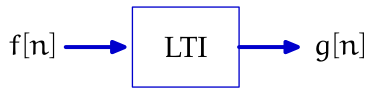
4 Recurring Delay (22 points)

A discrete-time, linear, time-invariant system has input $f[n]$ and output $g[n]$



and these signals are related by the following difference equation:

$$g[n] = f[n] - \frac{1}{2}g[n-3]$$

Part a. Assume that the output $g[n] = 0$ for $n < 0$, and find the output of the system when the input $f[n] = \delta[n]$. Enter an expression for your result in the following box.

$$g[n] = \begin{cases} \left(-\frac{1}{2}\right)^{n/3} & \text{if } n = 0, 3, 6, 9, \ldots \\ 0 & \text{otherwise} \end{cases}$$

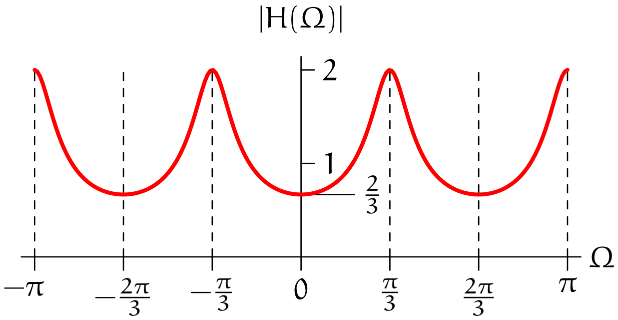
If $g[n] = 0$ for $n \leqslant 0$ and $f[n] = \delta[n]$ then

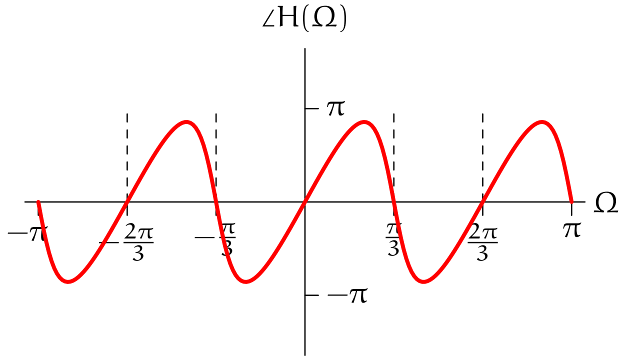
$$\begin{aligned}
g[0] &= f[0] - \tfrac{1}{2}g[-3] = 1 - 0 = 1 \\
g[1] &= f[1] - \tfrac{1}{2}g[-2] = 0 - 0 = 0 \\
g[2] &= f[2] - \tfrac{1}{2}g[-1] = 0 - 0 = 0 \\
g[3] &= f[3] - \tfrac{1}{2}g[0] \quad = 0 - \tfrac{1}{2} = -\tfrac{1}{2} \\
g[4] &= f[4] - \tfrac{1}{2}g[1] \quad = 0 - 0 = 0 \\
g[5] &= f[5] - \tfrac{1}{2}g[2] \quad = 0 - 0 = 0 \\
g[6] &= f[6] - \tfrac{1}{2}g[3] \quad = 0 + \tfrac{1}{4} = \tfrac{1}{4} \\
g[7] &= f[7] - \tfrac{1}{2}g[4] \quad = 0 - 0 = 0 \\
g[8] &= f[8] - \tfrac{1}{2}g[5] \quad = 0 - 0 = 0 \\
g[9] &= f[9] - \tfrac{1}{2}g[6] \quad = 0 + \tfrac{1}{8} = -\tfrac{1}{8} \\
&\ldots
\end{aligned}$$

In general,

$$g[n] = \begin{cases} \left(-\frac{1}{2}\right)^{n/3} & \text{if } n = 0, 3, 6, 9, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Part b. Sketch the frequency response of this system on the axes below, with magnitude on the left and phase on the right. Label all of the important points on your sketches.

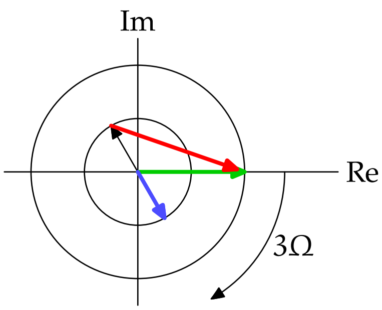




$$G(\Omega) = F(\Omega) - \frac{1}{2}e^{-j3\Omega}G(\Omega)$$

$$H(\Omega) = \frac{G(\Omega)}{F(\Omega)} = \frac{1}{1+\frac{1}{2}e^{-j3\Omega}} = \frac{1+\frac{1}{2}e^{j3\Omega}}{(1+\frac{1}{2}e^{-j3\Omega})(1+\frac{1}{2}e^{j3\Omega})} = \frac{1+\frac{1}{2}\cos(3\Omega)+\frac{1}{2}j\sin(3\Omega)}{\frac{5}{4}+\cos(3\Omega)}$$

$$H(\Omega) = \frac{1}{1+\frac{1}{2}e^{-j3\Omega}} = \frac{1}{1-\frac{1}{2}e^{j\pi}e^{-j3\Omega}} = \frac{1}{1-\frac{1}{2}e^{j(-3\Omega-\pi)}}$$



green arrow: 1

blue arrow: $\frac{1}{2}e^{-j3\Omega}$

black arrow: $\frac{1}{2}e^{-j(3\Omega-\pi)}$

red arrow: $1-\frac{1}{2}e^{-j(3\Omega-\pi)}$

2 Periodic Extension (20 points)

Let $f[n]$ represent a real-valued, discrete-time signal, and let $g[n]$ represent a periodically extended version of $f[n]$, as follows:

$$g[n] = \begin{cases} f[n] & \text{if } 0 \leqslant n < N \\ f[n-N] & \text{if } N \leqslant n < 2N \\ f[n-2N] & \text{if } 2N \leqslant n < 3N \\ f[n-mN] & \text{if } mN \leqslant n < (m+1)N \end{cases}$$

where m represents an integer that is greater than or equal to 3.

Let $F[k]$ represent the DFT of the first N samples of $f[n]$.

Let $G[k]$ represent the DFT of the first $2N$ samples of $g[n]$.

Part a. Determine an expression for $G[0]$ in terms of the DFT coefficients $F[k]$ as well as familiar constants such as π and e. Simplify your expression as much as possible, and enter your result in the box below.

$G[0] =$ $F[0]$

The first half of G is the same as F. Therefore the DC value of the first half of G is equal to the DC value of F.

The second half of G is the same as F. Therefore the DC value of the second half of G is equal to the DC value of F.

The DC value of G is the average of the DC values of its first and second halves. Therefore the DC value of G is equal to the DC value of F.

Part b. Determine an expression for $G[1]$ in terms of the DFT coefficients $F[k]$ as well as familiar constants such as π and e. Simplify your expression as much as possible, and enter your result in the box below.

$G[1] =$ 0

$$\begin{aligned} G[1] &= \sum_{n=0}^{2N-1} g[n]e^{\frac{-j2\pi kn}{2N}} \\ &= \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} + \sum_{n=N}^{2N-1} f[n-N]e^{\frac{-j2\pi kn}{2N}} \\ &= \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} + \sum_{m=0}^{N-1} f[m]e^{\frac{-j2\pi k(m+N)}{2N}} \\ &= \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} + \sum_{m=0}^{N-1} f[m]e^{\frac{-j2\pi km}{2N}} e^{\frac{-j2\pi kN}{2N}} \\ &= (1+e^{-j\pi k}) \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} = (1+e^{-j\pi}) \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi n}{2N}} \\ &= 0 \end{aligned}$$

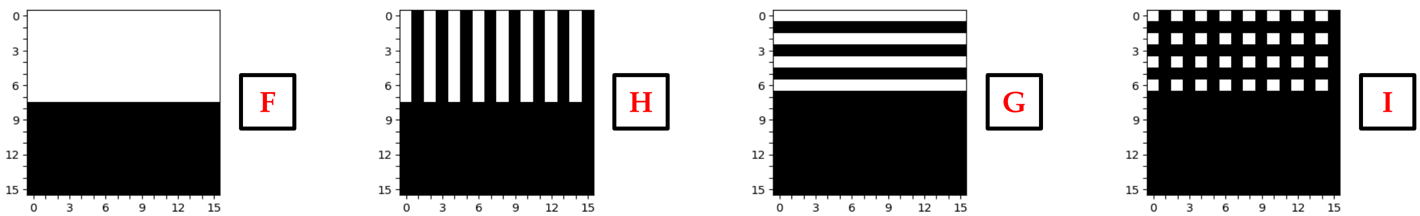
Part c. Determine an expression for $G[k]$ in terms of the DFT coefficients $F[k]$ as well as familiar constants such as π and e. Simplify your expression as much as possible, and enter your result in the box below.

$$G[k] = \begin{cases} F[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

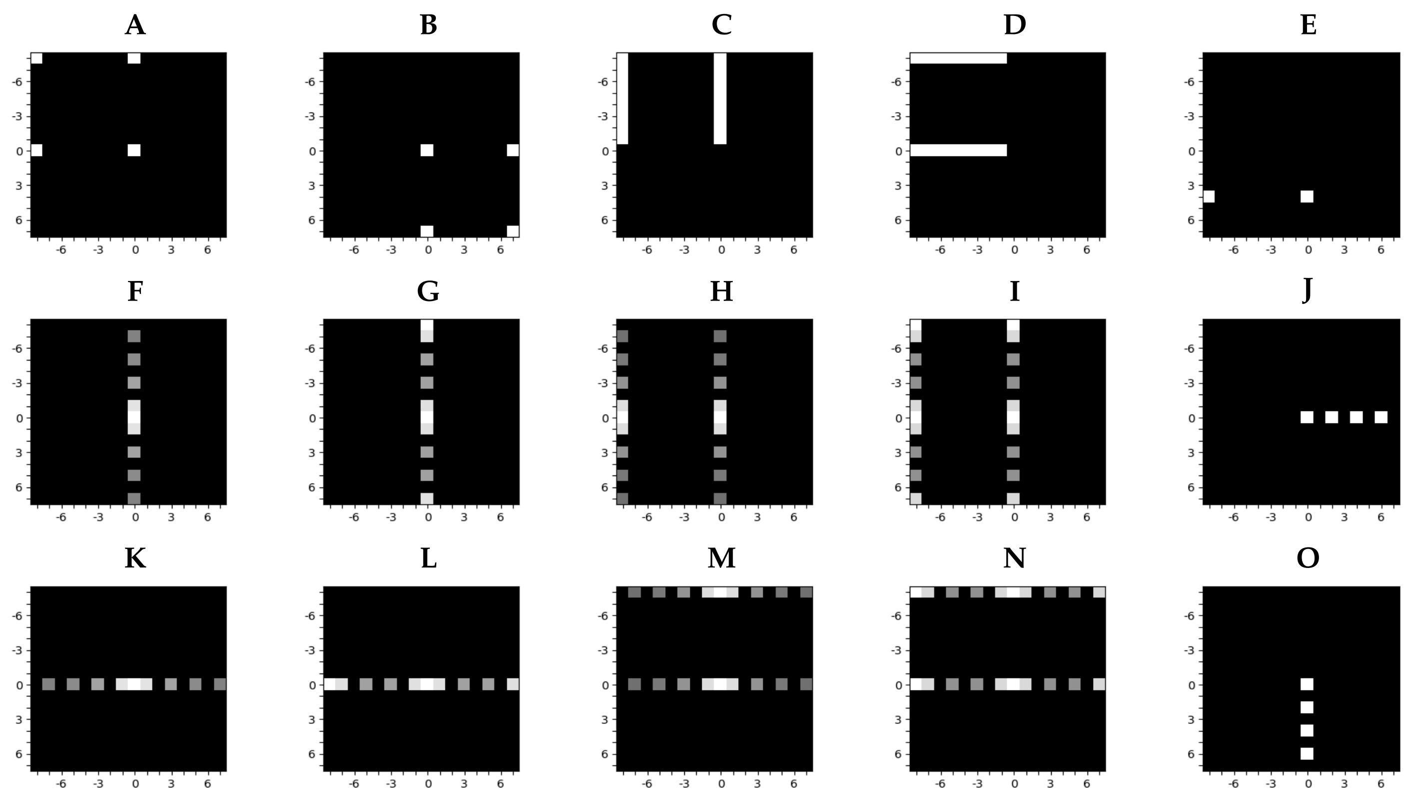
$$\begin{aligned}
G[k] &= \frac{1}{2N}\sum_{n=0}^{2N-1} g[n]e^{-\frac{j2\pi kn}{2N}} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} + \frac{1}{2N}\sum_{n=N}^{2N-1} f[n-N]e^{-\frac{j2\pi kn}{2N}} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} + \frac{1}{2N}\sum_{m=0}^{N-1} f[m]e^{-\frac{j2\pi k(m+N)}{2N}} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} + \frac{1}{2N}\sum_{m=0}^{N-1} f[m]e^{-\frac{j2\pi km}{2N}}e^{-\frac{j2\pi kN}{2N}} \\
&= \frac{1}{2N}\left(1 + e^{-\frac{j2\pi kN}{2N}}\right)\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} \\
&= \frac{1}{2N}\left(1 + (-1)^k\right)\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} \\
&= \left(\frac{1 + (-1)^n}{2}\right) F[k/2] \\
&= \begin{cases} F[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

5 2D Discrete Fourier Transforms (16 points)

Each of the following images contain 16×16 pixels that are either black (representing a value of 0) or white (representing a value of 1).



Determine which of the following images (**A-O**) shows the magnitude of the 2D DFT of each of the preceding images, and enter that letter in the corresponding box above.



In images A-O, white pixels represent the most positive magnitude in that image and black pixels a magnitude of 0. Notice that the zero-location inf images A-O is near the center of the image.

Left panel: Start by taking the 1D DFT of each row. For rows 0 through 7, the DFT is $\delta[k_c]$. For rows 8 through 15, the DFT is 0.

Now take the 1D DFT of each of the resulting columns. There is only one non-trivial column: $k_c = 0$. Values for this column are 1 if $r < 8$ and 0 otherwise.

$$F[k_r, 0] = \frac{1}{16}\sum_{r=0}^{7} e^{-j2\pi k_r r/16} = \frac{1}{16}\left(\frac{1-e^{-j2\pi k_r 8/16}}{1-e^{-j2\pi k_r/16}}\right) = \frac{1}{16}\left(\frac{1-(-1)^{k_r}}{1-e^{-j2\pi k_r/16}}\right)$$

Notice that the numerator is 0 for even values of k_r, and that the denominator is 0 at $k_r = 0$. The ratio is indeterminant at $k_r = 0$, but is easily seen to be $8/16$ from the original sum over r.

The only non-zero values of the 2D DFT result with $k_c = 0$ and $|k_r| = 0, 1, 3, 5,$ and 7. This corresponds to image F.

Second panel: Start by taking the 1D DFT of each row. For the first row, we get

$$F[k_c] = \sum_{\substack{c=0\\ \text{even}}}^{15} e^{-j2\pi k_c c/16} = \sum_{m=0}^{7} e^{-j4\pi k_c m/16} = \frac{1-e^{-j4\pi k_c 8/16}}{1-e^{-j4\pi k_c/16}}$$

The numerator of this expression is 0 for all values of k_c. The denominator is 0 at $k_c = 0$ and at $k_c = 8$. The result is $\delta[k_c] + \delta[k_c - 8]$ for the top 8 rows.

Now take the 1D DFT of each column. The results for $k_c = 0$ are the same as the previous part. The results for $k_c = -8$ match those for $k_c = 0$, i.e.,

$$F[k_r, 0] = F[k_r, -8]$$

for all k_c. It follows that the answer is image H.

An even easier way to find this answer is to realize that the second panel can be generated by zeroing all of the odd numbered columns of the first panel. We discussed zeroing the odd numbered columns in the lecture and recitation on MRI, where we saw that multiplying by even numbered stripes is equivalent to convolving by $\delta[k_c] + \delta[k_c - C/2]$. By carrying out this operation on image F we get image H.

Third panel: Multiplying an image by horizontal, even-numbered stripes is equivalent to convolving in the frequency domain by $\delta[k_r] + \delta[k_r - R/2]$. By carrying out this operation on image F we get image G.

Right panel: Image I results by zeroing both the odd numbered rows and columns of the left panel. The resulting 2D DFT is therefore equal to panel F convolved with $\delta[k_r] + \delta[k_r - R/2]$ and by $\delta[k_c] + \delta[k_c - C/2]$. By carrying out this operation on image F we get image I.