3 Impulsive Images (15 points)

Pixel values for three images ($f_1[r,c]$, $f_2[r,c]$, and $f_3[r,c]$) are shown below for $|r| \leqslant 2$ and $|c| \leqslant 2$. Pixel values outside the indicated regions are zero.

$f_1[r,c]$

r \ c	−2	−1	0	1	2
−2	0	0	0	1	0
−1	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	0
2	0	1	0	0	0

$f_2[r,c]$

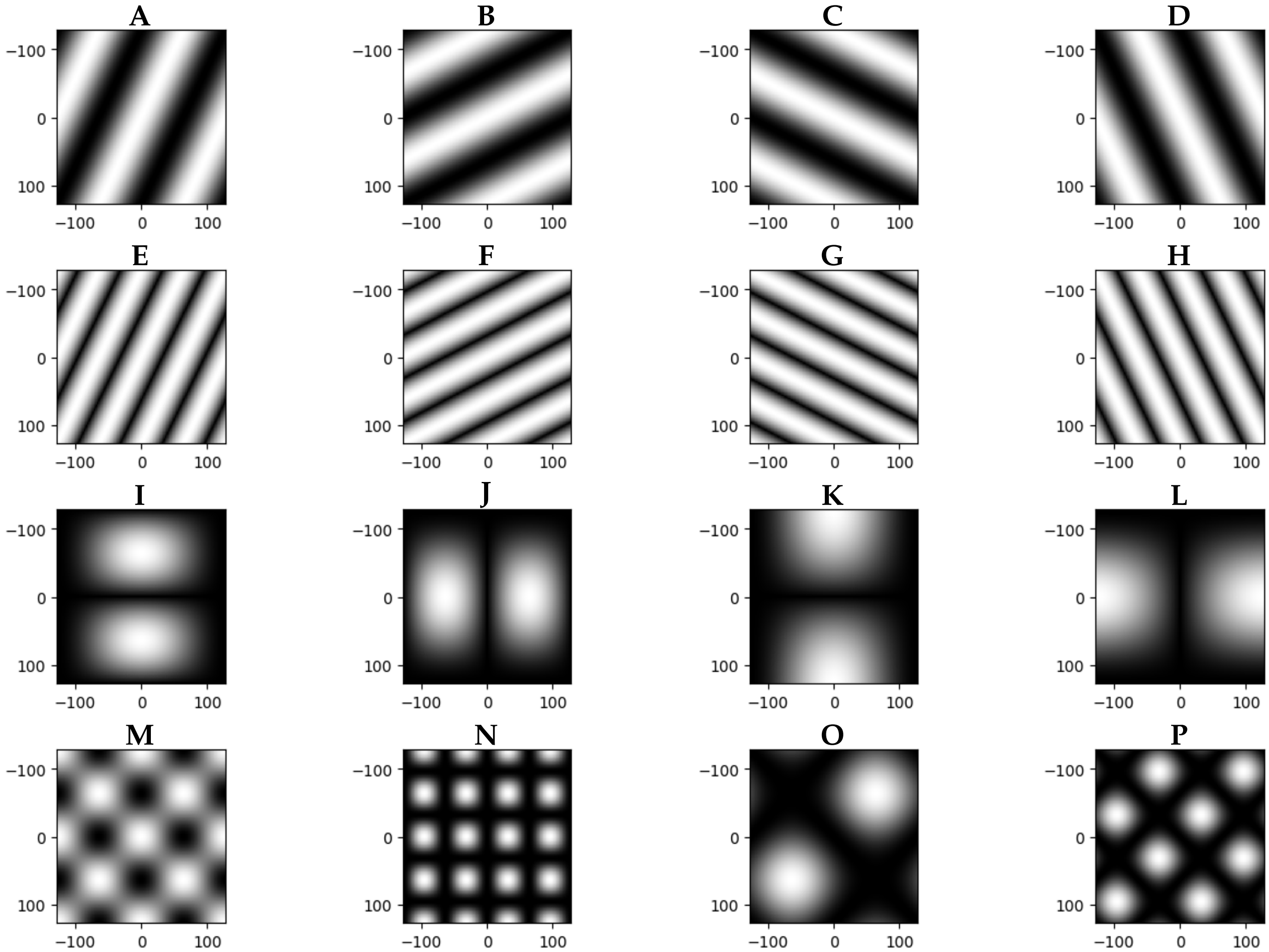
r \ c	−2	−1	0	1	2
−2	0	0	0	0	0
−1	0	−1	0	1	0
0	0	−2	0	2	0
1	0	−1	0	1	0
2	0	0	0	0	0

$f_3[r,c]$

r \ c	−2	−1	0	1	2
−2	0	0	1	0	0
−1	0	0	0	0	0
0	−1	0	4	0	1
1	0	0	0	0	0
2	0	0	−1	0	0

$|F_1[k_r,k_c]| = \boxed{G}$ $|F_2[k_r,k_c]| = \boxed{J}$ $|F_3[k_r,k_c]| = \boxed{P}$

Each of the panels below show the magnitude of a 2D DFT calculated for $-128 \leqslant r, c < 128$ and displayed with black and white representing the minimum and maximum magnitude in each image. Determine which of the panels below shows the magnitude of the 2D DFT for each of the images shown above, and enter its label (A-P) in the corresponding box above.



Let $R = C = 128 = N$.

$$f_1[r,c] = \delta[r{+}2, c{-}1] + \delta[r{-}2, c{+}1]$$

$$F_1[k_r,k_c] = \frac{1}{N^2}\sum_r\sum_c f_1[r,c]e^{-\frac{j2\pi}{N}\left(rk_r+ck_c\right)}$$

$$= \frac{1}{N^2}\left(e^{-\frac{j2\pi}{N}\left(-2k_r+1k_c\right)} + e^{-\frac{j2\pi}{N}\left(+2k_r-1k_c\right)}\right)$$

$$= \frac{2}{N^2}\cos\left(\frac{2\pi}{N}(2k_r{-}k_c)\right)$$

$$\Big|F_1[k_r,k_c]\Big| = \left|\frac{2}{N^2}\cos\left(\frac{2\pi}{N}(2k_r{-}k_c)\right)\right|$$

Notice that all points along the line

$$k_r = k_c/2$$

have the same value of $|F_1[k_r,k_c]|$, and that similar statements hold for lines that are parallel to $k_r = k_c/2$. Thus, pixels along downward sloping lines in the k_r-k_c plane have the same brightness.

Rows of $F_1[k_r,k_c]$ go through one period of brightness for $-C/2 \leqslant c < C/2$. However, rows of $|F_1[k_r,k_c]|$ go through **TWO** periods of brightness for $-C/2 \leqslant c < C/2$ because negative peaks of F_1 are positive peaks of $|F_1|$.

Similarly, columns of $F_1[k_r,k_c]$ go through two periods of brightness for $-R/2 \leqslant r < R/2$. And columns of $|F_1[k_r,k_c]|$ go through **FOUR** periods of brightness for $-R/2 \leqslant r < R/2$.

Therefore the answer is **G**.

$$f_2[r,c] = -\delta[r{+}1,c{+}1] + -2\delta[r,c{+}1] + -\delta[r{-}1,c{+}1] + \delta[r{+}1,c{-}1] + 2\delta[r,c{-}1] + \delta[r{-}1,c{-}1]$$

$$F_2[k_r,k_c] = \frac{1}{N^2}\sum_r\sum_c f_1[r,c]e^{-\frac{j2\pi}{N}\left(rk_r+ck_c\right)}$$

$$= \frac{1}{N^2}\Big(-e^{-\frac{j2\pi}{N}\left(-1k_r-1k_c\right)} - 2e^{-\frac{j2\pi}{N}\left(+0k_r-1k_c\right)} - e^{-\frac{j2\pi}{N}\left(+1k_r-1k_c\right)}$$
$$+ e^{-\frac{j2\pi}{N}\left(-1k_r+1k_c\right)} + 2e^{-\frac{j2\pi}{N}\left(+0k_r+1k_c\right)} + e^{-\frac{j2\pi}{N}\left(+1k_r+1k_c\right)}\Big)$$

$$= j\frac{4}{N^2}\Big(1+\cos(2\pi k_r/R)\Big)\sin(2\pi k_c/C)$$

$$\Big|F_2[k_r,k_c]\Big| = \frac{4}{N^2}\Big(1+\cos(2\pi k_r/R)\Big)\Big|\sin(2\pi k_c/C)\Big|$$

The cosine term is always positive. It reaches a peak at $k_r = k_c = 0$ and its closest zeros are at the top and bottom of the image.

The sine term is zero at $k_r = k_c = 0$, it has a peak at $k_c = C/4$ and a negative peak at $c_c = -C/4$. After taking the absolute value, the negative peak becomes a positive peak.

Therefore the answer is **J**.

$$f_3[r,c] = \delta[r{+}2,c] + -\delta[r{-}2,c] + -\delta[r,c{+}2] + \delta[r,c{-}2] + +4\delta[r,c]$$

$$F_3[k_r,k_c] = \frac{1}{N^2}\sum_r\sum_c f_1[r,c]e^{-\frac{j2\pi}{N}\left(rk_r+ck_c\right)}$$

$$= \frac{1}{N^2}\Big(+1e^{-\frac{j2\pi}{N}\left(-2k_r+0k_c\right)} - 1e^{-\frac{j2\pi}{N}\left(+2k_r+0k_c\right)} - 1e^{-\frac{j2\pi}{N}\left(+0k_r-2k_c\right)}$$
$$+1e^{-\frac{j2\pi}{N}\left(+0k_r+2k_c\right)} + 4e^{-\frac{j2\pi}{N}\left(+0k_r+0k_c\right)}\Big)$$

$$= \frac{1}{N^2}\Big(4 + j2\sin(4\pi k_r/N) - j2\sin(4\pi k_c/N)\Big)$$

If we break the DFT into 16 equal-sized segments (on a 4×4 grid), then $\sin(4\pi k_r/N)$ is positive in the top row and the third row of segments (left figure below) and $-\sin(4\pi k_c/N)$ is positive in the second and fourth columns of segments (center figure below). The sum is greatest in the diagonal pattern shown in the right figure below.

+	+	+	+
−	−	−	−
+	+	+	+
−	−	−	−

−	+	−	+
−	+	−	+
−	+	−	+
−	+	−	+

	+		+
−		−	
	+		+
−		−	

The magnitudes of the peak negative values of F_3 and the magnitudes of the peak negative values of F_3 are equal. Therefore the bright spots in $\left|F_3[k_r,k_c]\right|$ occur in the quadrants shown below.

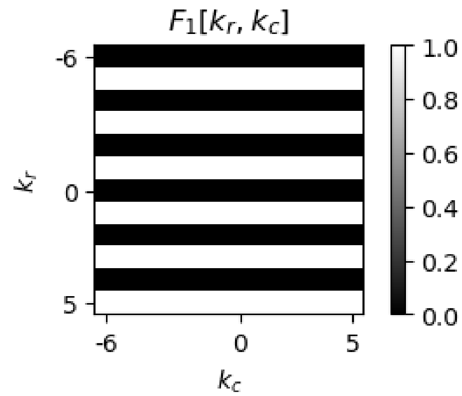
	+		+
+		+	
	+		+
+		+	

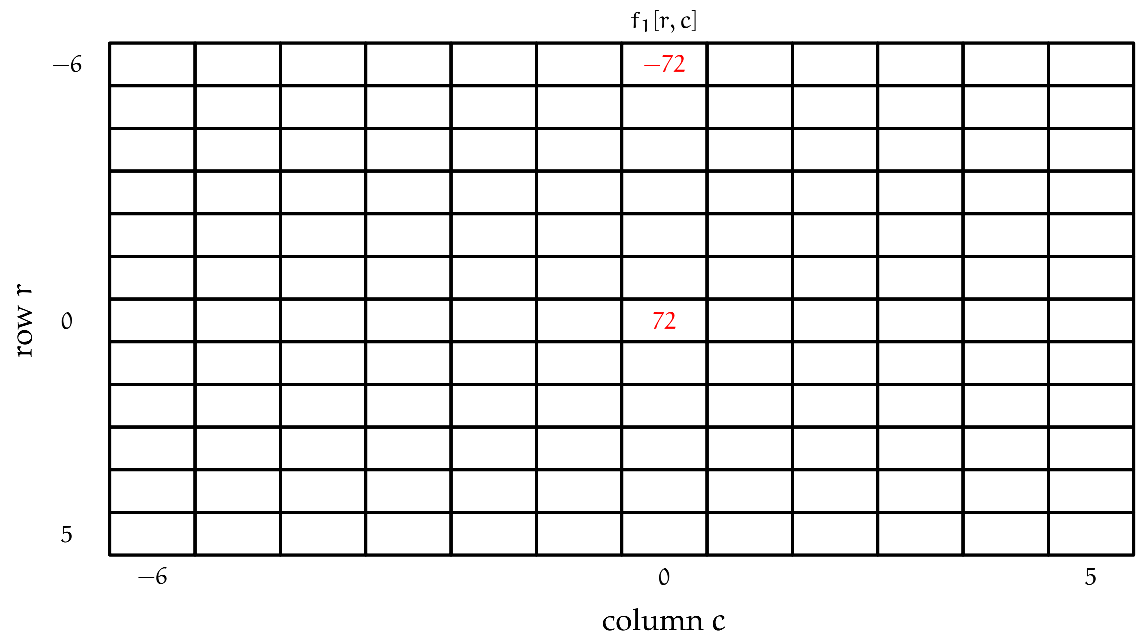
Therefore the answer is **P**.

3 Plaids (32 points)

Each part of this problem shows a 2D (12 × 12) Discrete Fourier Transform (DFT) $F_i[k_r, k_c]$ for which you must find the corresponding 2D signal $f_i[r, c]$. The values of the DFT $F_i[k_r, k_c]$ are real numbers that are displayed as brightnesses defined by the associated color bar.

Enter each non-zero value of $f_i[r, c]$ in the grid shown below the DFT. Empty cells in the grid will be taken as 0.





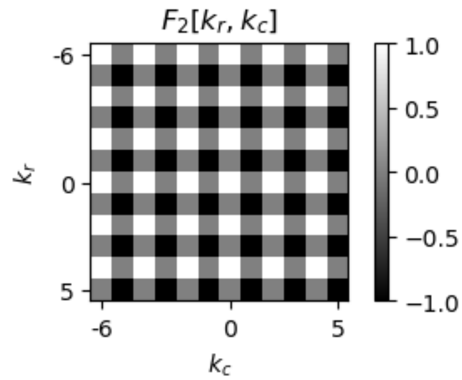
$F_1[k_r, k_c]$ can be written as $(1-\cos(\pi k_r))/2 = (1-e^{j\pi k_r})/2$. Substituting this into the synthesis equation yields

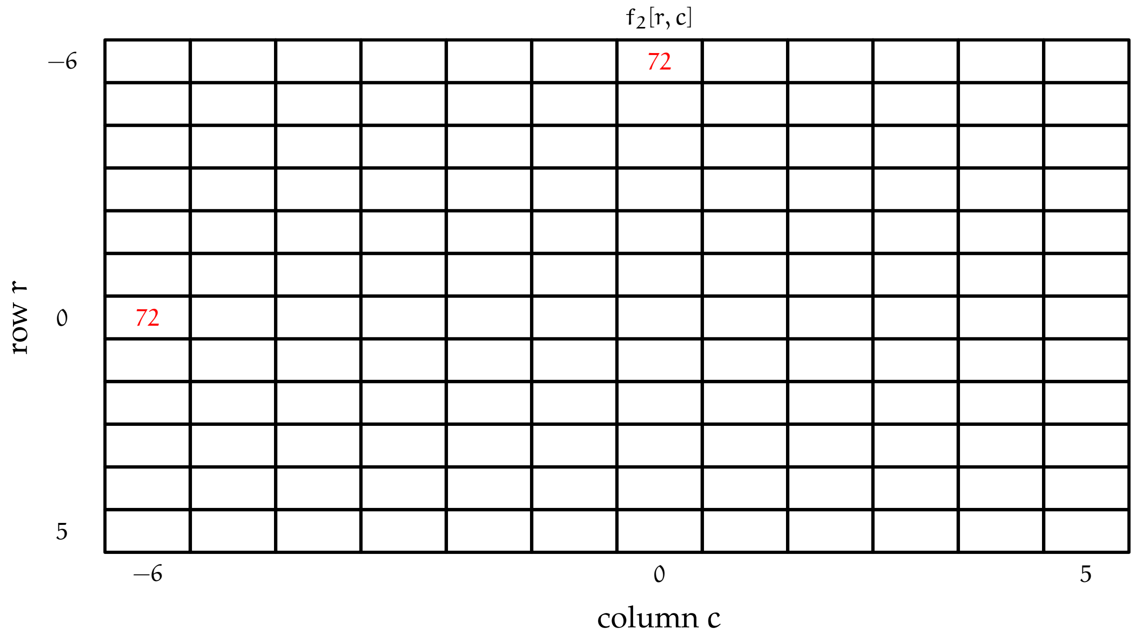
$$f_1[r, c] = \sum_{k_r=0}^{R-1} \sum_{k_c=0}^{C-1} \frac{1}{2}(1-e^{j\pi k_r})e^{j2\pi k_r r/R}e^{j2\pi k_c c/C}$$

$$= \frac{1}{2}\left(\sum_{k_r=0}^{R-1} e^{j2\pi k_r r/R}\right)\left(\sum_{k_c=0}^{C-1} e^{j2\pi k_c c/C}\right) - \frac{1}{2}\left(\sum_{k_r=0}^{R-1} e^{j2\pi k_r (r+R/2)/R}\right)\left(\sum_{k_c=0}^{C-1} e^{j2\pi k_c c/C}\right)$$

$$= \frac{1}{2}RC\,\delta[r]\delta[c] - \frac{1}{2}RC\,\delta[r+R/2]\delta[c] = \begin{cases} 72 & \text{if } k_r = k_c = 0 \\ -72 & \text{if } k_r = -6 \text{ and } k_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the following image, white, gray, and black represent pixel values of 1, 0, and -1, respectively.



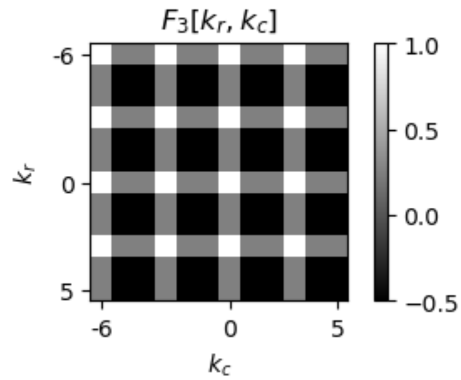


$F_2[k_r, k_c]$ can be written as $\frac{1}{2}(-1)^{k_r} + \frac{1}{2}(-1)^{k_c} = \frac{1}{2}e^{j\pi k_r} + \frac{1}{2}e^{j\pi k_c}$. Substituting this into the synthesis equation yields

$$f_2[r, c] = \sum_{k_r=0}^{R-1} \sum_{k_c=0}^{C-1} \left(\frac{1}{2}e^{j\pi k_r} + \frac{1}{2}e^{j\pi k_c}\right) e^{j2\pi k_r r/R} e^{j2\pi k_c c/C}$$

$$= \frac{RC}{2}\delta[r{+}R/2]\delta[c] + \frac{RC}{4}\delta[r]\delta[c{+}C/2] = \begin{cases} 72 & \text{if } r = -6 \text{ and } c = 0 \\ 72 & \text{if } r = 0 \text{ and } c = -6 \\ 0 & \text{otherwise} \end{cases}$$

In the following image, white, gray, and black represent pixel values of 1, 1/4, and −1/2, respectively.



$f_3[r, c]$

row r \ column c	−6						0				5	
−6												
							36					
0			36								36	
							36					
5												

column c

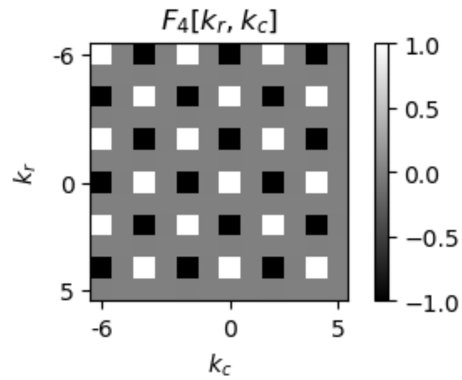
$F_3[k_r, k_c]$ can be written as $\frac{1}{2}\cos(2\pi k_r/3) + \frac{1}{2}\cos(2\pi k_c/3)$. Substituting this into the synthesis equation yields

$$f_3[r,c] = \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1}\left(\frac{1}{2}\cos(2\pi k_r/3) + \frac{1}{2}\cos(2\pi k_c/3)\right)e^{j2\pi k_r r/R}e^{j2\pi k_c c/C}$$

$$= \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1}\left(\frac{1}{4}e^{j2\pi k_r/3} + \frac{1}{4}e^{-j2\pi k_r/3} + \frac{1}{4}e^{j2\pi k_c/3} + \frac{1}{4}e^{-j2\pi k_c/3}\right)e^{j2\pi k_r r/R}e^{j2\pi k_c c/C}$$

$$= \frac{RC}{4}\delta[r+R/3]\delta[c] + \frac{RC}{4}\delta[r-R/3]\delta[c] + \frac{RC}{4}\delta[r]\delta[c+C/3] + \frac{RC}{4}\delta[r]\delta[c-C/3] + = \begin{cases} 36 & \text{if } r=-4 \text{ and } c=0 \\ 36 & \text{if } r=4 \text{ and } c=0 \\ 36 & \text{if } r=0 \text{ and } c=-4 \\ 36 & \text{if } r=0 \text{ and } c=4 \\ 0 & \text{otherwise} \end{cases}$$

In the following image, white, gray, and black represent pixel values of 1, 0, and −1, respectively.



$f_4[r, c]$

row r												
−6												
				36						36		
0												
				36						36		
5												

−6 0 5

column c

$F_4[k_r, k_c]$ can be written as $(\cos(2\pi k_r/4) * \cos(2\pi k_c/4))/2$. Substituting this into the synthesis equation yields

$$f_4[r,c] = \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1} \cos(2\pi k_c/4)\cos(2\pi k_r/4) e^{j2\pi k_r r/R} e^{j2\pi k_c c/C}$$

$$= \sum_{k_r=0}^{R-1}\sum_{k_c=0}^{C-1} \left(\frac{1}{2}e^{j2\pi k_r/4} + \frac{1}{2}e^{-j2\pi k_r/4}\right)\left(\frac{1}{2}e^{j2\pi k_c/3} + \frac{1}{2}e^{-j2\pi k_c/3}\right) e^{j2\pi k_r r/R} e^{j2\pi k_c c/C}$$

$$= \frac{RC}{4}\delta[r+R/4]\delta[c] + \frac{RC}{4}\delta[r-R/4]\delta[c] + \frac{RC}{4}\delta[r]\delta[c+C/4] + \frac{RC}{4}\delta[r]\delta[c-C/4] = \begin{cases} 36 & \text{if } r=-3 \text{ and } c=0 \\ 36 & \text{if } r=3 \text{ and } c=0 \\ 36 & \text{if } r=0 \text{ and } c=-3 \\ 36 & \text{if } r=0 \text{ and } c=3 \\ 0 & \text{otherwise} \end{cases}$$