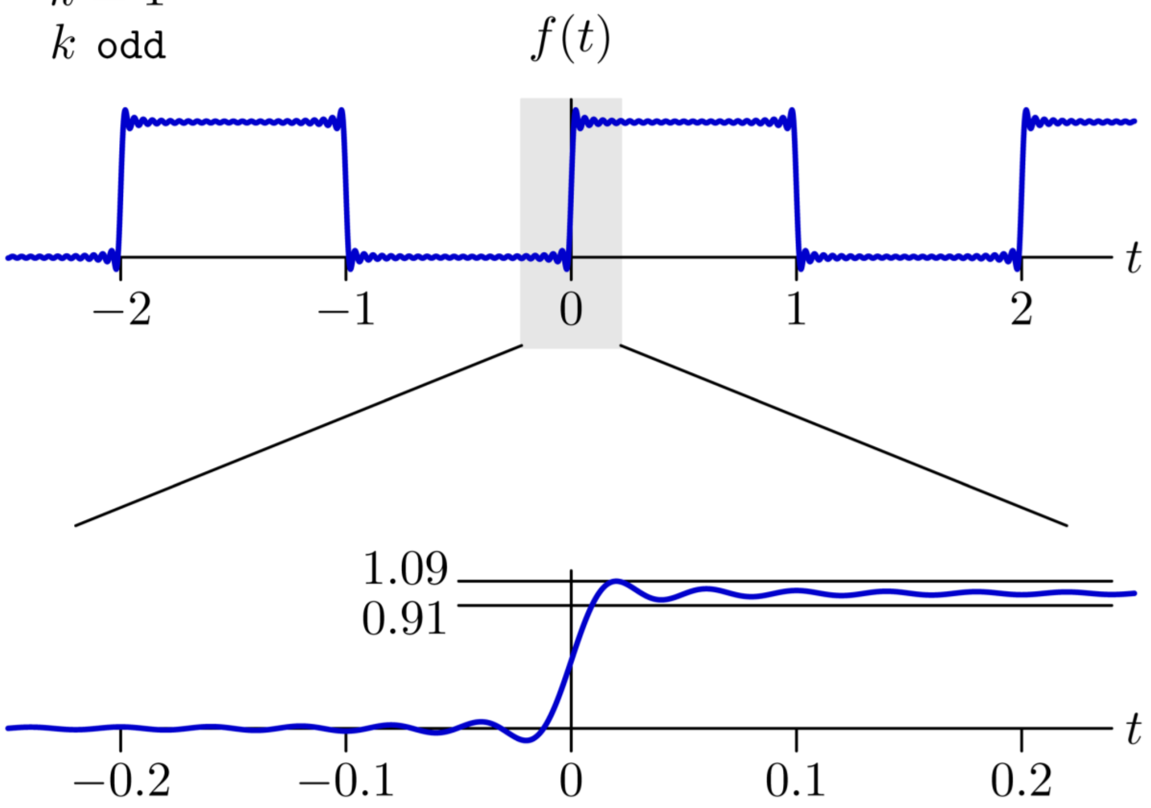
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

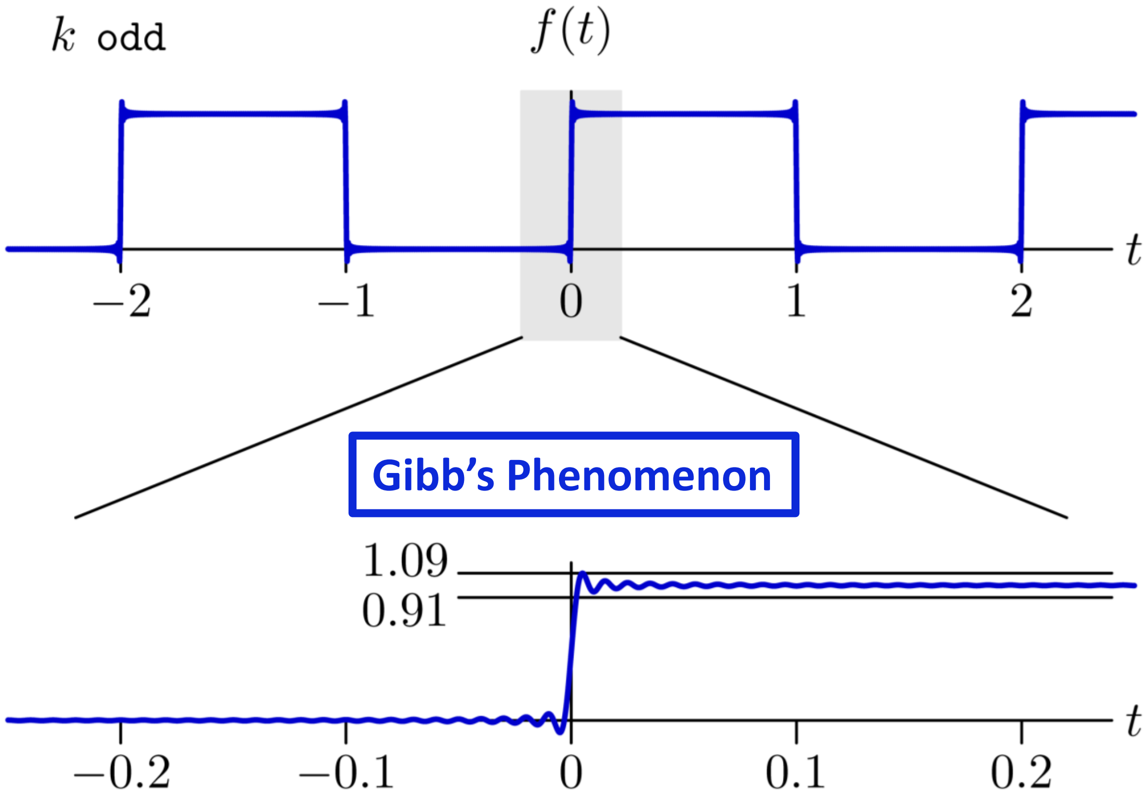
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{49} \frac{2}{k\pi}\sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{199} \frac{2}{k\pi} \sin(k\pi t)$$



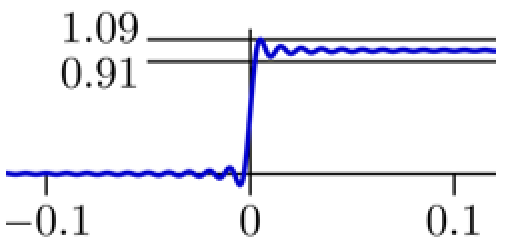
Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.

Convergence of Fourier Series

If there is a **step discontinuity** in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as t approaches t_0 from the left and from the right.

Let $f_K(t)$ represent the **partial sum** of the Fourier series using just N terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$



As $K \to \infty$,

- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number ϵ shrinks to zero.

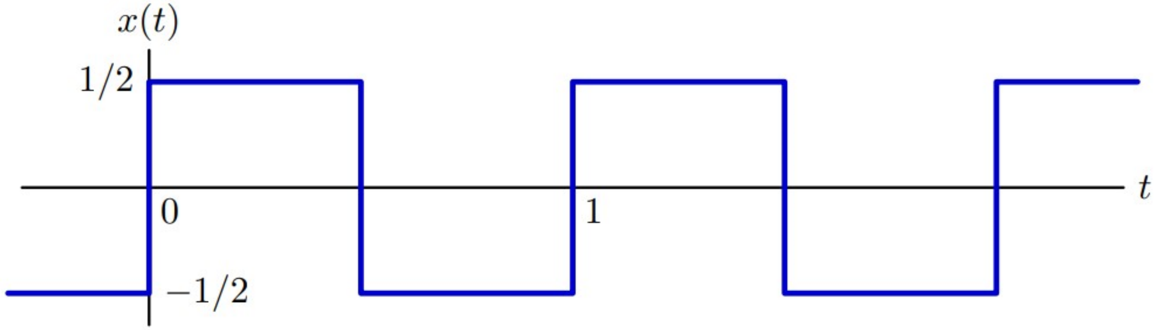
We refer to this type of overshoot as **Gibb's Phenomenon**.

Can any periodic signals be represented by Fourier Series

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

Jean-Baptiste Joseph Fourier

Joseph-Louis Lagrange



Dirichlet conditions:

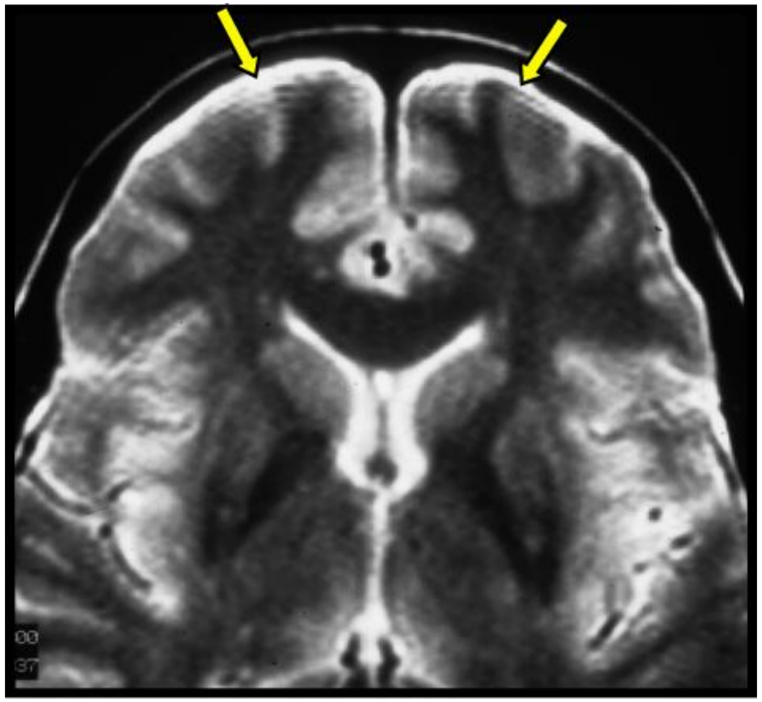
- Over any period, f(t) absolutely integrable;
- In any finite interval of time, f(t) is of bounded variation
- In any finite interval of time, there are only a finite number of discontinuities, each discontinuity is finite

Who was right?

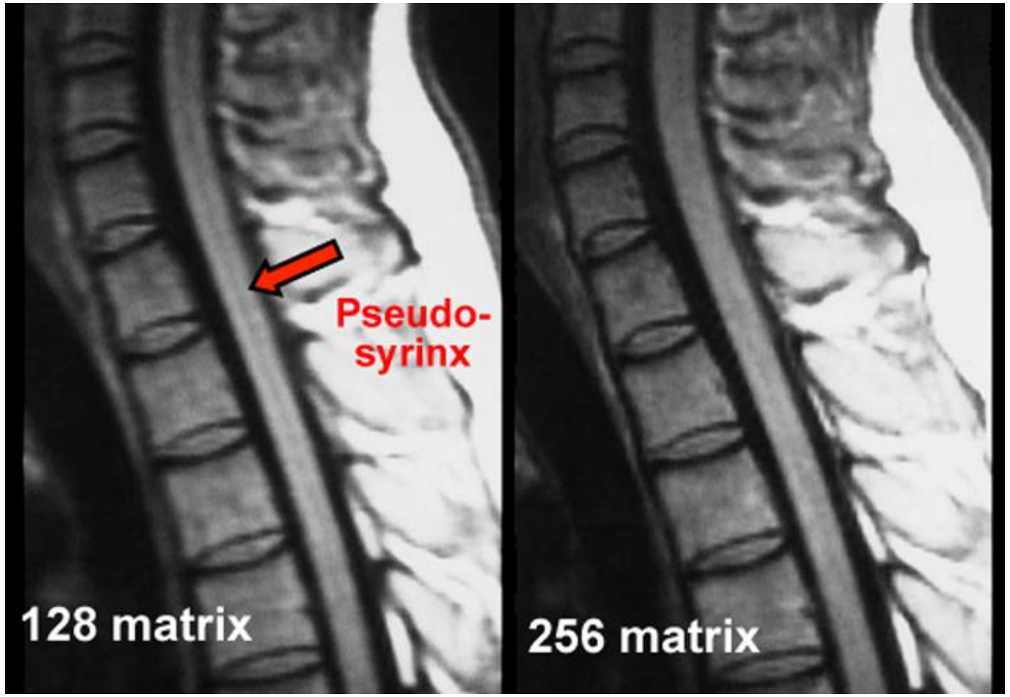
In a way both had their point. The series representation of a discontinuous function converges, but no uniformly.

Gibb's Phenomenon

Gibbs artifacts in MRI

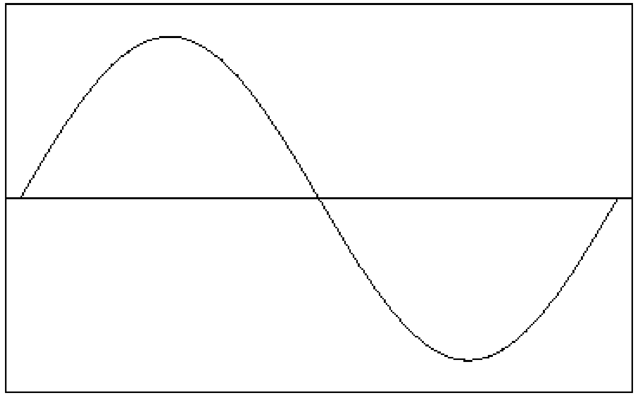


Decreasing artifacts with more frequency components



Q1: Why these happens?

Q2: How to alleviate Gibbs artifacts?



https://mriquestions.com/gibbs-artifact.html

Summary

- We examined the convergence of Fourier Series
  - Functions with discontinuous slopes well represented
  - Functions with discontinuous values generate ripples
    - Gibb’s phenomenon.
- Sinusoidal functions have interesting mathematical properties
  - Harmonically related sinusoids are orthogonal over [0, T).
- Sines and cosines play important roles in physics
  - Especially in the physics of waves.

Recitation

- If the 3rd letter of your kerberos is in the range of a-k, stay in: 4-370
- Otherwise, please go to: 4-237