# 6.300: Signal Processing

## Signal Processing

- Overview of Subject
- Signals: Definitions, Examples, and Operations
- Time and Frequency Representations
- Fourier Series

Homework #1 is posted on our course website:

http://sigproc.mit.edu

It is due one week from today (on Thursday, September 17) at 2pm

*September 10, 2026*

# 6.300: Signal Processing

**Signals** are mathematical functions that contain and convey information.

Examples:

- MP3 representation of a sound
- JPEG representation of a picture
- MRI image of a brain



**Signal Processing** develops the use of signals as abstractions:

- **identifying** signals in physical, mathematical, computation contexts,
- **analyzing** signals to understand the information they contain, and
- **manipulating** signals to modify the information they contain.

## Teaching and Learning Activities

**Lectures** (TR 2 pm): **introduce** signal processing theory.
– **Lecture notes** serve as our primary **technical reference**.[1]

**Recitations** (TR 3 pm) **use** signal processing methods to **solve problems**.

**Homework**: **practice** using signal processing methods.
– First homework is posted: due next Thursday (September 17) at 2 pm.

**Office Hours** (TBD): **work with peers and staff** to understand the technical content and to complete homework assignments.

**Two In-Class Quizzes** and a **Final Exam** (scheduled by MIT Registrar).

**Course Website:** **http:sigproc.mit.edu**

**Questions:** **6.300-instructors@mit.edu**

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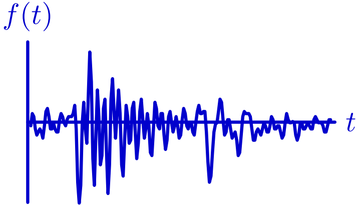
[1] Supplemental references
**Langton and Levin:** *The Intuitive Guide to Fourier Analysis & Spectral Estimation*
**Oppenheim and Willsky:** *Signals and Systems*
**Oppenheim and Schafer:** *Discrete-Time Signal Processing*
**Jae Lim:** *Two-Dimensional Signal and Image Processing*

# Check Yourself

Manipulating signals to modify the information they contain.




Listen to the following four manipulated signals:

$$f_1(t),\ f_2(t),\ f_3(t),\ f_4(t).$$

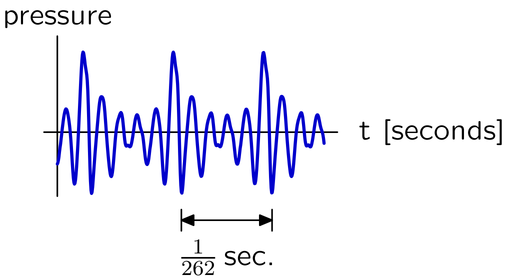
How many of the following relations are true?

- $f_1(t) = f(2t)$
- $f_2(t) = -f(t)$
- $f_3(t) = f(2t)$
- $f_4(t) = \frac{1}{3}f(t)$

* speech signal synthesized by Robert Donovan

# Musical Sounds as Signals

Musical sounds can also be represented as signals.
However, relation between waveform and information can be complicated.

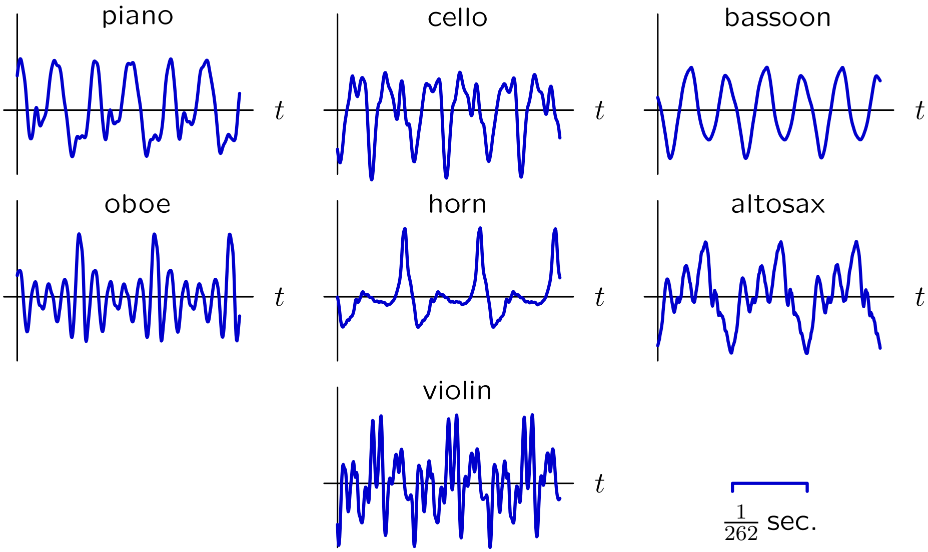



The function representation of this signal highlights some information: e.g., the waveform is approximately periodic with a period of 1/262 seconds. Therefore the frequency is approximately 262 Hz: i.e., middle C.

Other information (such as which instrument was played) is less obvious.

# Musical Sounds as Signals

All of the following sounds have the same pitch, but they sound different.



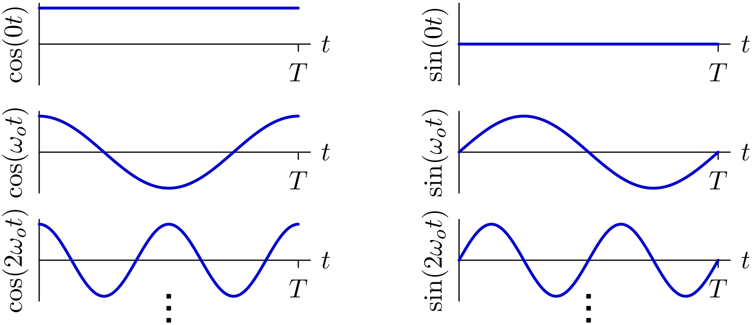

It's not clear how the audible differences relate to the time waveforms.

(audio clips from from http://theremin.music.uiowa.edu)

# Musical Signals as Sums of Sinusoids

An effective way to expose more information about musical sounds is to compute their **frequency** content (as a sum of frequency components):

$$f(t) = \sum_{k=0}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) \quad \text{where} \quad \omega_o = 2\pi/T$$



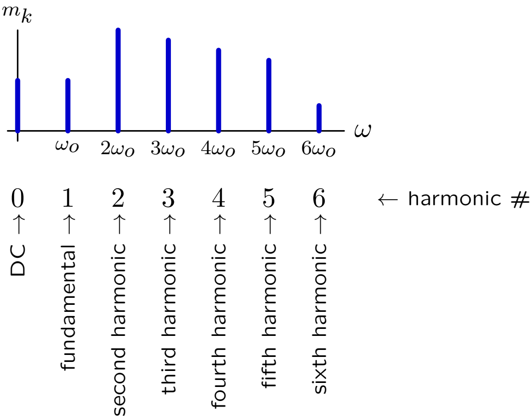

Since $f(t)$ is periodic in time (with period $T$), we only need to include sinusoids that are periodic in that same period (i.e., sinusoids at frequencies that are integer multiples of the **fundamental frequency** $\omega_o = 2\pi/T$).

# Harmonic Structure

Sine and cosine components for each integer $k$ in the following equation

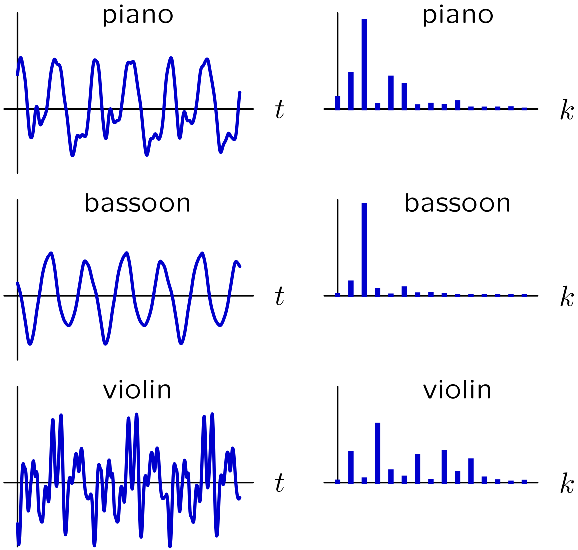
$$f(t) = \sum_{k=0}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) = \sum_{k=0}^{\infty} m_k \cos\left(k\omega_o t - \phi_k\right)$$

can be combined trigonometrically as a **single phase-shifted cosine** which can be characterized by a **magnitude** $m_k$ (given by $m_k^2 = c_k^2 + d_k^2$) and **phase** (aka angle) $\phi_k$ (given by $\tan\phi_k = d_k/c_k$).



# Harmonic Structure

The harmonic structures of notes from different instruments are different.



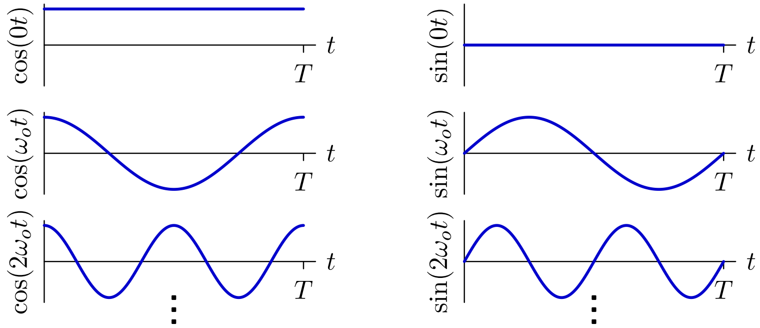
# Finding Fourier Representations of Signals

**Fourier series** are representations of signals based on sums of sinusoids:

$$f(t) = \sum_{k=0}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

where $\omega_o = 2\pi/T$ represents the fundamental frequency.

The **basis functions** for Fourier series are harmonically related sinusoids:




Q1: Under what conditions can a function be expressed as a Fourier series?

Q2: How can we find the coefficients $c_k$ and $d_k$.

# Finding Fourier Representations of Signals

Under what conditions can a function be expressed as a Fourier series?

Fourier series can only represent **periodic** signals.

Definition: a signal $f(t)$ is periodic in $T$ if

$$f(t) = f(t+T)$$

for all $t$.

Note: if a signal is periodic in $T$, then it is also periodic in $2T$, $3T$, ...

The smallest positive value of $T$ for which $f(t) = f(t+T)$ for all $t$ is sometimes called the **fundamental period**.

If there are no positive values of $T$ for which $f(t) = f(t+T)$ for all values of $t$, then the signal is said to be **aperiodic**.

# Calculating Fourier Coefficients

How can we find the coefficients $c_k$ and $d_k$? Start with $c_0$.

Key idea: simplify by integrating over the fundamental period $T$.
Start with the general form:[2]

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

Integrate both sides over $T$:

$$\int_0^T f(t)\, dt = \int_0^T c_0\, dt + \int_0^T \left( \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) \right) dt$$

$$= T c_0 + \sum_{k=1}^{\infty} \left( c_k \int_0^T \cos(k\omega_o t)\, dt + d_k \int_0^T \sin(k\omega_o t)\, dt \right) = T c_0$$

All but the first term integrates to zero, leaving

$$c_0 = \frac{1}{T} \int_0^T f(t)\, dt.$$

The $c_0$ term is equal to the average ("DC") value of $f(t)$.

2 Notice that $d_0$ can always be taken to be 0 since $\sin(k\omega_o t) = 0$ if $k = 0$.

# Calculating Fourier Coefficients

Isolate the $c_l$ term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

$$\begin{aligned}\int_0^T f(t) \cos(l\omega_o t)\, dt = \int_0^T c_0 \cos(l\omega_o t)\, dt \\ + \sum_{k=1}^{\infty} \int_0^T c_k \cos(k\omega_o t) \cos(l\omega_o t)\, dt \\ + \sum_{k=1}^{\infty} \int_0^T d_k \sin(k\omega_o t) \cos(l\omega_o t)\, dt\end{aligned}$$

A product of sinusoids can be expressed as sum and difference frequencies.

$$\cos(k\omega_o t) \cos(l\omega_o t) = \frac{1}{2} \cos((k-l)\omega_o t) + \frac{1}{2} \cos((k+l)\omega_o t)$$

$$\sin(k\omega_o t) \cos(l\omega_o t) = \frac{1}{2} \sin((k-l)\omega_o t) + \frac{1}{2} \sin((k+l)\omega_o t)$$

## Calculating Fourier Coefficients

Isolate the $c_l$ term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

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$$\begin{aligned}
\int_0^T f(t) \cos(l\omega_o t)\, dt = &\int_0^T c_0 \cos(l\omega_o t)\, dt \\
&+ \sum_{k=1}^{\infty} \int_0^T c_k \left( \frac{1}{2} \cos((k{-}l)\omega_o t) + \frac{1}{2} \cos((k{+}l)\omega_o t) \right) dt \\
&+ \sum_{k=1}^{\infty} \int_0^T d_k \left( \frac{1}{2} \sin((k{-}l)\omega_o t) + \frac{1}{2} \sin((k{+}l)\omega_o t) \right) dt
\end{aligned}$$

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# Calculating Fourier Coefficients

Isolate the $c_l$ term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = c_0 + \sum_{k=1}^{\infty}\Big(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\Big)$$

$$\begin{aligned}\int_0^T f(t)\cos(l\omega_o t)\,dt &= \int_0^T \cancelto{0}{c_0\cos(l\omega_o t)\,dt} \\ &\quad+ \sum_{k=1}^{\infty}\int_0^T c_k\left(\frac{1}{2}\cos((k{-}l)\omega_o t) + \frac{1}{2}\cos((k{+}l)\omega_o t)\right)\,dt \\ &\quad+ \sum_{k=1}^{\infty}\int_0^T d_k\left(\frac{1}{2}\sin((k{-}l)\omega_o t) + \frac{1}{2}\sin((k{+}l)\omega_o t)\right)\,dt\end{aligned}$$

The $c_0$ term is zero because the integral of $\cos(l\omega_o t)$ over $T$ is zero.

# Calculating Fourier Coefficients

Isolate the $c_l$ term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

$$\begin{aligned}
\int_0^T f(t)\cos(l\omega_o t)\,dt = & \int_0^T c_0 \cancelto{\mathbf{0}}{\cos(l\omega_o t)}\,dt \\
& + \sum_{k=1}^{\infty} \int_0^T c_k \left( \frac{1}{2} \cancelto{\mathbf{\frac{T}{2}c_l}}{\cos((k{-}l)\omega_o t)} + \frac{1}{2} \cancelto{\mathbf{0}}{\cos((k{+}l)\omega_o t)} \right) dt \\
& + \sum_{k=1}^{\infty} \int_0^T d_k \left( \frac{1}{2} \sin((k{-}l)\omega_o t) + \frac{1}{2} \sin((k{+}l)\omega_o t) \right) dt
\end{aligned}$$

If $k = l$, then $\cos((k{-}l)\omega_o t) = 1$ and the integral is $\frac{T}{2}c_l$.
All of the other $\cos((k-l)\omega_o t)$ terms in the sum integrate to zero.
All of the $\cos((k+l)\omega_o t)$ terms in the integrate to zero.

# Calculating Fourier Coefficients

Isolate the $c_l$ term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

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\int_0^T f(t)\cos(l\omega_o t)\,dt = & \int_0^T \cancelto{0}{c_0 \cos(l\omega_o t)}\,dt \\
& + \sum_{k=1}^{\infty} \int_0^T c_k \left( \frac{1}{2} \cancelto{\frac{T}{2}c_l}{\cos((k-l)\omega_o t)} + \frac{1}{2} \cancelto{0}{\cos((k+l)\omega_o t)} \right) dt \\
& + \sum_{k=1}^{\infty} \int_0^T d_k \left( \frac{1}{2} \cancelto{0}{\sin((k-l)\omega_o t)} + \frac{1}{2} \cancelto{0}{\sin((k+l)\omega_o t)} \right) dt
\end{aligned}$$

If $k = l$, then $\sin((k-l)\omega_o t = 0$ and the integral is $0$.
All of the other $d_k$ terms are harmonic sinusoids that integrate to $0$.

The only non-zero term on the right side is $\frac{T}{2}c_l$.
We can solve to get an expression for $c_l$ as

$$c_l = \frac{2}{T} \int_0^T f(t)\cos(l\omega_o t)\,dt$$

# Calculating Fourier Coefficients

Analogous reasoning allows us to calculate the $d_k$ coefficients, but this time multiplying by $\sin(l\omega_o t)$ before integrating.

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

$$\begin{aligned}
\int_0^T f(t) \sin(l\omega_o t)\, dt = & \int_0^T c_0 \sin(l\omega_o t)\, dt \\
& + \sum_{k=1}^{\infty} \int_0^T c_k \cos(k\omega_o t) \sin(l\omega_o t)\, dt \\
& + \sum_{k=1}^{\infty} \int_0^T d_k \sin(k\omega_o t) \sin(l\omega_o t)\, dt
\end{aligned}$$

A single term remains after integrating, allowing us to solve for $d_l$ as

$$d_l = \frac{2}{T} \int_0^T f(t) \sin(l\omega_o t)\, dt$$

## Calculating Fourier Coefficients

Summarizing ...

If $f(t)$ is expressed as a Fourier series

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

the Fourier coefficients are given by

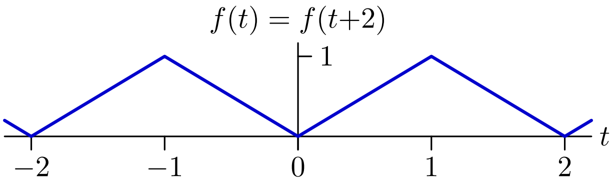
$$c_0 = \frac{1}{T} \int_T f(t)\, dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_o t)\, dt; \;\; k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_o t)\, dt; \;\; k = 1, 2, 3, \ldots$$

# Example of Analysis

Find the Fourier series coefficients for the following triangle wave:




$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt = 2\int_0^1 t\cos(\pi kt)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$$d_k = 0 \quad \text{(by symmetry)}$$

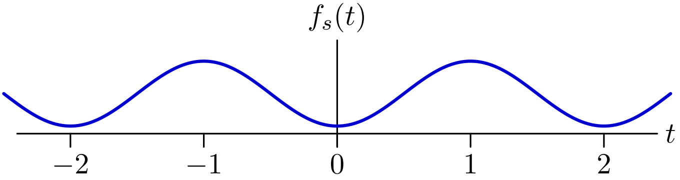
# Example of Synthesis

Generate $f(t)$ from the Fourier coefficients shown on the previous slide.

Let $f_s(t)$ represent the function synthesized from the Fourier coefficients.

$$f_s(t) = c_0 - \sum_{k=1}^{\infty}\Big(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\Big) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2}\cos(k\pi t)$$

$$f_s(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{1} \frac{4}{\pi^2 k^2}\cos(k\pi t)$$



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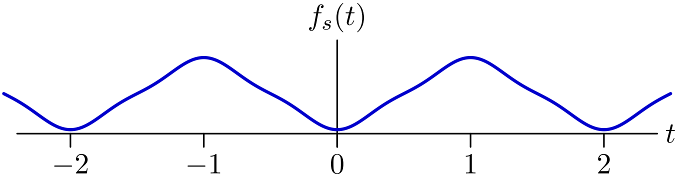
$$f_s(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k\ \texttt{odd}}}^{3} \frac{4}{\pi^2 k^2}\cos(k\pi t)$$



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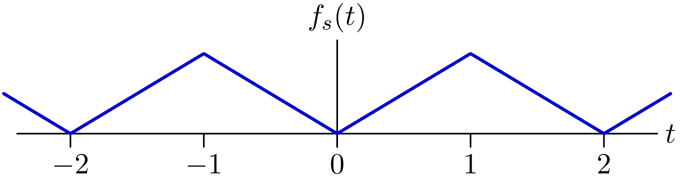
$$f_s(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

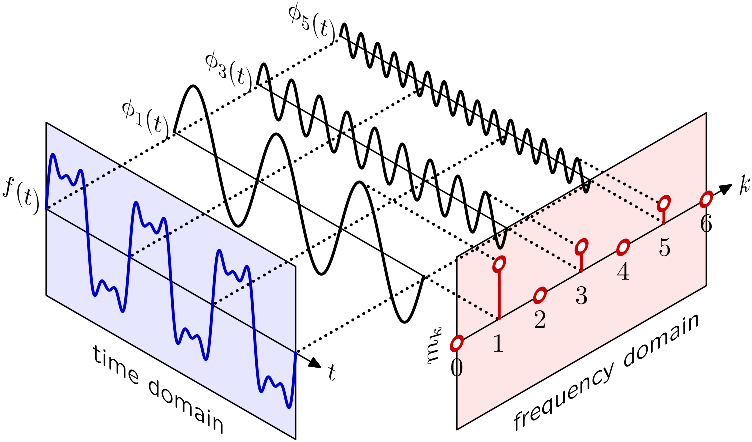



The synthesized function $f_s(t) \to f(t)$ as the number of terms increases.

# Summary: Two Views of the Same Signal

Harmonic expansion provides an alternative view of the signal.

$$f(t) = \sum_{k=0}^{\infty} \Big( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$




We can view a signal as a **function of time** $f(t)$, or as a **sum of harmonics**.
Both views are useful. Together, they form the basis of **Fourier analysis.**