

6.300: Signal Processing

Discrete Fourier Transform 2

- Frequency Resolution
- Circular Convolution

October 29, 2026

Last Time

Define the Discrete Fourier Transform (DFT).

Compare the DFT to other Fourier representations.

analysis

synthesis

DTFS:
$$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n}$$

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$$

DTFT:
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega$$

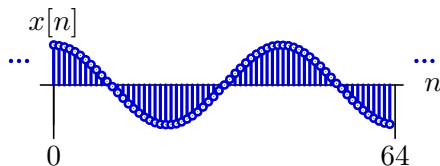
DFT:
$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi k}{N} n}$$

$$x[n] = \sum_{n=\langle N \rangle} X[k] e^{j \frac{2\pi k}{N} n}$$

The DFT is a powerful Fourier representation that is easy to compute, but results can be tricky to interpret.

Interpreting the DFT

The 64-point DFT of $x[n] = \cos \frac{3\pi n}{64}$



has multiple frequency “components” that we might not expect since $x[n]$ is a single sinusoid.



Fourier series (DTFS) and Fourier transforms (DTFT) offer different, complementary interpretations of this surprising result.

DTFS Interpretation: Periodic Extension

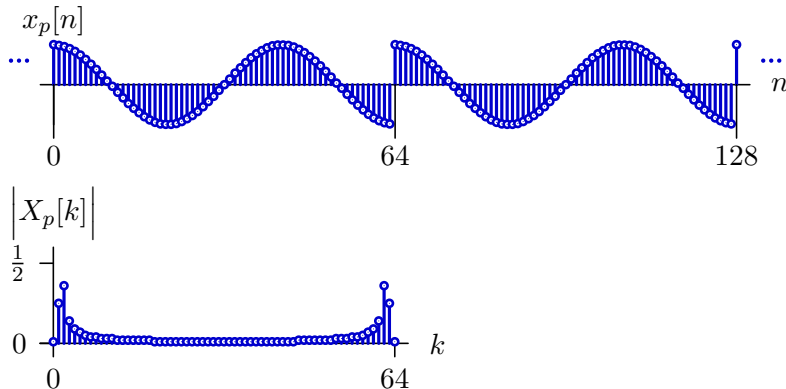
The 64-point **DFT** of $x[n]$:

$$x[n] = \cos \frac{3\pi n}{64}$$

is equal to the 64-point **DTFS** of $x_p[n]$:

$$x_p[n] = x[n \bmod 64]$$

where $x_p[n]$ represents the 64-point **periodic extension** of $x[n]$.

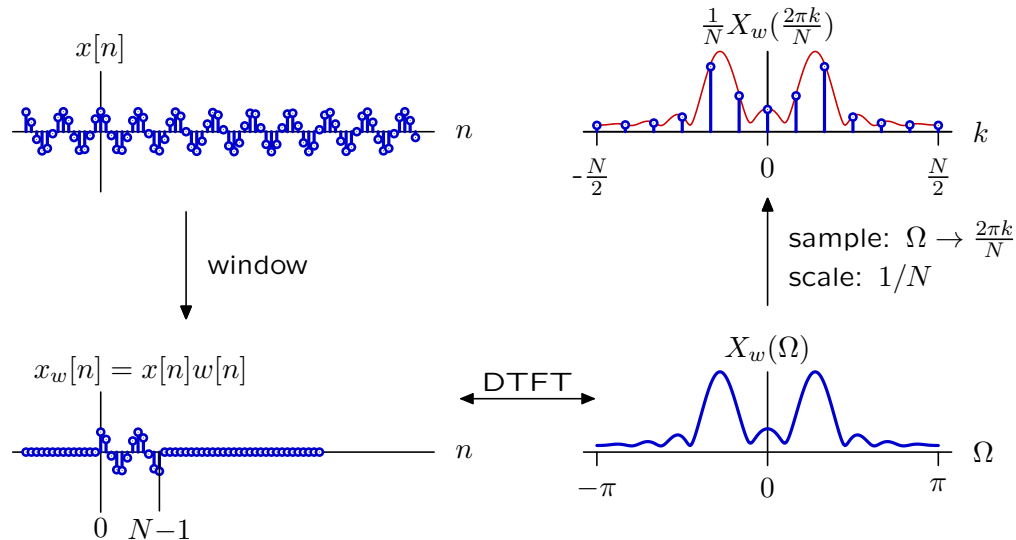


From this perspective, periodic extension introduces additional frequency components that are needed to generate the step discontinuity in $x_p[n]$.

Effects of DFT Window Length

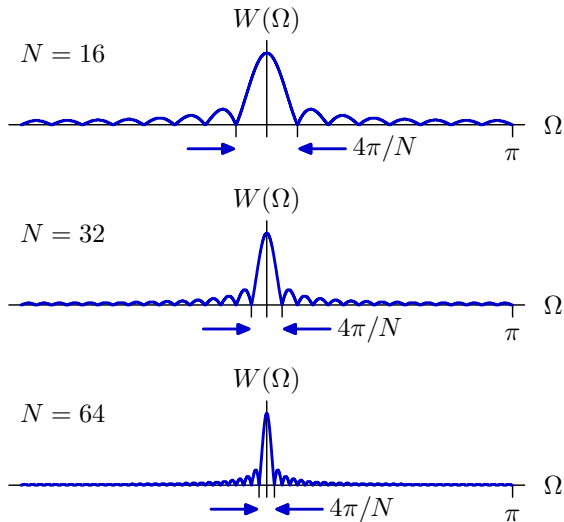
Decreasing the length of the window (N) increases frequency spread.

$N = 12$



Frequency Resolution

Frequency spread is fundamental to the way the DFT works. Longer windows provide finer frequency resolution.

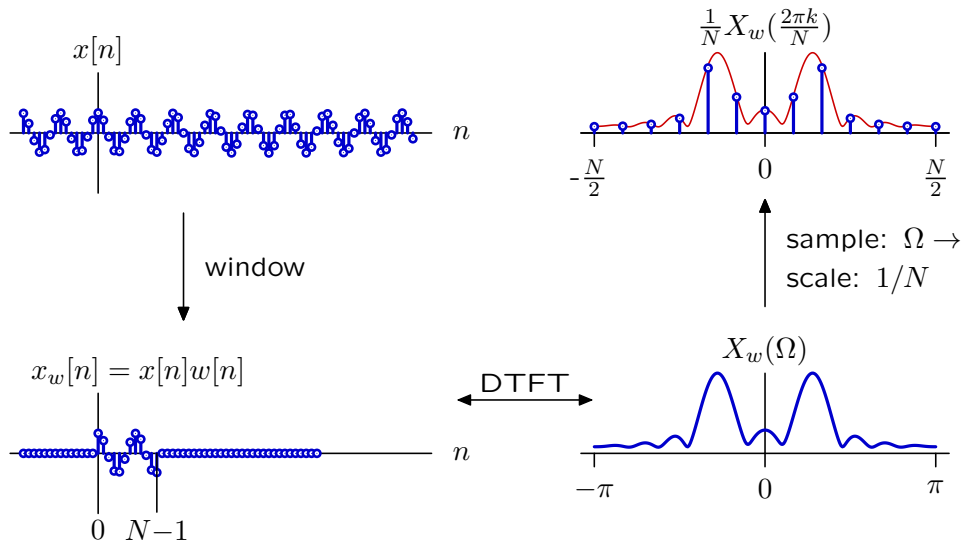


The width of the central lobe is inversely related to window length.

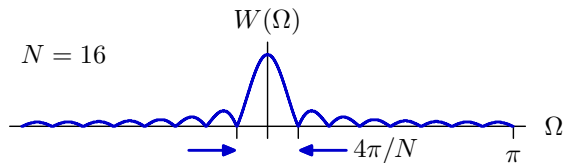
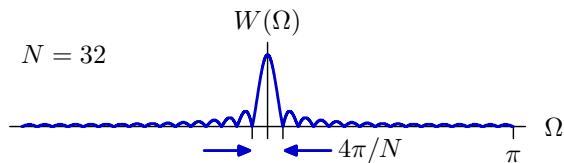
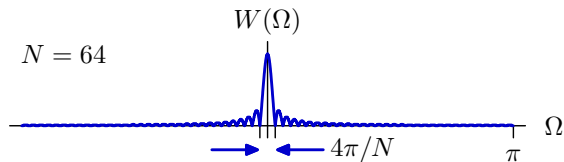
Length of Analysis Window N

The DFT provides a parameter (N) to customize performance.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



How should we choose N ?

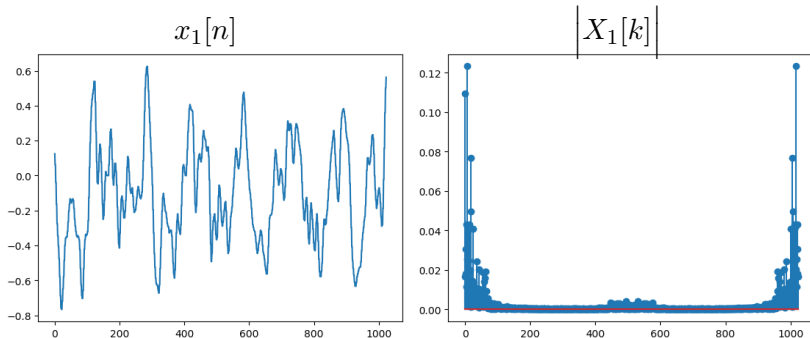
Frequency Analysis

Example: Determine the frequency content of the following sound.

cello: DEb3.wav ($f_s = 44,100$ Hz)

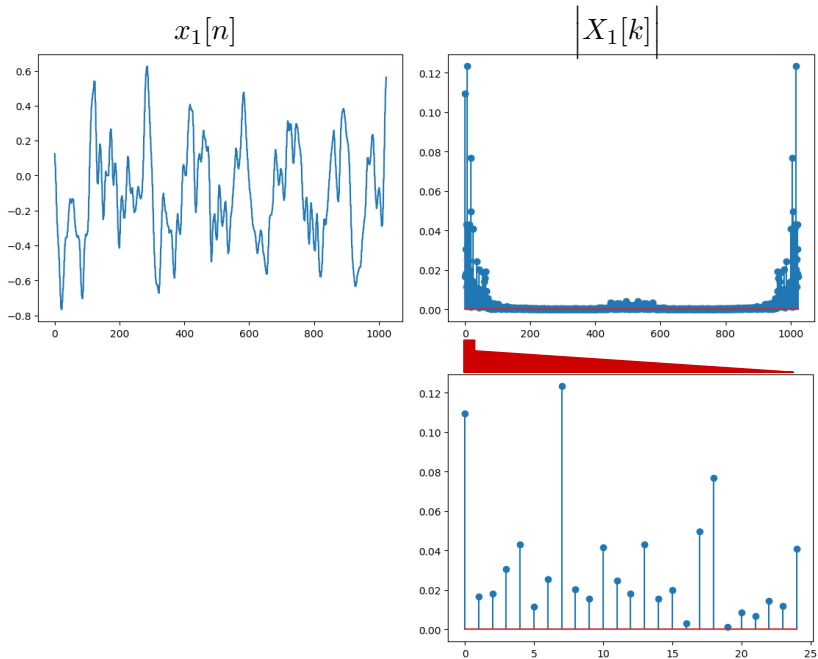
Frequency Analysis

Extract 1024 samples from the time signal and calculate DFT.



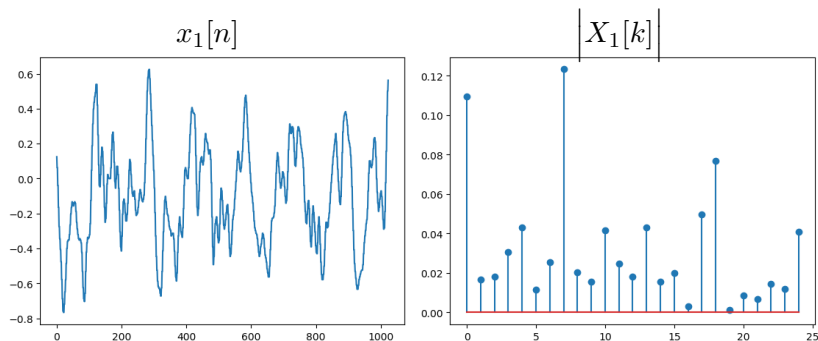
Frequency Analysis

Information about pitch is at low frequencies. Zoom in on $k = 0$ to 24.



Check Yourself

The magnitude of the DFT is largest at $k = 7$ ($f_s = 44100$ Hz).

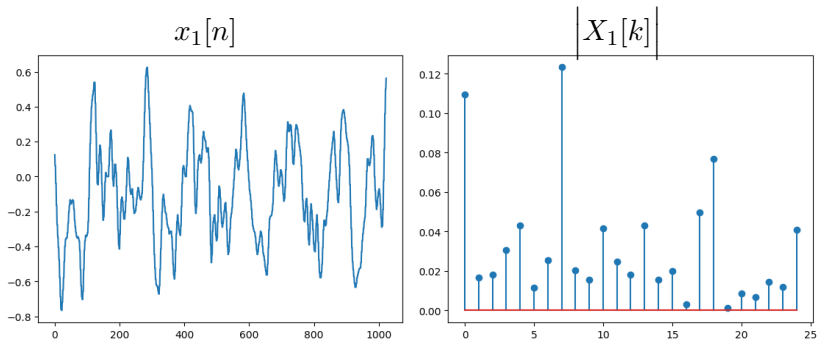


What is the corresponding frequency in Hz?

1. 293.66 Hz
2. 301.46 Hz
3. 146.83 Hz
4. 150.73 Hz
5. None of the above

Frequency Analysis

Information about pitch is at low frequencies. Zoom in on $k = 0$ to 24.



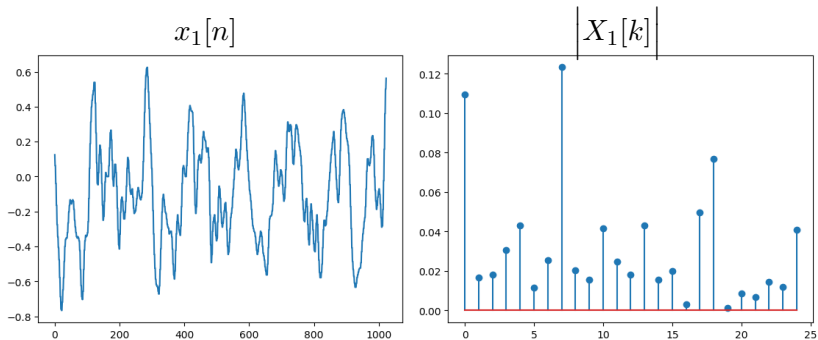
The DFT provides integer resolution in k . Therefore, the peak at $k = 7$ is likely to be off by as much as $\pm \frac{1}{2}$.

$$\Delta f = \frac{\Delta k}{N} f_s = \frac{1/2}{1024} \times 44100 \approx 21.5 \text{ Hz}$$

Thus the frequency of the biggest peak is $280 < f_o < 323$, easily including both D (293.66 Hz) and E-flat (311.13 Hz).

Increasing Frequency Resolution

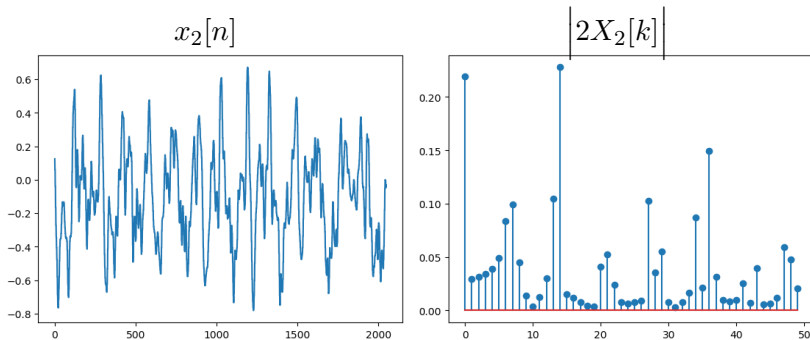
In order to increase **frequency resolution**, we need to include more data.



With $N = 1024$, the peak is at $k = 7$.

Increasing Frequency Resolution

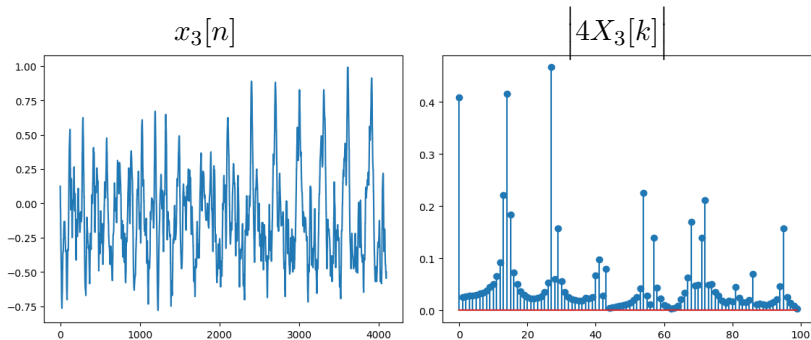
Lengthen by a factor of 2 ($N=2048$).



With $N = 2048$, the peak is at $k = 14$.

Increasing Frequency Resolution

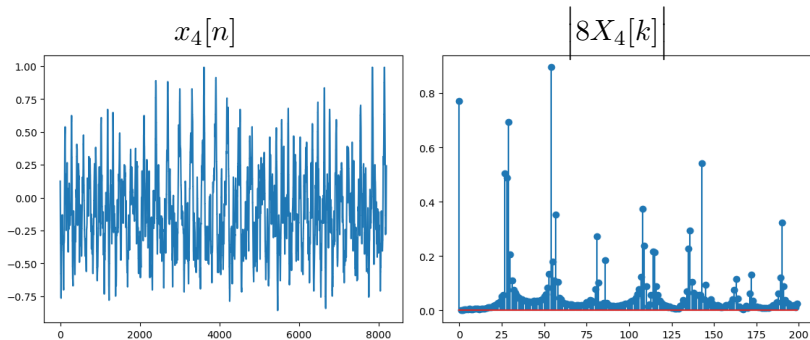
Lengthen by a factor of 4 ($N=4096$).



With $N = 4096$, the peak is at $k = 27$.

Increasing Frequency Resolution

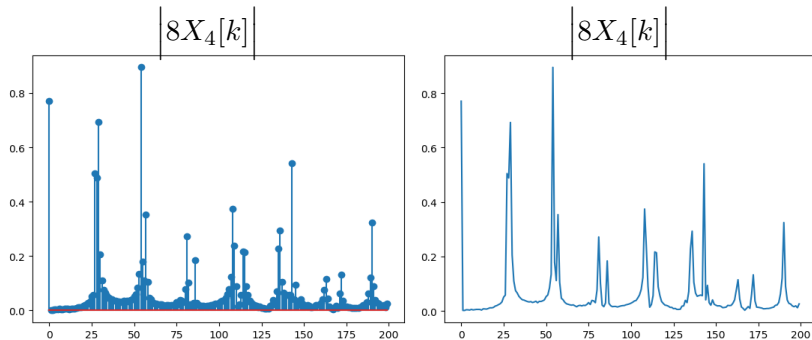
Lengthen by a factor of 8 ($N=8192$).



With $N = 8192$, the peak is at $k = 54$.

Increasing Frequency Resolution

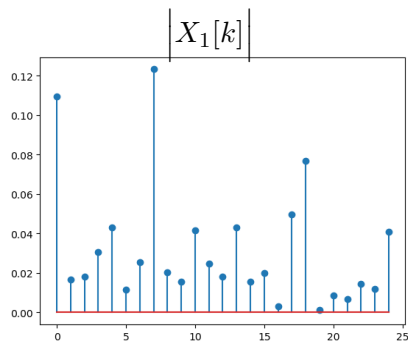
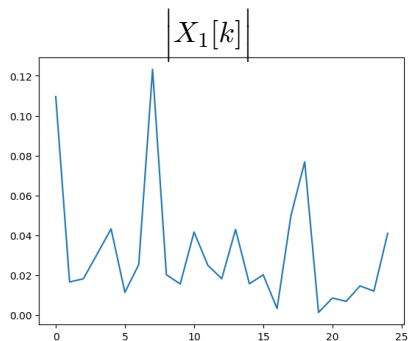
Lengthen by a factor of 8 (N=8192).



Use line plot instead of stem plot.

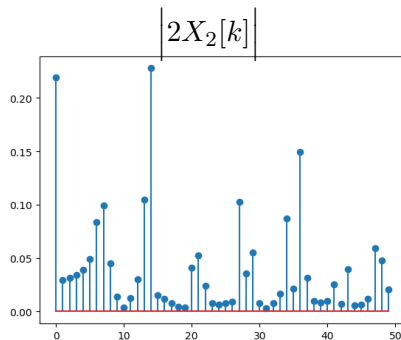
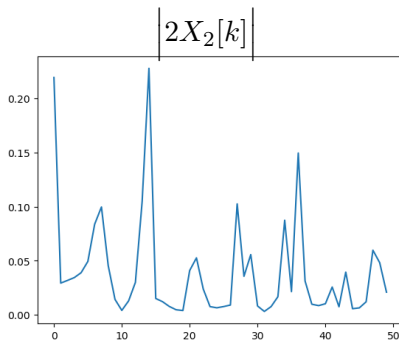
Increasing Frequency Resolution

Original (N=1024).



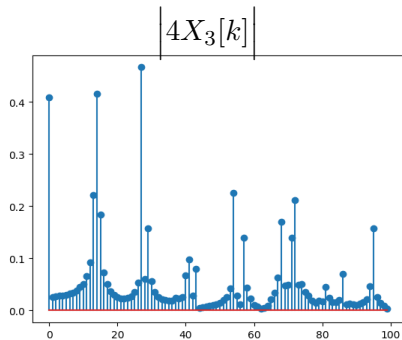
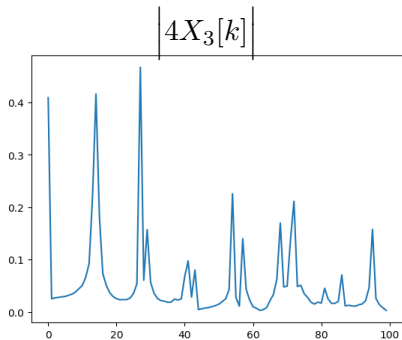
Increasing Frequency Resolution

Lengthen by a factor of 2 (N=2048).



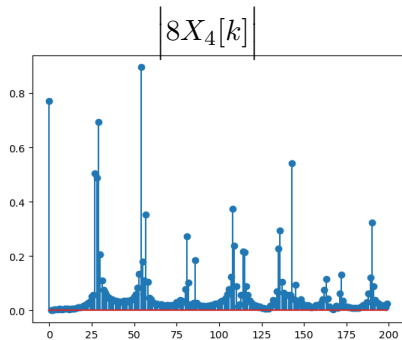
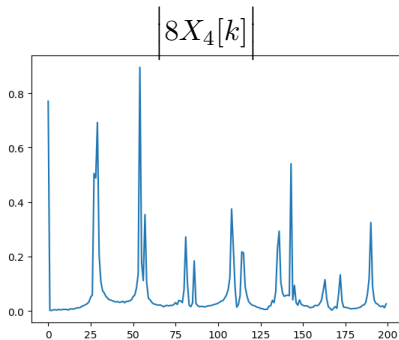
Increasing Frequency Resolution

Lengthen by a factor of 4 (N=4096).



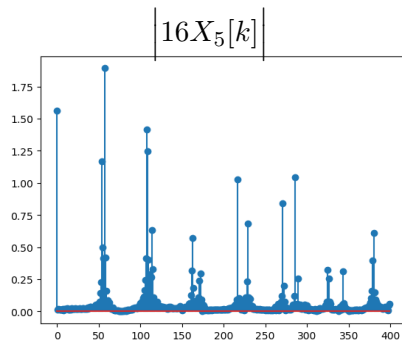
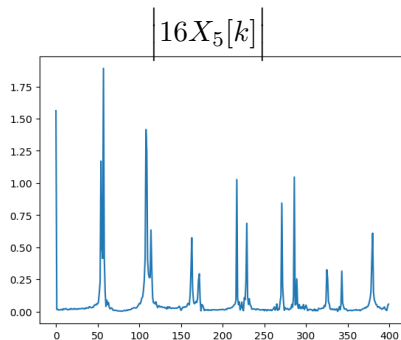
Increasing Frequency Resolution

Lengthen by a factor of 8 ($N=8192$).



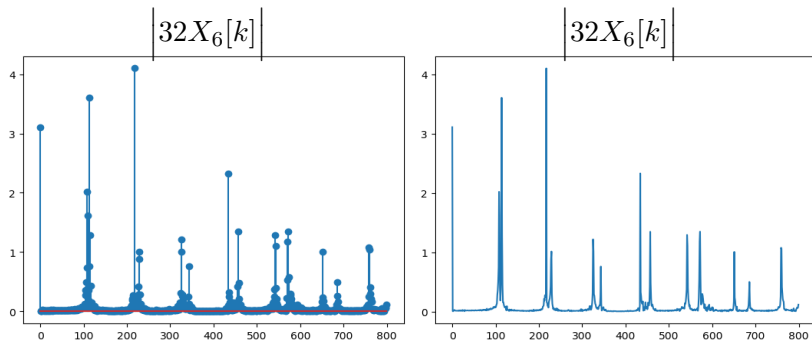
Increasing Frequency Resolution

Lengthen by a factor of 16 ($N=16,384$).



Increasing Frequency Resolution

Lengthen by a factor of 32 ($N=32,768$).



Clear peaks at $k = 217$ and $k = 228$ ($f = 292.04$ Hz and $f = 306.85$ Hz).
→ close to D (293.66 Hz) and E-flat (311.13 Hz): both notes are present!

Anything else?

Implementing Convolution with DFT

In addition to being useful for characterizing the frequency content of a signal, the DFT can also be used to implement convolution.

Recall the convolution result for the DTFT.

If

$$f_a[n] \xrightarrow{\text{DTFT}} F_a(\Omega)$$

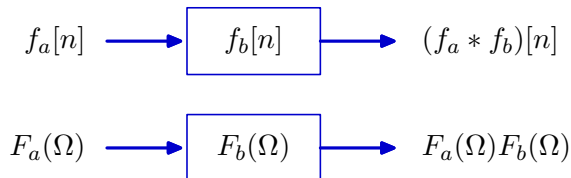
and

$$f_b[n] \xrightarrow{\text{DTFT}} F_b(\Omega)$$

then

$$(f_a * f_b)[n] \xrightarrow{\text{DTFT}} F_a(\Omega)F_b(\Omega)$$

This property is the basis of the **filtering** view of a system:



Regular Convolution

Why does multiplication in frequency correspond to convolution in time?

Let $F(\Omega) = F_a(\Omega) \times F_b(\Omega)$. Find $f[n]$.

$$\begin{aligned} f[n] &= \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{2\pi} F_a(\Omega) F_b(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{2\pi} F_a(\Omega) \left(\sum_{m=-\infty}^{\infty} f_b[m] e^{-j\Omega m} \right) e^{j\Omega n} d\Omega \\ &= \sum_{m=-\infty}^{\infty} f_b[m] \underbrace{\frac{1}{2\pi} \int_{2\pi} F_a(\Omega) e^{j\Omega(n-m)} d\Omega}_{f_a[n-m]} \\ &= \sum_{m=-\infty}^{\infty} f_b[m] f_a[n-m] \equiv (f_a * f_b)[n] \end{aligned}$$

Multiplying in frequency is equivalent to convolving in time.

Implementing Convolution with DFT

The argument for the DFT is similar to the one for the DTFT.

Let $F[k] = F_a[k] \times F_b[k]$. Find $f[n]$.

$$\begin{aligned} f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n} \\ &= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \underbrace{\left(\sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \right)}_{f_a[n-m]?} \end{aligned}$$

No! $f_a[n]$ is only defined for $0 \leq n < N$, but $n-m$ can fall outside that range.

Implementing Convolution with DFT

The argument for the DFT is similar to the one for the DTFT.

Let $F[k] = F_a[k] \times F_b[k]$. Find $f[n]$.

$$\begin{aligned} f[n] &= \sum_{k=0}^{N-1} F[k] e^{j \frac{2\pi k}{N} n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j \frac{2\pi k}{N} n} \\ &= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j \frac{2\pi k}{N} m} \right) e^{j \frac{2\pi k}{N} n} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \underbrace{\left(\sum_{k=0}^{N-1} F_a[k] e^{j \frac{2\pi k}{N} (n-m)} \right)}_{f_a[(n-m) \bmod N]} \end{aligned}$$

Since k is integer, the complex exponential is periodic in $n-m$, period N . Therefore the parenthesized expression is $f_a[(n-m) \bmod N]$, and

$$f[n] = \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N] \equiv \frac{1}{N} (f_a \circledast f_b)[n]$$

Implementing Convolution with DFT

Multiplying DFTs is equivalent to **circular** convolution in time.

If

$$f_a[n] \xrightarrow{\text{DFT}} F_a[k]$$

and

$$f_b[n] \xrightarrow{\text{DFT}} F_b[k]$$

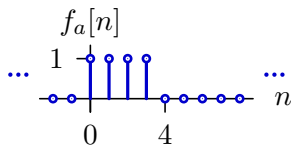
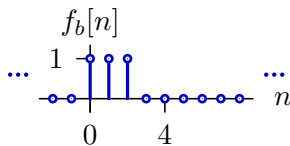
then

$$\frac{1}{N}(f_a \circledast f_b)[n] \xrightarrow{\text{DFT}} F_a[k]F_b[k]$$

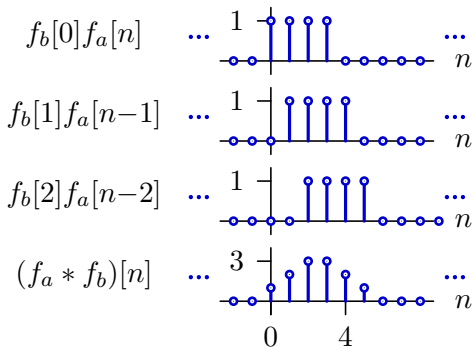
where

$$(f_a \circledast f_b)[n] = \sum_{m=0}^{N-1} f_b[m]f_a[(n-m) \bmod N]$$

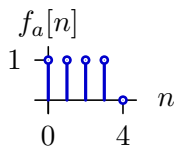
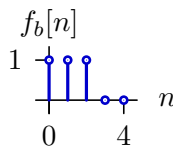
Superposition View of Conventional Convolution



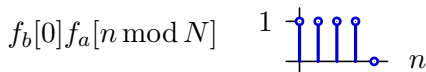
$$(f_a * f_b)[n] = \sum_{m=-\infty}^{\infty} f_b[m]f_a[n-m] = f_b[0]f_a[n] + f_b[1]f_a[n-1] + f_b[2]f_a[n-2]$$



Superposition View of Circular Convolution (N=5)



$$\begin{aligned} (f_a \circledast f_b)[n] &= \sum_{m=0}^{N-1} f_b[m] f_a[(n-m) \bmod N] \\ &= f_b[0] f_a[n \bmod N] + f_b[1] f_a[(n-1) \bmod N] + f_b[2] f_a[(n-2) \bmod N] \end{aligned}$$



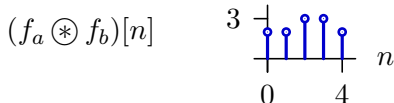
$f_a[0] \quad f_a[1] \quad f_a[2] \quad f_a[3] \quad f_a[4]$



$f_a[4] \quad f_a[0] \quad f_a[1] \quad f_a[2] \quad f_a[3]$

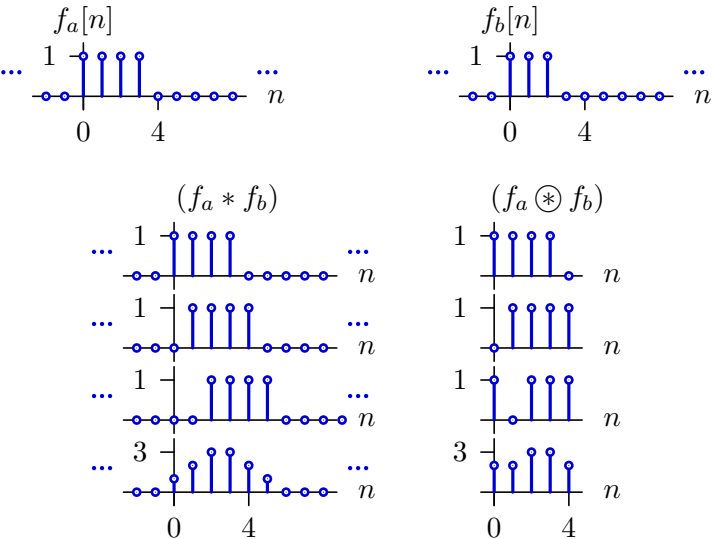


$f_a[3] \quad f_a[4] \quad f_a[0] \quad f_a[1] \quad f_a[2]$



Side By Side

The parts of the conventional convolution that would fall outside the DFT window “alias” to points inside the DFT window.



Summary

Today we discussed two important issues in using the DFT.

- Frequency resolution – how the length of a signal determines the ability to discriminate frequencies using the DFT.
- Circular Convolution – how the DFT can be used to carry out time domain operations.