6.300 Signal Processing

Week 3, Lecture A:
Sampling and Aliasing

- Continuous signals → discrete signals
- Sampling effect
- Quantization effect

September 22, 2026

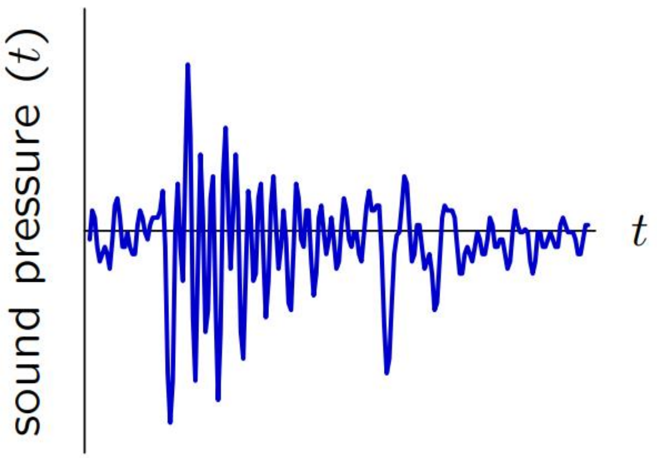
Continuous and Discrete signals

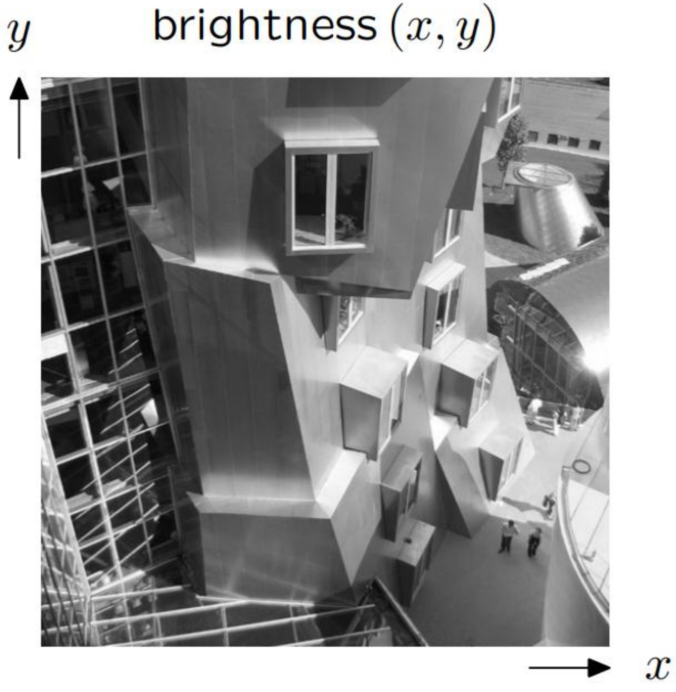
Physical signals are often of continuous domain:

- continuous spatial coordinates (in meters)
- continuous values

Computations manipulate functions of discrete domain:

- discrete spatial coordinates (in pixels)
- discrete values

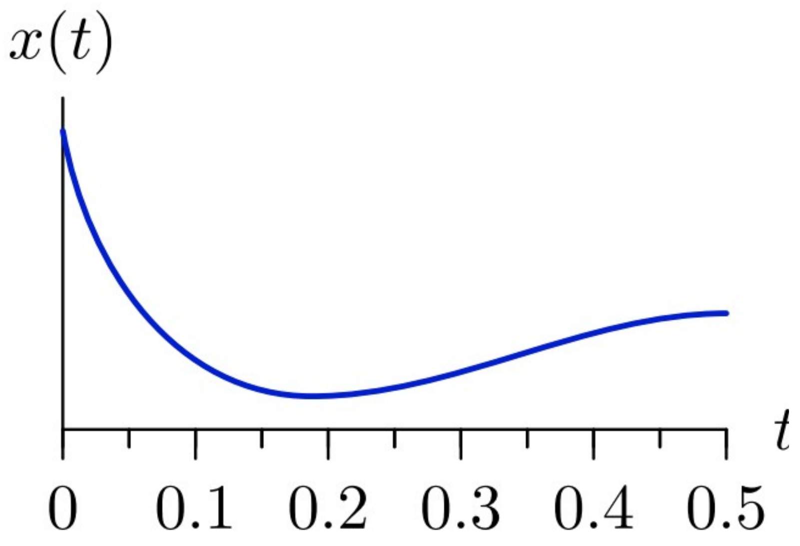


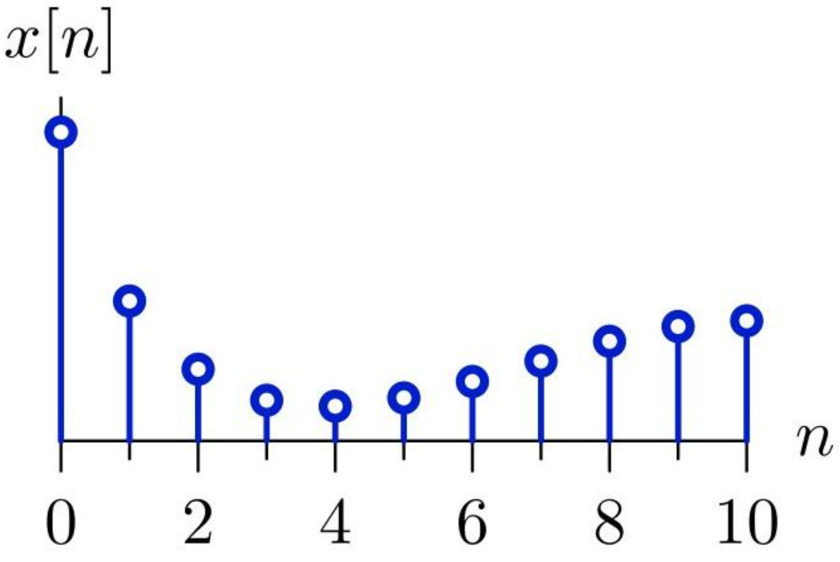


Today: from continuous to discrete signals

- Signal processing requires the operation: continuous → discrete
- Today: Understand the relationship between continuous signals to discrete signals.
- Question: How to convert continuous signals to discrete signals?

Sampling & Quantization

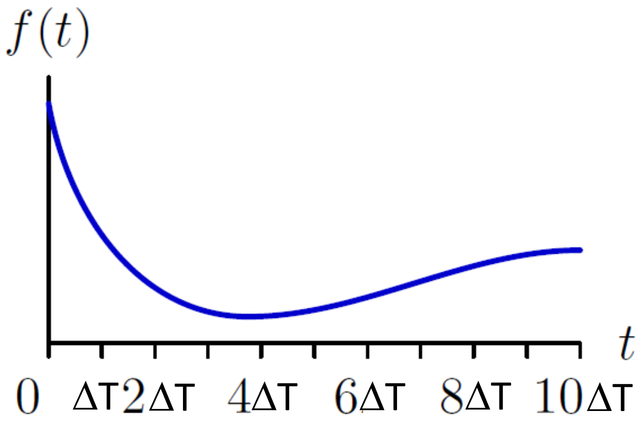


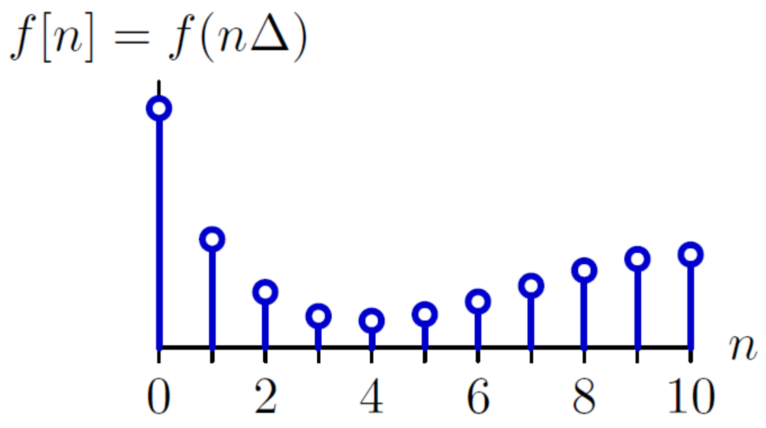


Sampling

Sampling refers to the process by which a continuous-time signal $f(t)$ is converted to a discrete-time signal $f[n]$.

We use parentheses to denote functions of continuous domain (e.g., $f(t)$) and square brackets to denote functions of discrete domain (e.g., $f[n]$).





$$\Delta = \text{sampling interval}$$
$$f_s = \frac{1}{\Delta} = \text{sampling frequency}$$

How does sampling affect the information contained in a signal?

Effect of sampling are easily heard

- How does sampling affect the information contained in a signal?

Sampling Music

$$f_s = \frac{1}{\Delta}$$

- $f_s = 44.1$ kHz
- $f_s = 22$ kHz
- $f_s = 11$ kHz
- $f_s = 5.5$ kHz
- $f_s = 2.8$ kHz

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Effect of sampling are easily seen

Sampling images: original 4112 x 3088

Effect of sampling are easily seen

Sampling images: undersample 2x

Effect of sampling are easily seen

Sampling images: undersample 8x

Effect of sampling are easily seen

Sampling images: undersample 16x

Effect of sampling are easily seen

Sampling images: undersample 32x

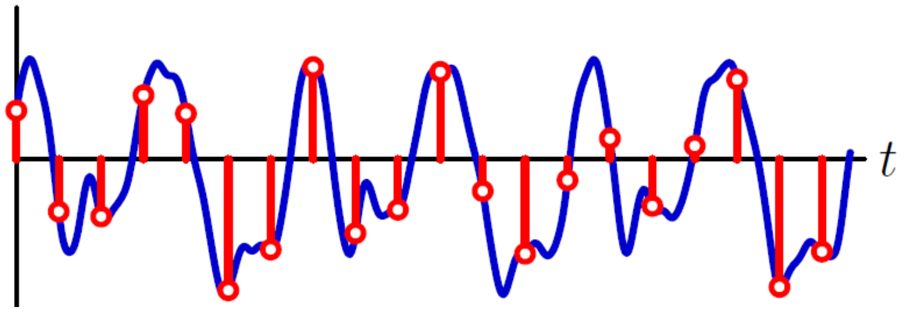
- Information loss
- Distortion

Effect of sampling

We would like to sample in a way that preserves **information**.

However, information is generally **lost** in the sampling process.

Example: samples (red) provide no information about intervening values.



Effect of sampling

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However, information is generally **lost** in the sampling process.

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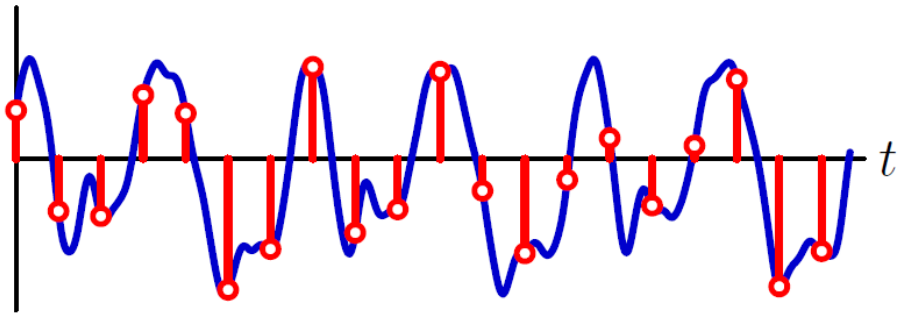
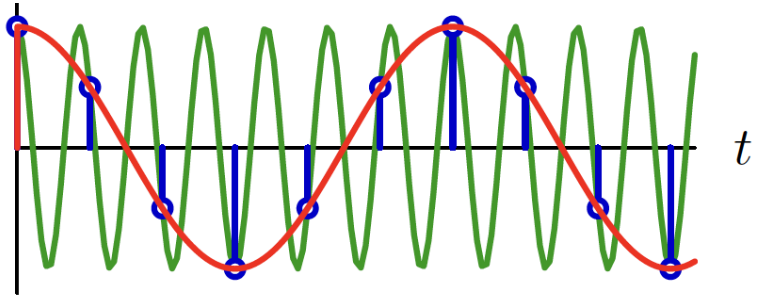


Furthermore, information that is retained by sampling can be misleading.

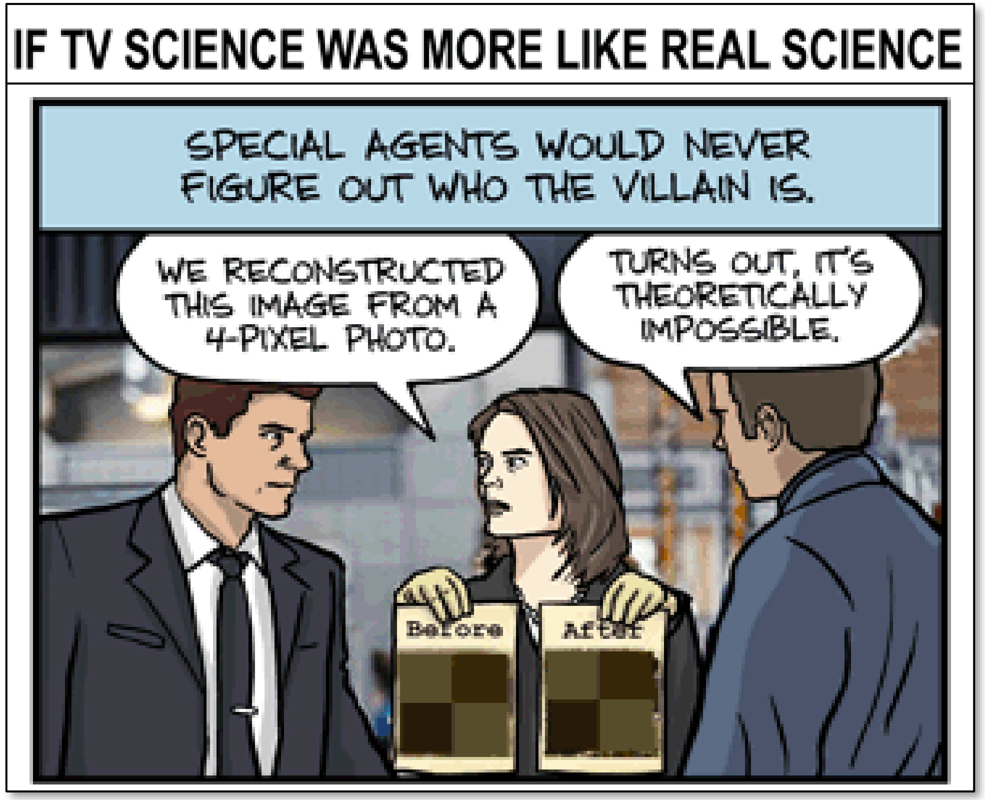
Example: samples can suggest patterns not contained in the original.



Samples (blue) of the original high-frequency signal (green)

The artifacts of sampling in real life

Loss of information



Aliasing

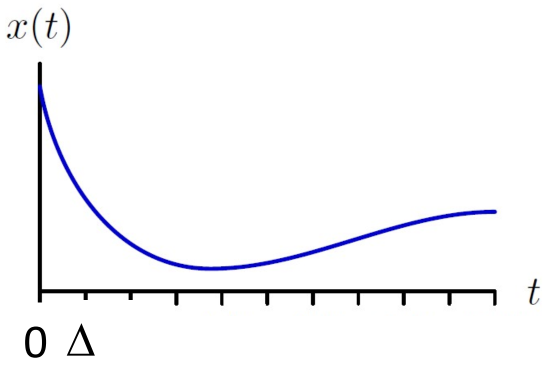
When fps = rpm

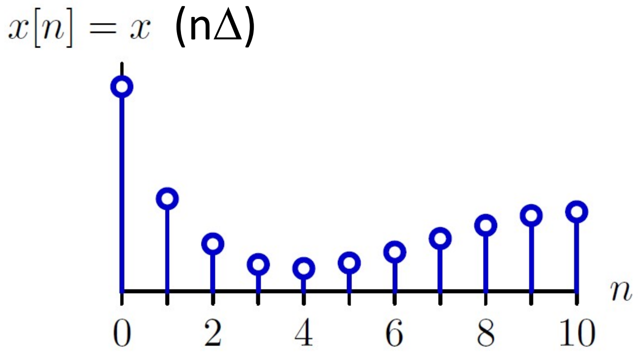
https://www.youtube.com/watch?v=_xTjyV8F6XU

To mitigate artifacts, we need to understand sampling

What is sampling:

- Continuous coordinate → discrete coordinate
- Measuring the value every Δ seconds



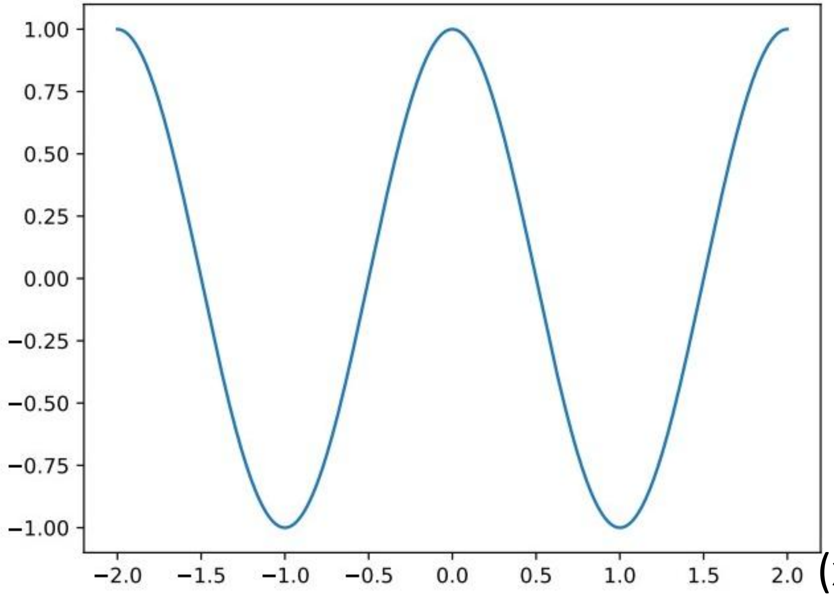


Parameters that matter here:

$$\Delta \ \text{(seconds / sample)} = \text{sampling interval}$$
$$f_s \ \text{(samples / second)} = \text{sampling rate}$$

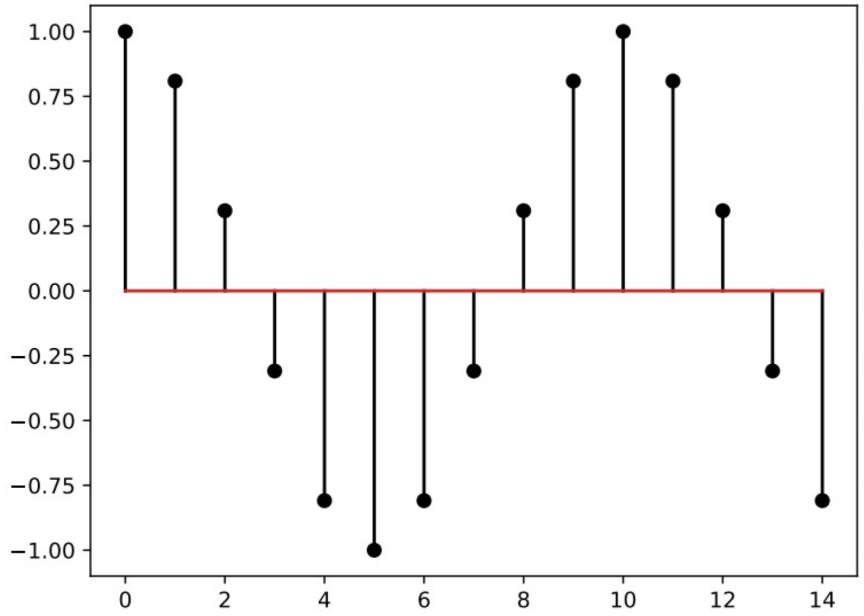
Let's use sinusoids an example from CT to DT

$x(t) = \cos(\omega t)$



($\times 10^{-3}$ sec)

$x[n] = \cos(\Omega n)$



Question: what is the relationship between ω and Ω?

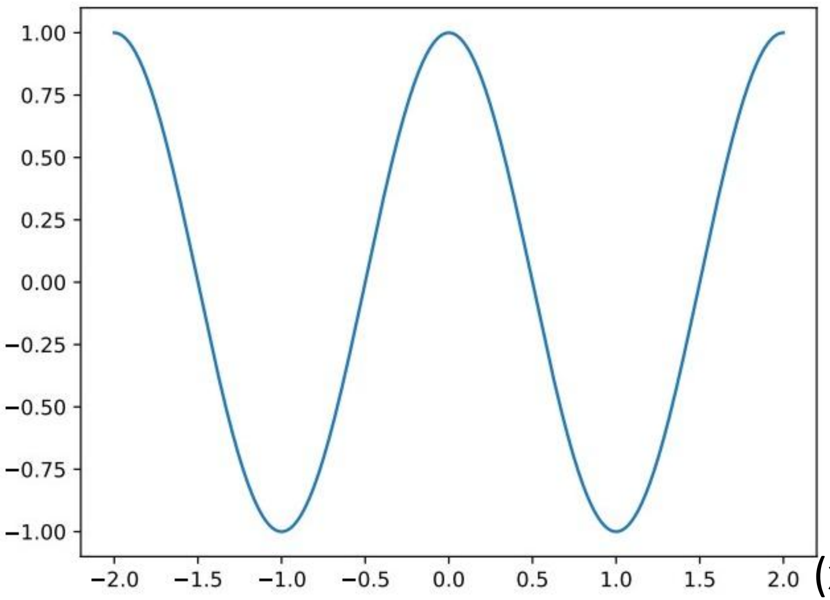
Sample $x(t) = \cos(\omega t)$ every Δ seconds to obtain $x[n]$:

$$x[n] = x(n\Delta) = \cos(\omega n \Delta) = \cos\Big((\omega\Delta)n\Big)$$

$\Omega = \omega\Delta = \omega/f_s$

Check yourself

$x(t) = \cos(\omega t)$



(x10^{-3} sec)

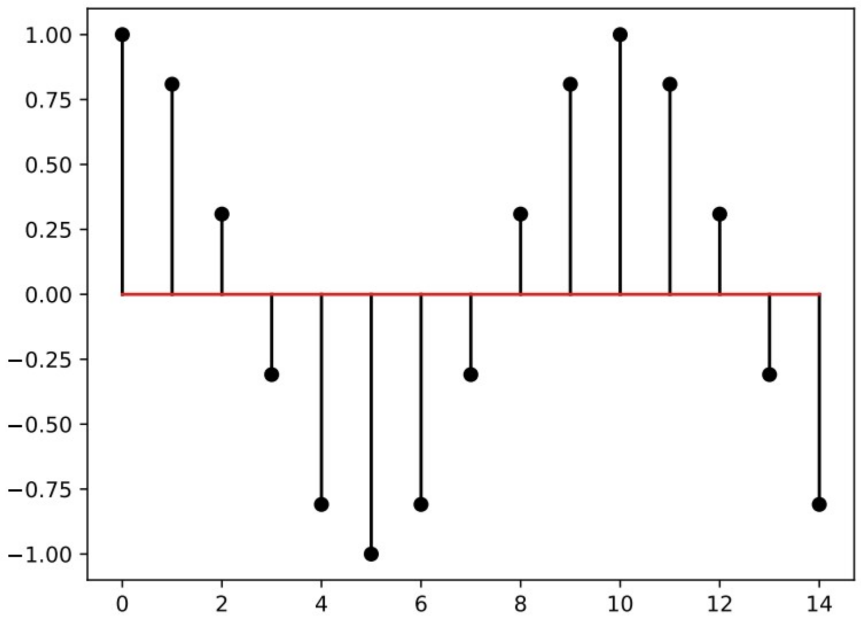
$\Omega = \omega / f_s$

$\Longrightarrow$

Δ= 2 x10^{-3} sec/10

$f_s = \frac{1}{\Delta} = 5000$

$x[n] = \cos(\Omega n)$



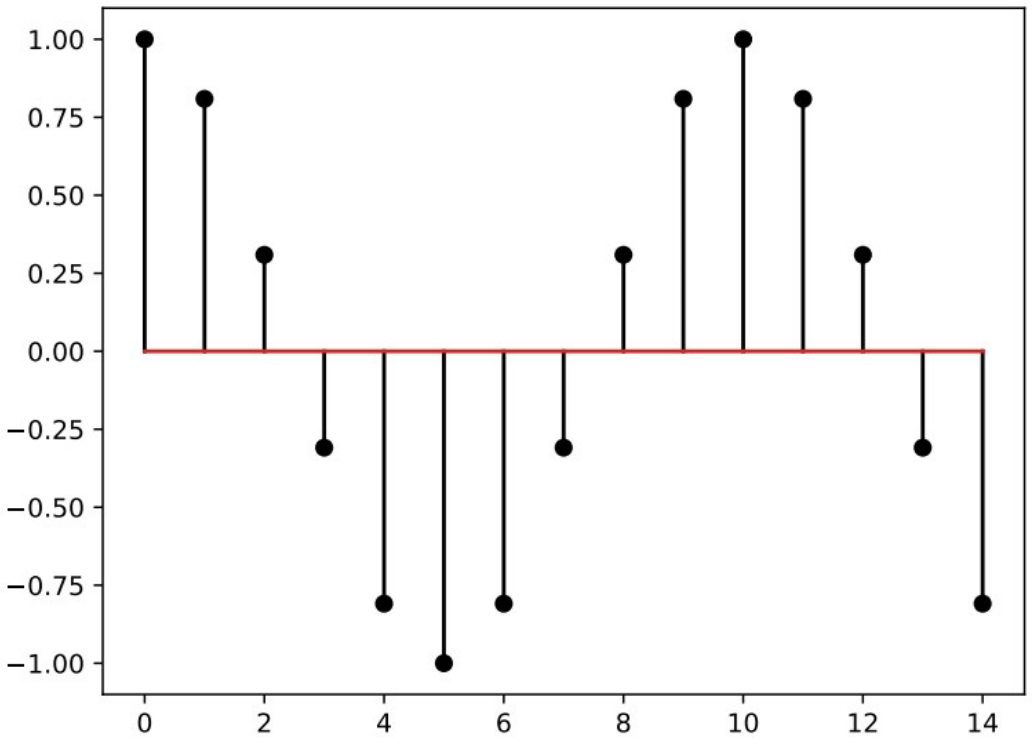
Question: We have the CT signal shown on the left and its corresponding sampled DT signal shown on the right. What is ω and what is Ω? What is the sampling rate f_s ?

$\omega = 2\pi/(2 \text{ x}10^{-3} \text{ sec}) = 1000\pi$

$\Omega = 2\pi/10 = 0.2\pi$

Intro to aliasing

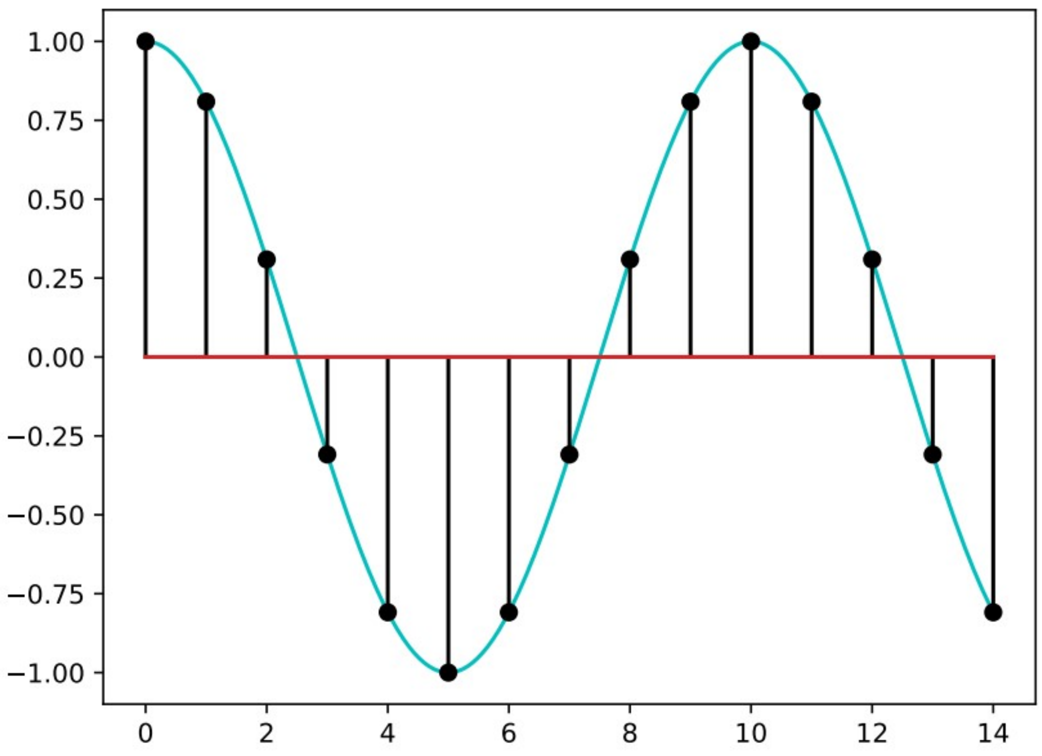
$$x[n] = \cos(\Omega n)$$



- The same list of data can actually mean different DT signals! And can be used to obtain CT signals of different frequency!

Intro to aliasing

$$x[n] = \cos(\Omega n)$$

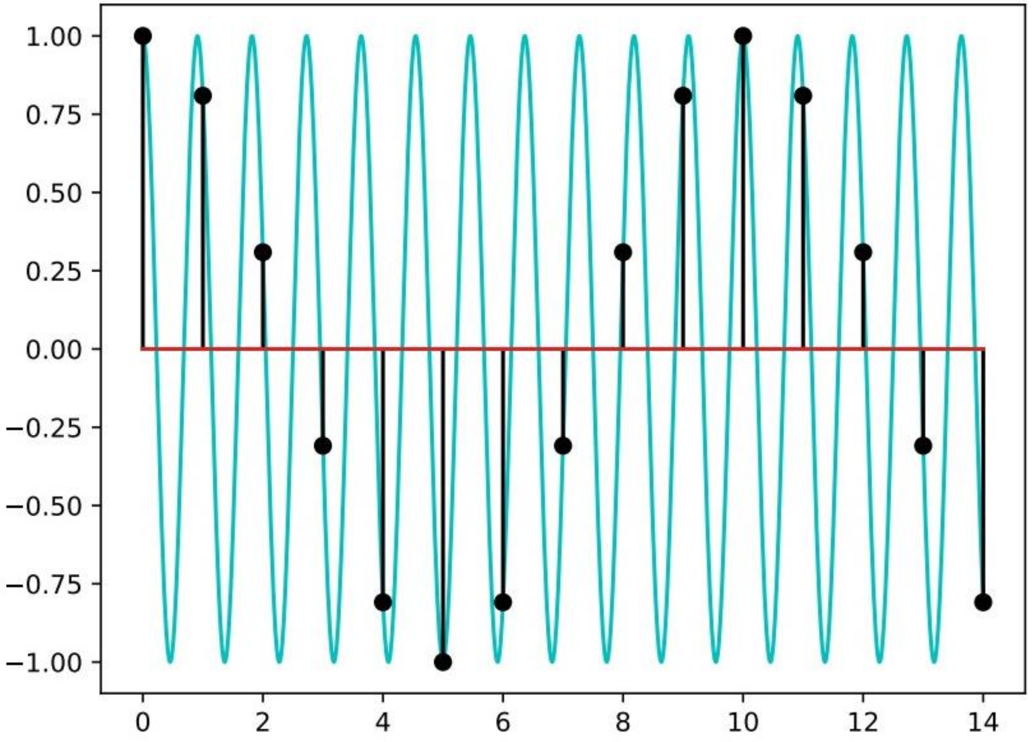


1 cycle (2π radians) in 10 samples: $\Omega = 0.2\pi$

- The same list of data can actually mean different DT signals! And can be used to obtain CT signals of different frequency!

Intro to aliasing

$$x[n] \; = \cos(\Omega n)$$

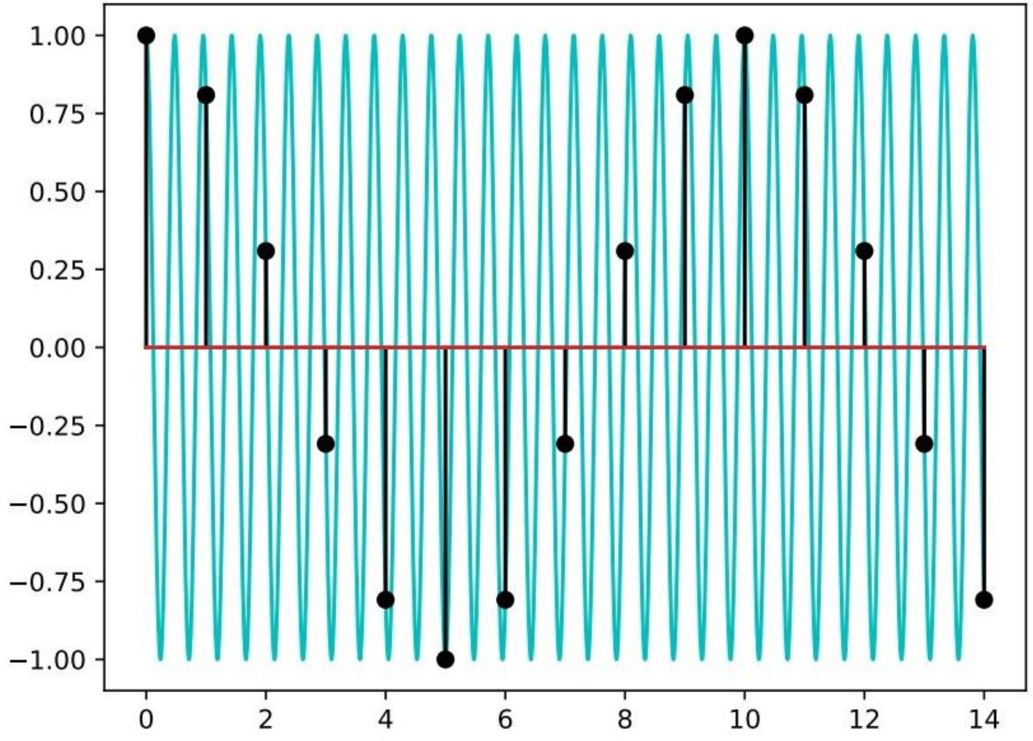


- The same list of data can actually mean different DT signals! And can be used to obtain CT signals of different frequency!

11 cycles (22π radians) in 10 samples: $\Omega = 2.2\pi$

Intro to aliasing

$$x[n] = \cos(\Omega n)$$



- ❑ The same list of data can actually mean different DT signals! And can be used to obtain CT signals of different frequency!

21 cycles (42π radians) in 10 samples: $\Omega = 4.2\pi$

Q: Why can this happen?

Why aliasing can happen

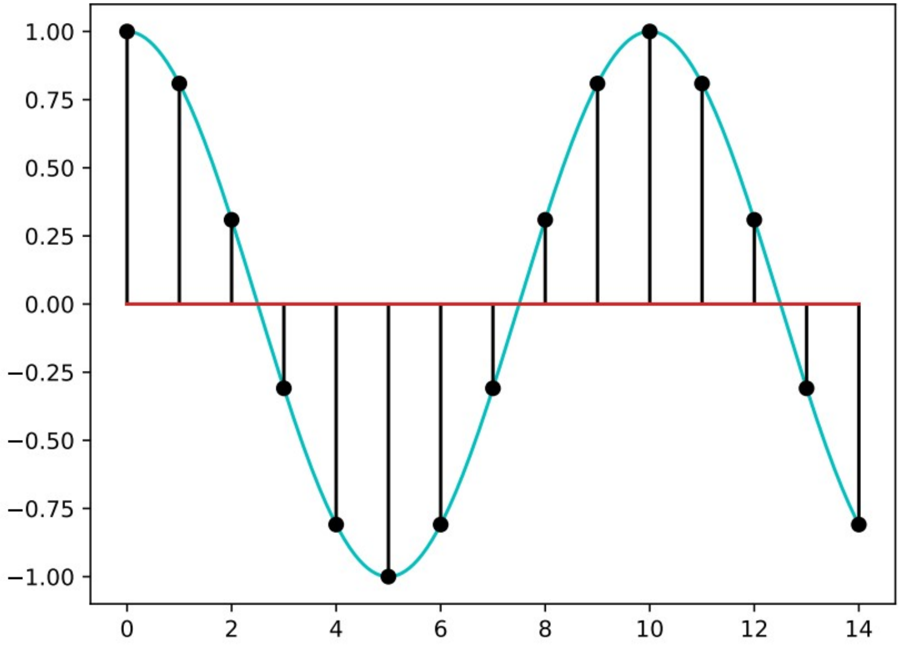
$x(t) = \cos(\omega t)$ v.s. $x[n] = \cos(\Omega n)$

Q: What happens when we increase ω?

The frequency of the sound increases as ω increases

Q: What happens when we increase Ω:

$\cos((\Omega + 2\pi)n) = \cos(\Omega n + 2\pi n) = \cos(\Omega n)$



- Compared to CT signals: n is always an integer
  - We only have values at integer multiples of Ω
  - There are multiple Ω values that lead to the exact same set of discrete points
- Consequences
  - This graph could be described by an infinite number of different Ω values
  - Hence aliasing: the same signal can be described by different "names".

Aliasing

In our example, we had:

ω_1 =1000π, f=500Hz

ω_2 =11000π, f=5500Hz

ω_3 =21000π, f=10500Hz

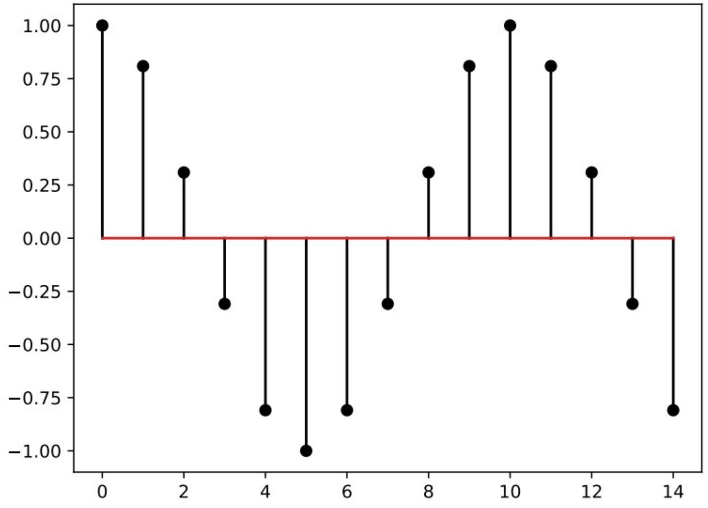
$f_s = 5000$

$\omega = \Omega \cdot fs$

$$x_1[n] = \cos(0.2\pi n)$$

$$x_2[n] = \cos(2.2\pi n)$$

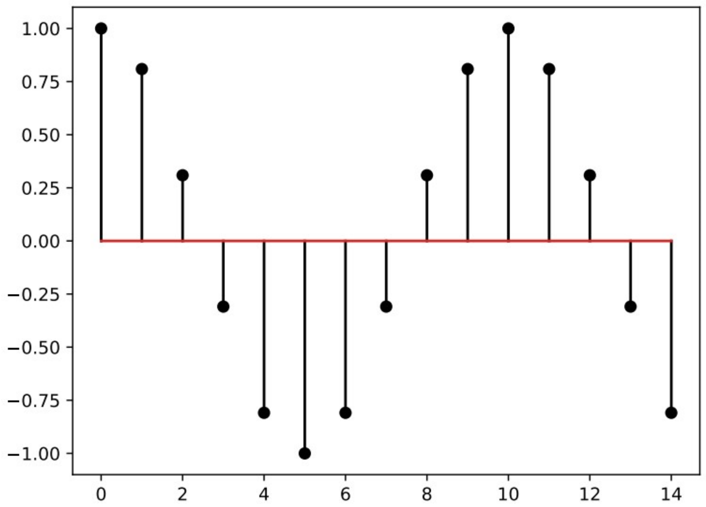
$$x_3[n] = \cos(4.2\pi n)$$



These all represent the exact same signal! They are all *aliases* for that signal.

Consider we obtained $x_1[n]$, $x_2[n]$, $x_3[n]$ by sampling from continuous time signals, if a sampling rate fs of 5000 Hz was used, what frequency (ω or f) should the original continuous time signal has?

Base band



Multiple frequencies Ω that we could use to refer to this signal

We can remove the ambiguity of which frequency is represented by a set of samples by choosing the one in the range $0 \leq \Omega \leq \pi$.

We call that range of frequencies the **base band** of frequencies, and the value of Ω that falls in that range is often referred to as the *principle alias*.

Maximum Frequency

If we limit our attention to frequencies in the base band, then there is a maximum possible discrete frequency $\Omega_{\text{max}} = \pi$.

Note this difference from CT, where there is no maximum frequency.

Why considering $\Omega_{\text{max}} = \pi$ is important?

Let's compute maximum frequency: Nyquist frequency

If a CT signal $x(t) = \cos \omega t$ is sampled at times $t = n\Delta$, the resulting DT signal is $x[n] = \cos \Omega n$ where

$$\Omega = \omega \, \Delta$$

If we restrict DT frequencies to the range $0 \leq \Omega \leq \pi$, then the corresponding CT frequencies are in the range $0 \leq \omega \leq \omega_N$ where

$$\omega_N = \frac{\pi}{\Delta}$$

and

$$f_N = \frac{\omega_N}{2\pi} = \frac{1}{2\Delta} = \frac{1}{2} f_s$$

The Nyquist frequency is basically half the sampling rate.

Aliasing and Anti-Aliasing

Sampling Music

$$f_N = \frac{1}{2} f_s$$

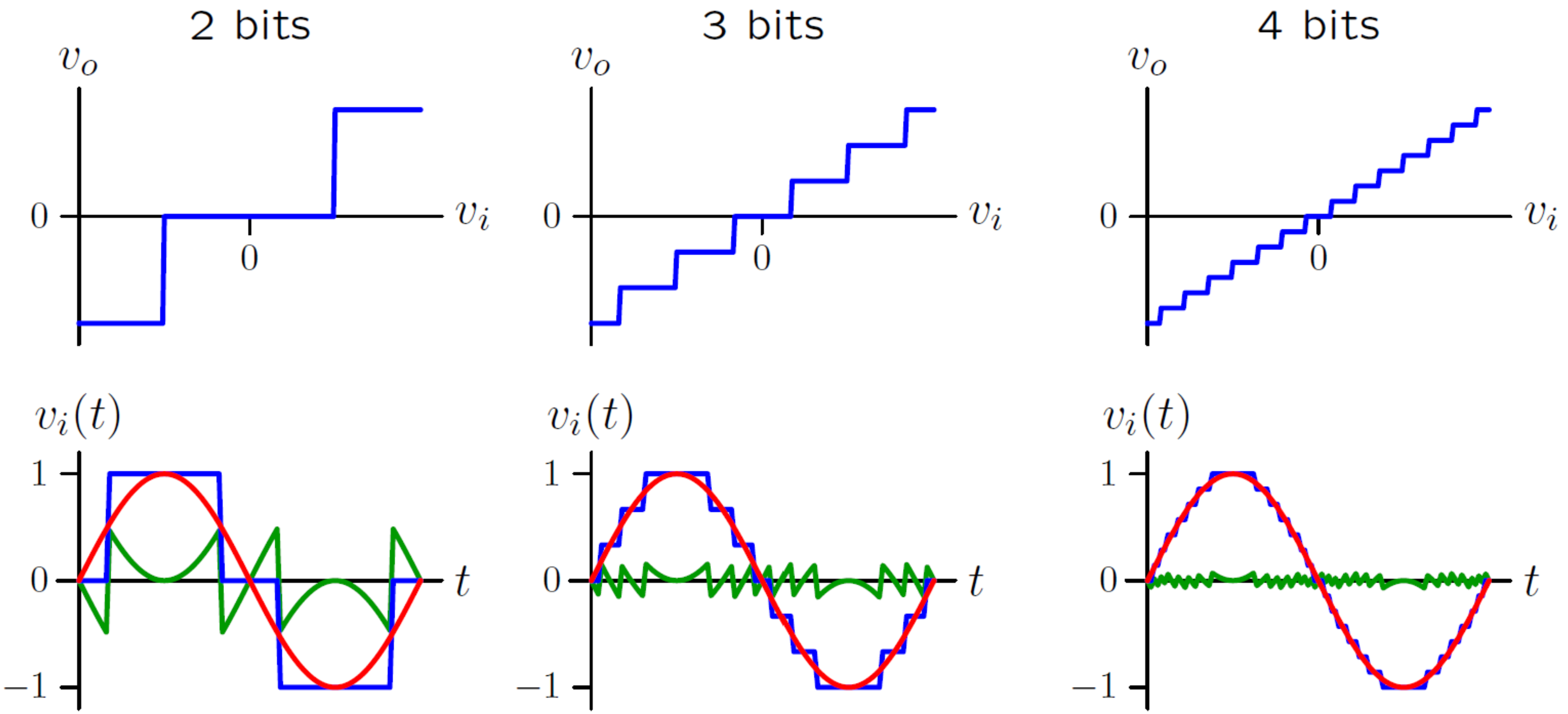
J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

	No anti-aliasing	with anti-aliasing
• f_s = 5.5kHz		
• f_s = 2.8kHz		

- If there are frequencies in the CT signal that are greater than the Nyquist frequency, they will alias to frequencies in the base band (that really don't have anything to do with the original frequency!)
- Anti-aliasing to prevent distortions: remove frequencies above Nyquist frequency before sampling so that they won't alias into the base band

Quantization

The information content of a signal depends not only with sample rate but also with the number of bits used to represent each sample.



Bit rate = (# bits/sample)×(# samples/sec)

Check yourself

We hear sounds that range in amplitude from 1,000,000 to 1.

How many bits are needed to represent this range?

1. 5 bits
2. 10 bits
3. 20 bits
4. 30 bits
5. 40 bits

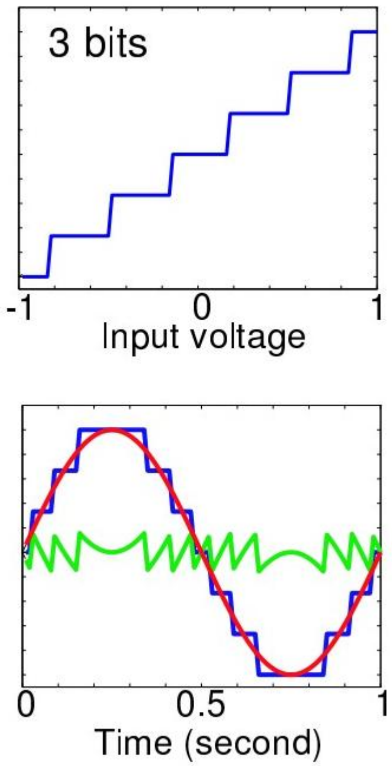
bits	range
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
11	2,048
12	4,096
13	8,192
14	16,384
15	32,768
16	65,536
17	131,072
18	262,144
19	524,288
20	1,048,576

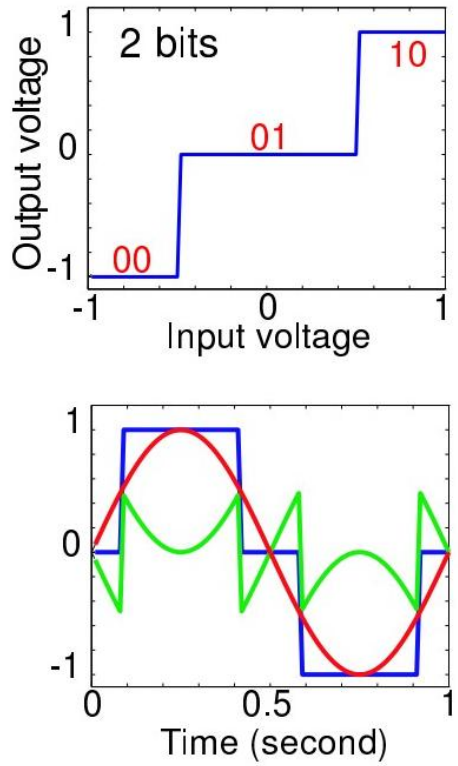
Quantization Demonstration in Music

16 bits/sample

3 bits/sample

2 bits/sample





J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Quantization of Images

8 bit 100%

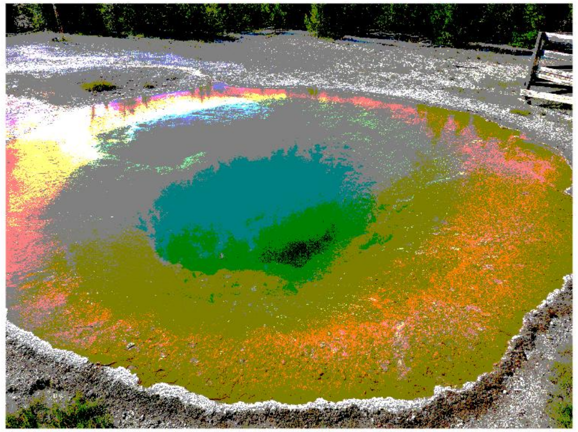
5 bit 63%

4 bit 50%

3 bit 38%

2 bit 25%

1 bit 13%



Quantization and Dithering

16 bits/sample

3 bits/sample

2 bits/sample

With Dithering

With Dithering

Dithering: add a small amount of random noise before quantization, which actually makes the quantization error less correlated with the signal, producing more “natural” noise instead of distortion.

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Summary

- Converting continuous time signals into discrete signals
  - Sampling
  - Quantization
- Important new concepts:
  - Sampling rate (sampling frequency) f_s
  - Base band, Nyquist frequency
  - Aliasing and Anti-aliasing
  - Dithering

For recitation:

- If the 3rd letter of your kerberos is in the range of a-k, please stay in: 4-370
- Otherwise, please go to: 4-237