

# 6.300: Signal Processing

## Discrete Fourier Transform 2

- Frequency Resolution
- Circular Convolution

*October 29, 2026*

## Last Time

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**Define** the Discrete Fourier Transform (DFT).

**Compare** the DFT to other Fourier representations.

**analysis**

**synthesis**

**DTFS:** 
$$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n}$$

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$$

**DTFT:** 
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega$$

**DFT:** 
$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi k}{N} n}$$

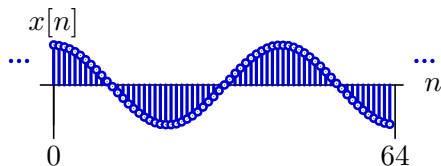
$$x[n] = \sum_{n=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$$

The DFT is a powerful Fourier representation that is easy to compute, but results can be tricky to interpret.

## Interpreting the DFT

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The 64-point DFT of  $x[n] = \cos \frac{3\pi n}{64}$



has multiple frequency “components” that we might not expect since  $x[n]$  is a single sinusoid.



Fourier series (DTFS) and Fourier transforms (DTFT) offer different, complementary interpretations of this surprising result.

## DTFS Interpretation: Periodic Extension

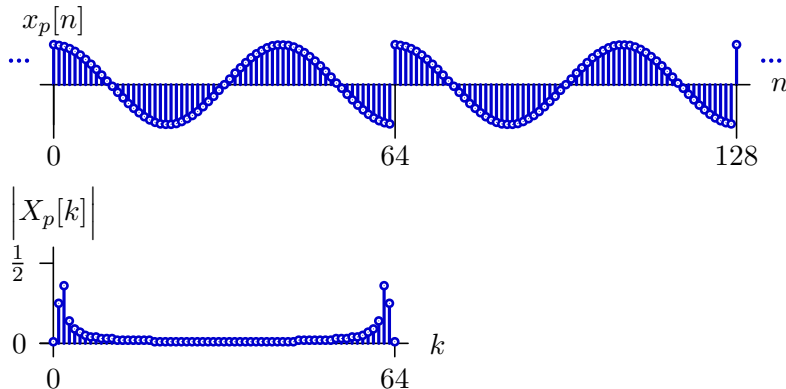
The 64-point **DFT** of  $x[n]$ :

$$x[n] = \cos \frac{3\pi n}{64}$$

is equal to the 64-point **DTFS** of  $x_p[n]$ :

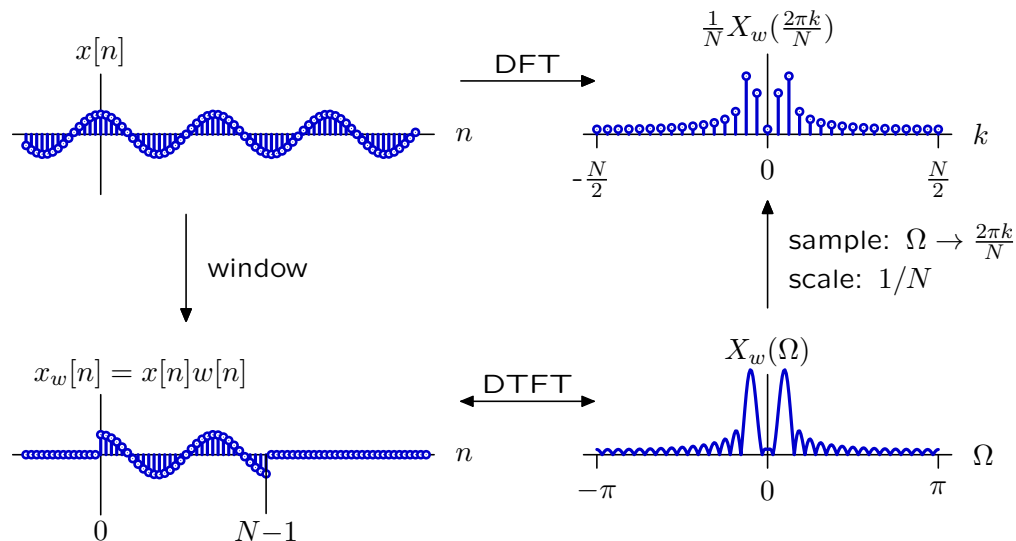
$$x_p[n] = x[n \bmod 64]$$

where  $x_p[n]$  represents the 64-point **periodic extension** of  $x[n]$ .



From this perspective, periodic extension introduces additional frequency components that are needed to generate the step discontinuity in  $x_p[n]$ .

## DTFT Interpretation: Windowing

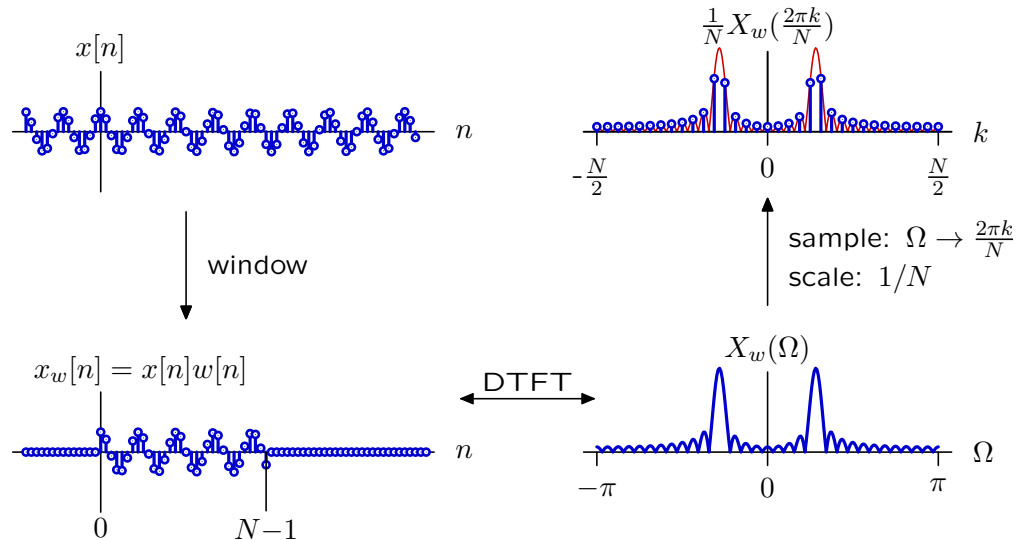


**Windowing** that is implicit with the DTFT decreases frequency resolution. From this perspective, new frequency components result because energy in nearby frequency bins spreads ("leaks") into adjacent frequency bins.

## Effects of DFT Window Length

We can control the amount/extent of frequency leakage with  $N$ .

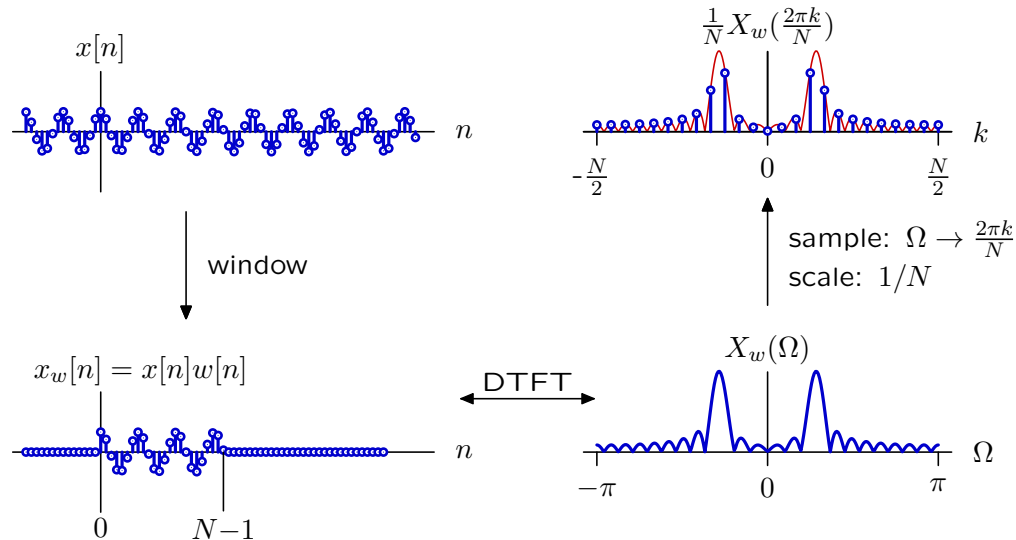
$$N = 32$$



## Effects of DFT Window Length

Decreasing the length of the window ( $N$ ) increases frequency spread.

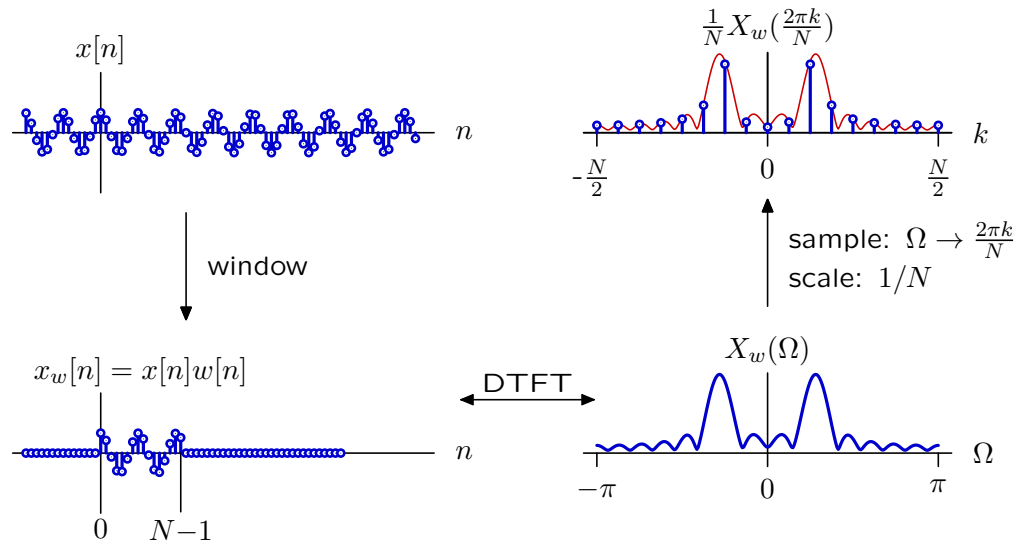
$N = 24$



## Effects of DFT Window Length

Decreasing the length of the window ( $N$ ) increases frequency spread.

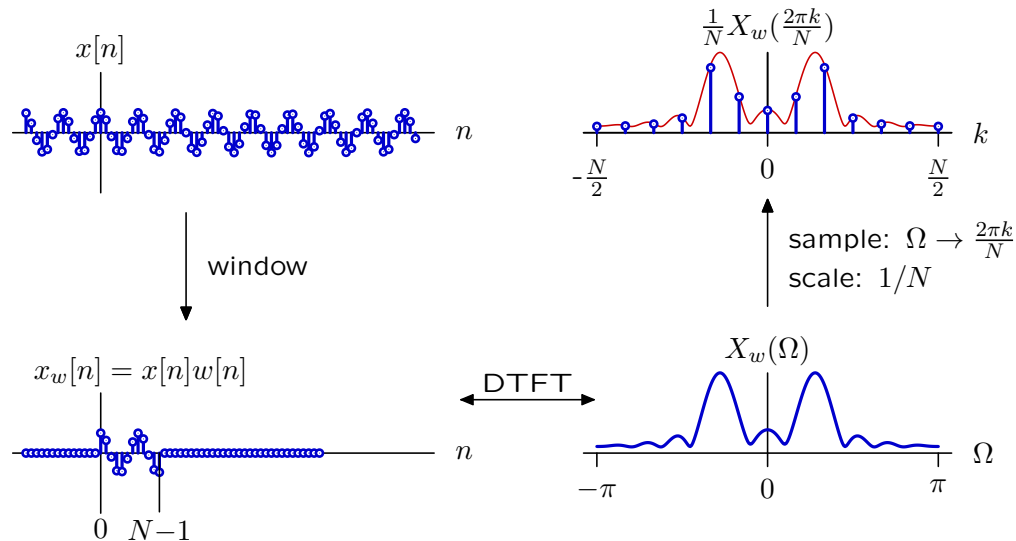
$N = 16$



## Effects of DFT Window Length

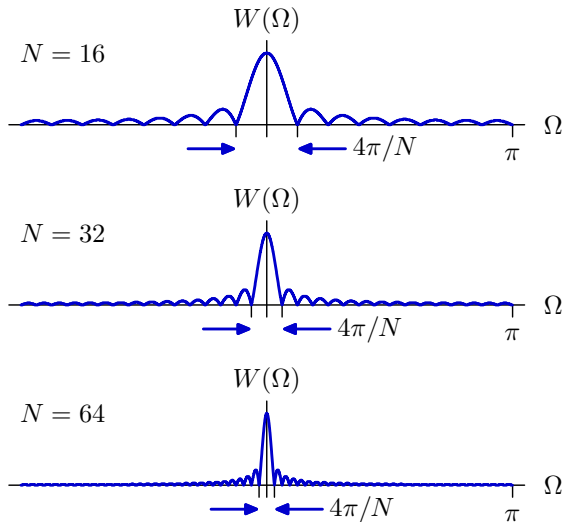
Decreasing the length of the window ( $N$ ) increases frequency spread.

$N = 12$



## Frequency Resolution

Frequency spread is fundamental to the way the DFT works. Longer windows provide finer frequency resolution.



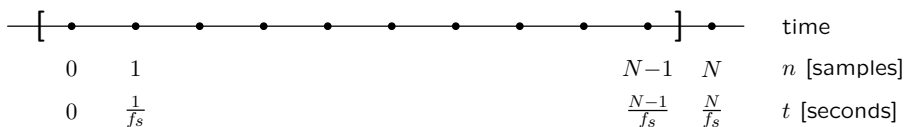
The width of the central lobe is inversely related to window length.

## Length of Analysis Window N

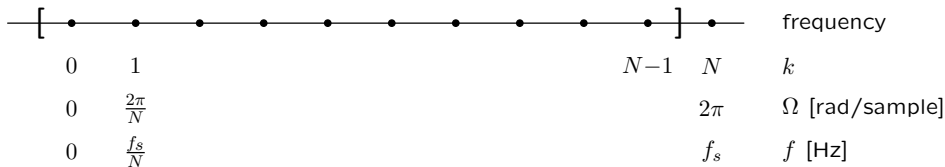
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The DFT provides a parameter ( $N$ ) to customize performance.

The time window is divided into  $N$  samples numbered  $n = 0$  to  $N-1$ .



Discrete frequencies are similarly numbered as  $k = 0$  to  $N-1$ .



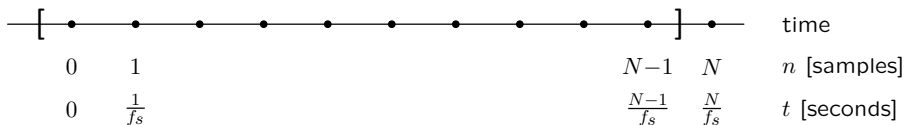
$N$  determines both the length of the window in time and the frequency resolution of the result.

## Length of Analysis Window $N$

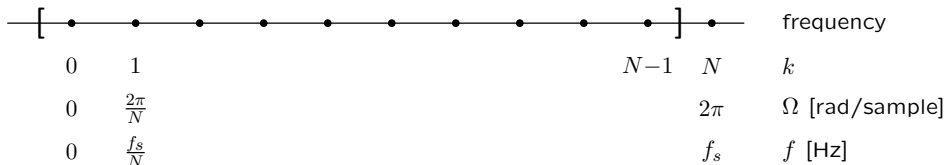
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The DFT provides a parameter ( $N$ ) to customize performance.

The time window is divided into  $N$  samples numbered  $n = 0$  to  $N-1$ .



Discrete frequencies are similarly numbered as  $k = 0$  to  $N-1$ .



How should we choose  $N$ ?

## Frequency Analysis

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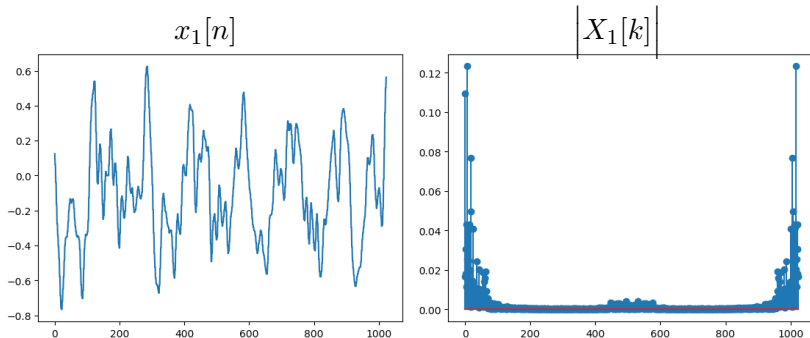
Example: Determine the frequency content of the following sound.

cello: DEb3.wav ( $f_s = 44,100$  Hz)

## Frequency Analysis

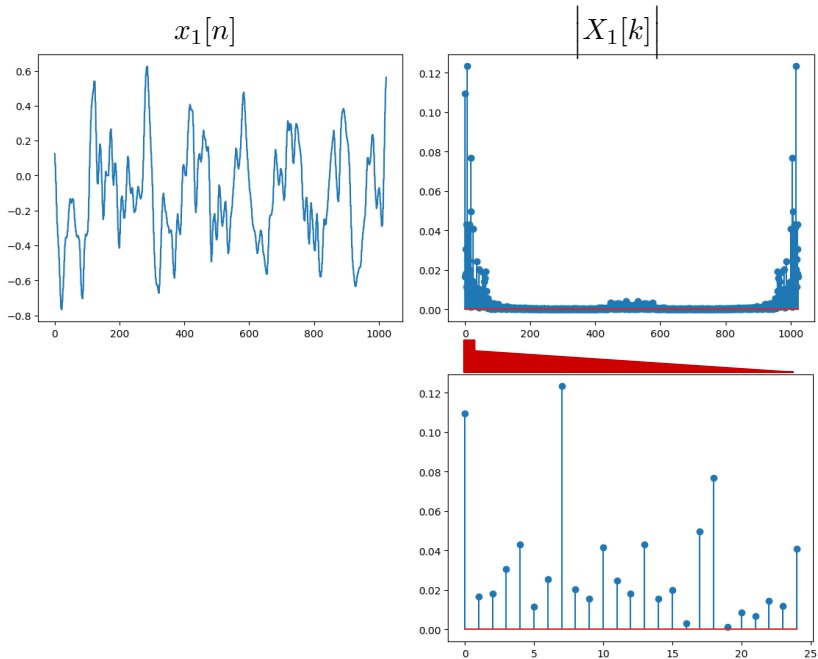
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Extract 1024 samples from the time signal and calculate DFT.



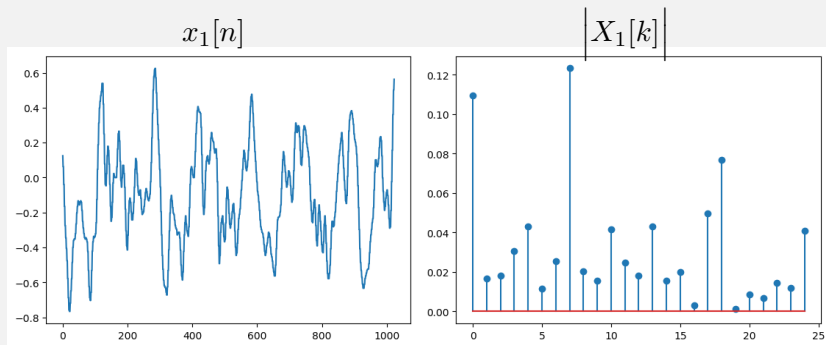
# Frequency Analysis

Information about pitch is at low frequencies. Zoom in on  $k = 0$  to 24.



## Check Yourself

The magnitude of the DFT is largest at  $k = 7$  ( $f_s = 44100$  Hz).

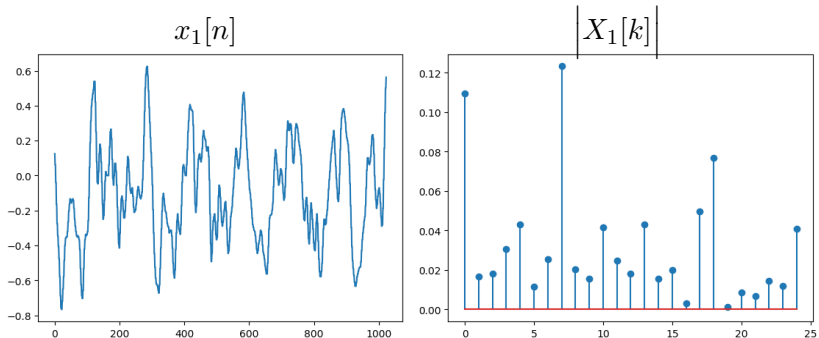


What is the corresponding frequency in Hz?

1. 293.66 Hz
2. 301.46 Hz
3. 146.83 Hz
4. 150.73 Hz
5. None of the above

## Check Yourself

The magnitude of the DFT is largest at  $k = 7$ .



What is the corresponding frequency in Hz?

Use proportional reasoning:

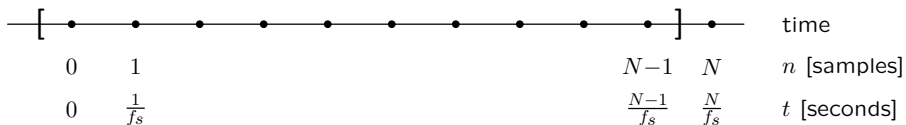
$$\frac{f_o}{f_s} = \frac{k_o}{N}$$

## Length of Analysis Window $N$

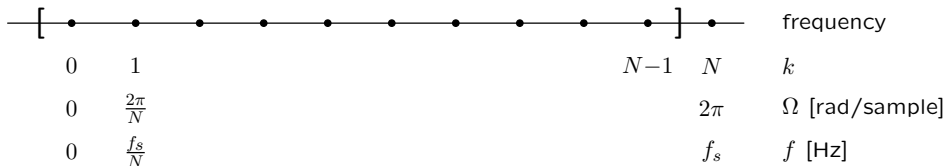
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The DFT provides a parameter ( $N$ ) to customize performance.

The time window is divided into  $N$  samples numbered  $n = 0$  to  $N-1$ .

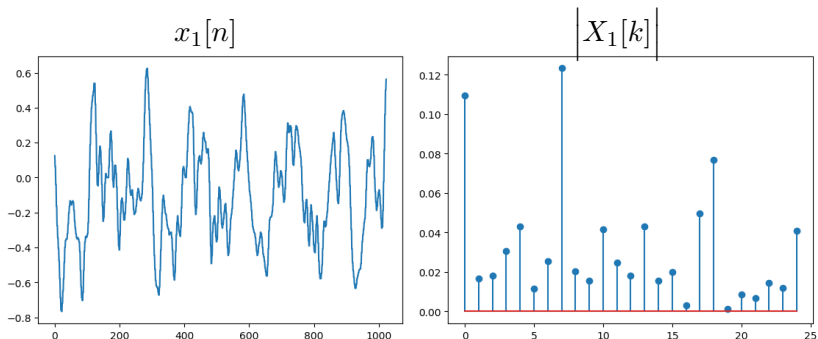


Discrete frequencies are similarly numbered as  $k = 0$  to  $N-1$ .



## Check Yourself

The magnitude of the DFT is largest at  $k = 7$ .



What is the corresponding frequency in Hz?

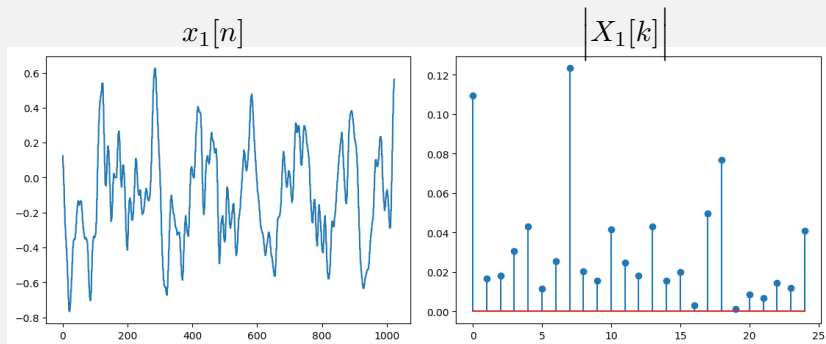
Use proportional reasoning:

$$\frac{f_o}{f_s} = \frac{k_o}{N} \rightarrow f_o = \frac{k_o}{N} f_s = \frac{7}{1024} \times 44100 \approx 301.46 \text{ Hz}$$

This frequency is between D4 (293.66 Hz) and E-flat-4 (311.13 Hz).

## Check Yourself

The magnitude of the DFT is largest at  $k = 7$  ( $f_s = 44100$  Hz).

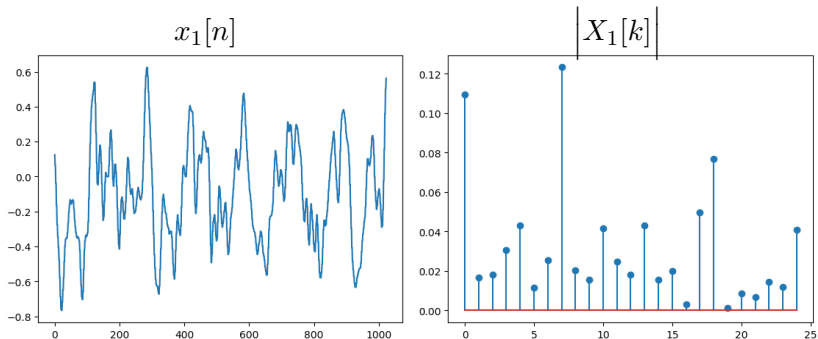


What is the corresponding frequency in Hz? 2

1. 293.66 Hz
2. 301.46 Hz
3. 146.83 Hz
4. 150.73 Hz
5. None of the above

## Frequency Analysis

Information about pitch is at low frequencies. Zoom in on  $k = 0$  to 24.



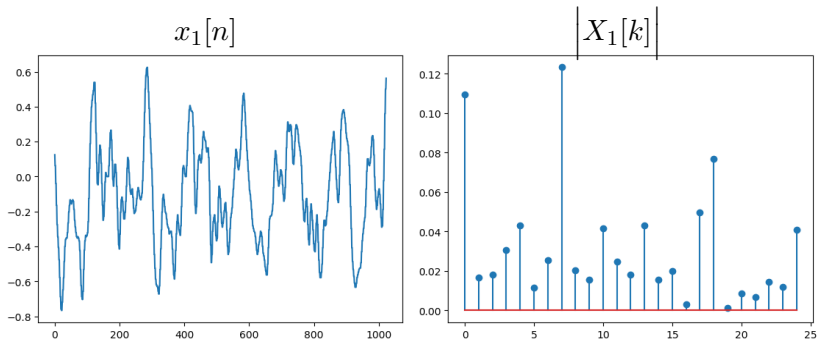
The DFT provides integer resolution in  $k$ . Therefore, the peak at  $k = 7$  is likely to be off by as much as  $\pm \frac{1}{2}$ .

$$\Delta f = \frac{\Delta k}{N} f_s = \frac{1/2}{1024} \times 44100 \approx 21.5 \text{ Hz}$$

Thus the frequency of the biggest peak is  $280 < f_o < 323$ , easily including both D (293.66 Hz) and E-flat (311.13 Hz).

## Increasing Frequency Resolution

In order to increase **frequency resolution**, we need to include more data.

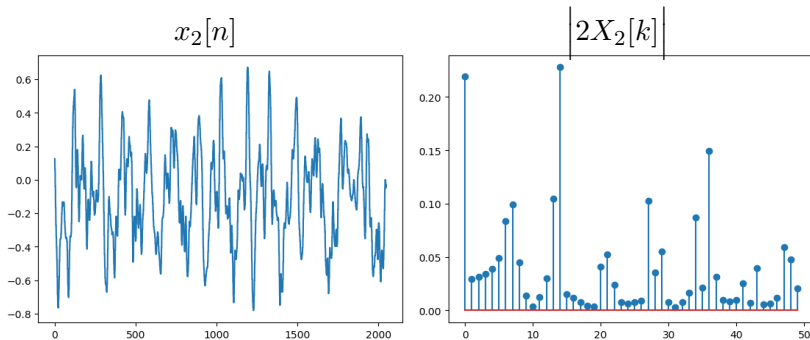


With  $N = 1024$ , the peak is at  $k = 7$ .

## Increasing Frequency Resolution

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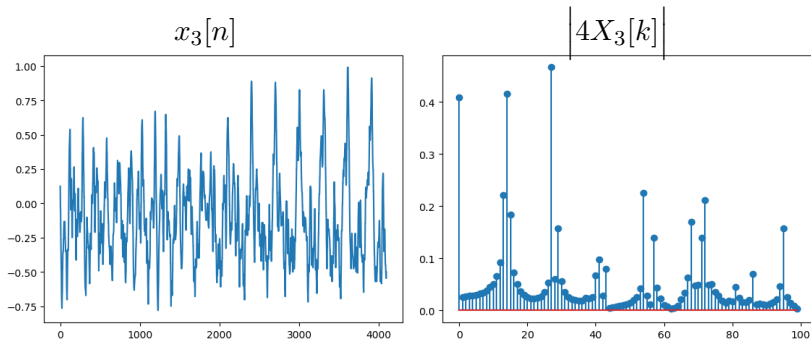
Lengthen by a factor of 2 ( $N=2048$ ).



With  $N = 2048$ , the peak is at  $k = 14$ .

## Increasing Frequency Resolution

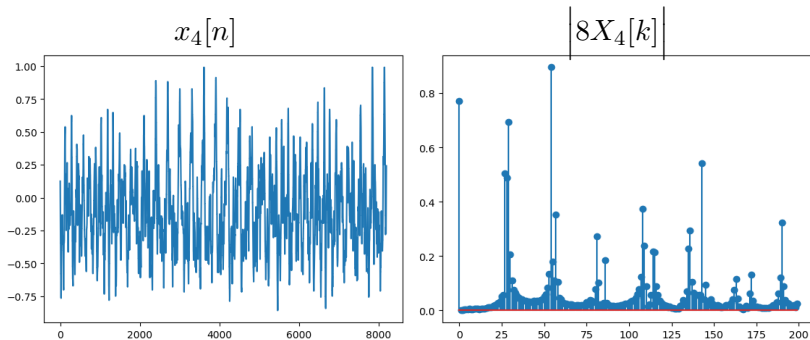
Lengthen by a factor of 4 ( $N=4096$ ).



With  $N = 4096$ , the peak is at  $k = 27$ .

## Increasing Frequency Resolution

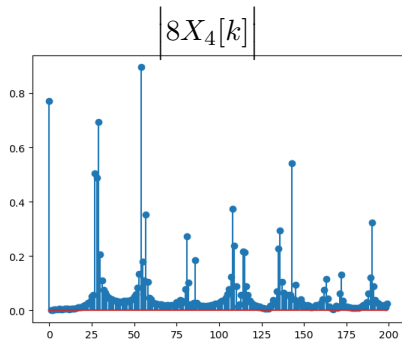
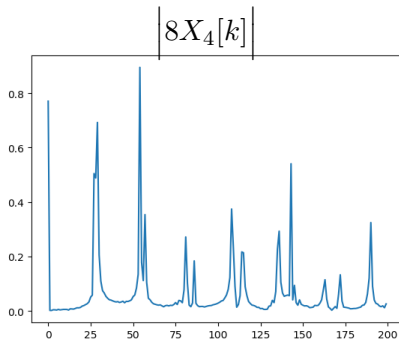
Lengthen by a factor of 8 ( $N=8192$ ).



With  $N = 8192$ , the peak is at  $k = 54$ .

## Increasing Frequency Resolution

Lengthen by a factor of 8 (N=8192).

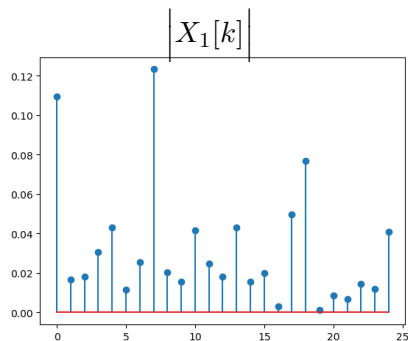
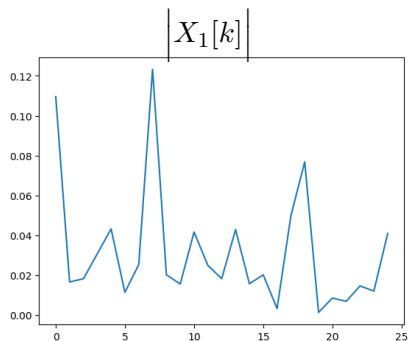


Use line plot instead of stem plot.

## Increasing Frequency Resolution

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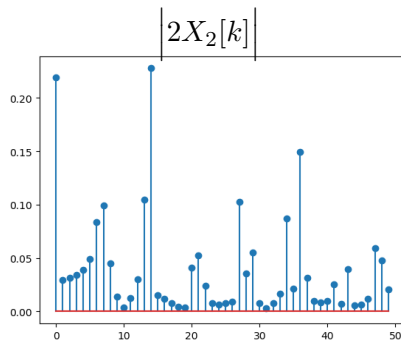
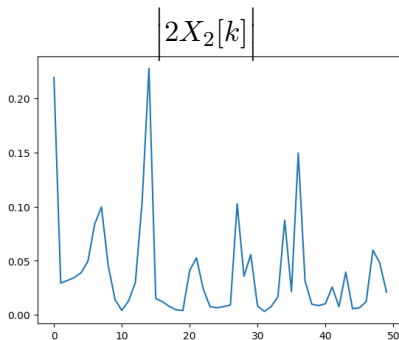
Original (N=1024).



## Increasing Frequency Resolution

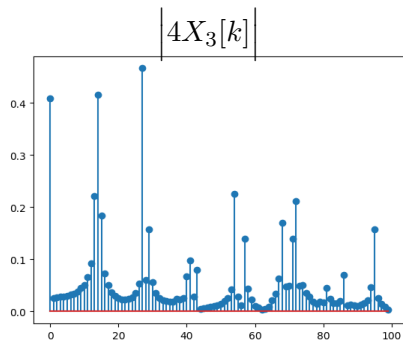
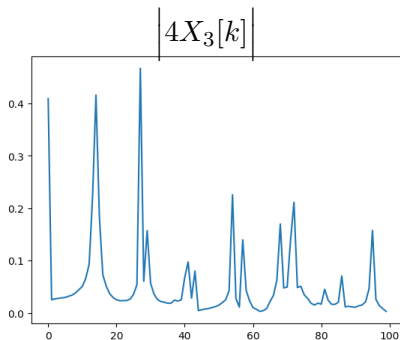
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Lengthen by a factor of 2 (N=2048).



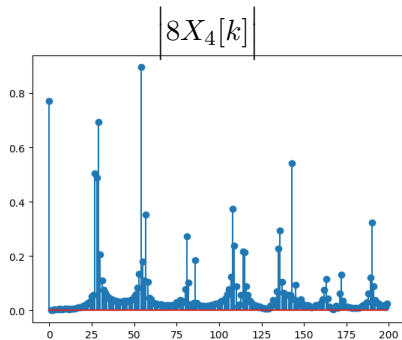
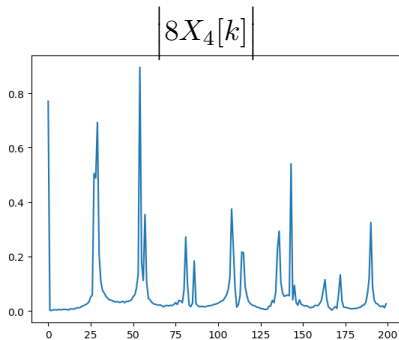
## Increasing Frequency Resolution

Lengthen by a factor of 4 ( $N=4096$ ).



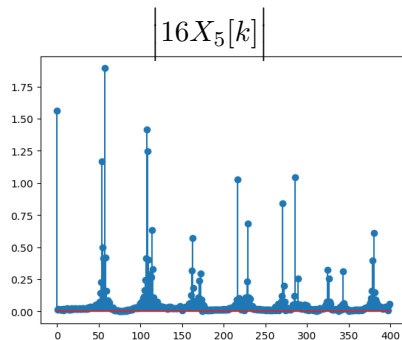
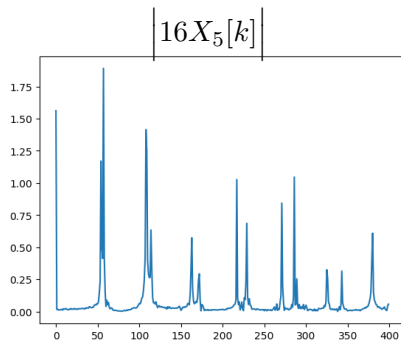
## Increasing Frequency Resolution

Lengthen by a factor of 8 ( $N=8192$ ).



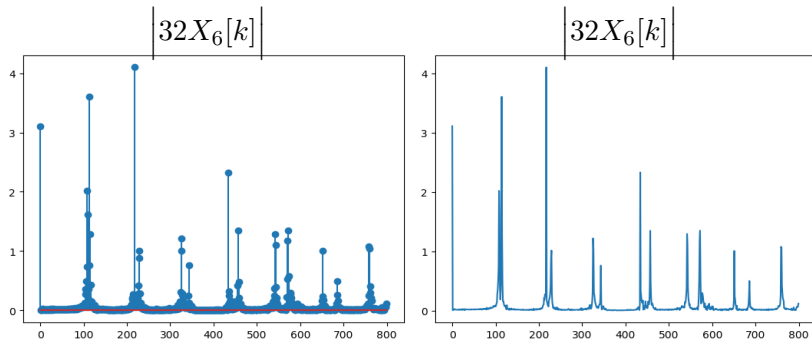
## Increasing Frequency Resolution

Lengthen by a factor of 16 ( $N=16,384$ ).



## Increasing Frequency Resolution

Lengthen by a factor of 32 ( $N=32,768$ ).

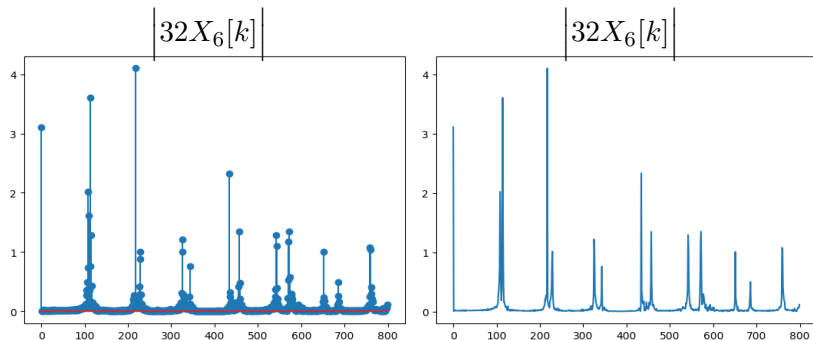


Clear peaks at  $k = 217$  and  $k = 228$  ( $f = 292.04$  Hz and  $f = 306.85$  Hz).  
→ close to D (293.66 Hz) and E-flat (311.13 Hz): both notes are present!

Anything else?

## Increasing Frequency Resolution

Lengthen by a factor of 32 ( $N=32,768$ ).



Clear peaks at  $k = 217$  and  $k = 228$  ( $f = 292.04$  Hz and  $f = 306.85$  Hz).  
→ close to D (293.66 Hz) and E-flat (311.13 Hz): both notes are present!

Notice that these are the second harmonics of lower frequencies.  
→ an octave lower than was suggested by the analysis with  $N = 1024$ .

The fundamental components were not clearly resolved with  $N = 1024$  but are clear with  $N = 32,768$ .

## Implementing Convolution with DFT

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In addition to being useful for characterizing the frequency content of a signal, the DFT can also be used to implement convolution.

Recall the convolution result for the DTFT.

If

$$f_a[n] \xrightarrow{\text{DTFT}} F_a(\Omega)$$

and

$$f_b[n] \xrightarrow{\text{DTFT}} F_b(\Omega)$$

then

$$(f_a * f_b)[n] \xrightarrow{\text{DTFT}} F_a(\Omega)F_b(\Omega)$$

## Implementing Convolution with DFT

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In addition to being useful for characterizing the frequency content of a signal, the DFT can also be used to implement convolution.

Recall the convolution result for the DTFT.

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$$f_a[n] \xrightarrow{\text{DTFT}} F_a(\Omega)$$

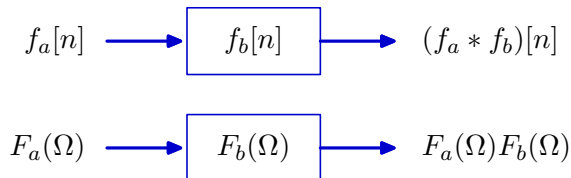
and

$$f_b[n] \xrightarrow{\text{DTFT}} F_b(\Omega)$$

then

$$(f_a * f_b)[n] \xrightarrow{\text{DTFT}} F_a(\Omega)F_b(\Omega)$$

This property is the basis of the **filtering** view of a system:



## Regular Convolution

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Why does multiplication in frequency correspond to convolution in time?

Let  $F(\Omega) = F_a(\Omega) \times F_b(\Omega)$ . Find  $f[n]$ .

$$\begin{aligned} f[n] &= \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{2\pi} F_a(\Omega) F_b(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{2\pi} F_a(\Omega) \left( \sum_{m=-\infty}^{\infty} f_b[m] e^{-j\Omega m} \right) e^{j\Omega n} d\Omega \\ &= \sum_{m=-\infty}^{\infty} f_b[m] \underbrace{\frac{1}{2\pi} \int_{2\pi} F_a(\Omega) e^{j\Omega(n-m)} d\Omega}_{f_a[n-m]} \\ &= \sum_{m=-\infty}^{\infty} f_b[m] f_a[n-m] \equiv (f_a * f_b)[n] \end{aligned}$$

Multiplying in frequency is equivalent to convolving in time.

## Implementing Convolution with DFT

---

The argument for the DFT is similar to the one for the DTFT.

Let  $F[k] = F_a[k] \times F_b[k]$ . Find  $f[n]$ .

$$\begin{aligned} f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n} \\ &= \sum_{k=0}^{N-1} F_a[k] \left( \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \underbrace{\left( \sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \right)}_{f_a[n-m]?} \end{aligned}$$

No!  $f_a[n]$  is only defined for  $0 \leq n < N$ , but  $n-m$  can fall outside that range.

## Implementing Convolution with DFT

---

The argument for the DFT is similar to the one for the DTFT.

Let  $F[k] = F_a[k] \times F_b[k]$ . Find  $f[n]$ .

$$\begin{aligned} f[n] &= \sum_{k=0}^{N-1} F[k] e^{j \frac{2\pi k}{N} n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j \frac{2\pi k}{N} n} \\ &= \sum_{k=0}^{N-1} F_a[k] \left( \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j \frac{2\pi k}{N} m} \right) e^{j \frac{2\pi k}{N} n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \underbrace{\left( \sum_{k=0}^{N-1} F_a[k] e^{j \frac{2\pi k}{N} (n-m)} \right)}_{f_a[(n-m) \bmod N]} \end{aligned}$$

Since  $k$  is integer, the complex exponential is periodic in  $n-m$ , period  $N$ . Therefore the parenthesized expression is  $f_a[(n-m) \bmod N]$ , and

$$f[n] = \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N] \equiv \frac{1}{N} (f_a \circledast f_b)[n]$$

## Implementing Convolution with DFT

---

Multiplying DFTs is equivalent to **circular** convolution in time.

If

$$f_a[n] \xrightarrow{\text{DFT}} F_a[k]$$

and

$$f_b[n] \xrightarrow{\text{DFT}} F_b[k]$$

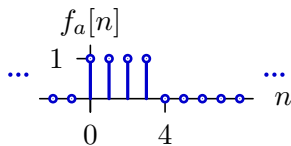
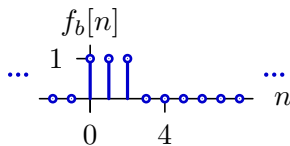
then

$$\frac{1}{N}(f_a \circledast f_b)[n] \xrightarrow{\text{DFT}} F_a[k]F_b[k]$$

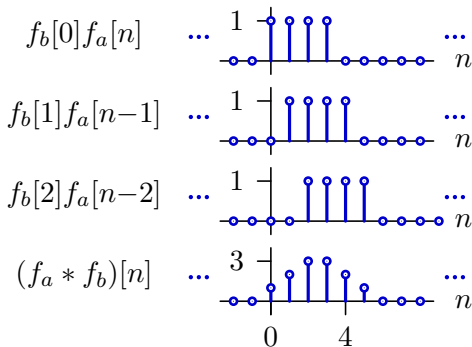
where

$$(f_a \circledast f_b)[n] = \sum_{m=0}^{N-1} f_b[m]f_a[(n-m) \bmod N]$$

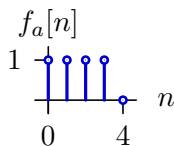
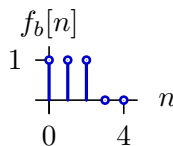
# Superposition View of Conventional Convolution



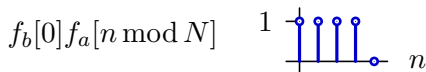
$$(f_a * f_b)[n] = \sum_{m=-\infty}^{\infty} f_b[m]f_a[n-m] = f_b[0]f_a[n] + f_b[1]f_a[n-1] + f_b[2]f_a[n-2]$$



# Superposition View of Circular Convolution (N=5)



$$\begin{aligned} (f_a \circledast f_b)[n] &= \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N] \\ &= f_b[0] f_a[n \bmod N] + f_b[1] f_a[(n-1) \bmod N] + f_b[2] f_a[(n-2) \bmod N] \end{aligned}$$



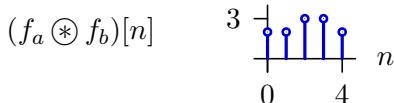
$f_a[0] \quad f_a[1] \quad f_a[2] \quad f_a[3] \quad f_a[4]$



$f_a[4] \quad f_a[0] \quad f_a[1] \quad f_a[2] \quad f_a[3]$

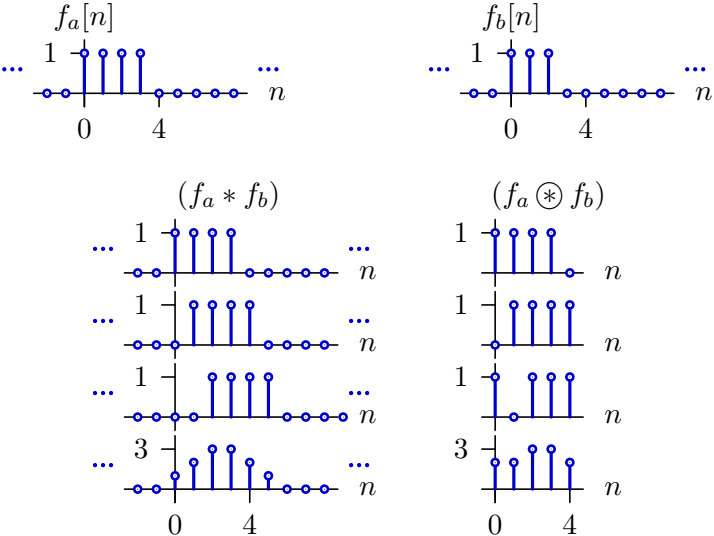


$f_a[3] \quad f_a[4] \quad f_a[0] \quad f_a[1] \quad f_a[2]$



# Side By Side

The parts of the conventional convolution that would fall outside the DFT window “alias” to points inside the DFT window.



## Summary

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Today we discussed two important issues in using the DFT.

- Frequency resolution – how the length of a signal determines the ability to discriminate frequencies using the DFT.
- Circular Convolution – how the DFT can be used to carry out time domain operations.