6.3000: Signal Processing

Discrete Cosine Transform

$$X_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right)$$

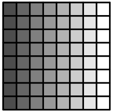
$$x[n] = X_C[0] + 2\sum_{k=1}^{N-1} X_C[k]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right)$$

November 25, 2025

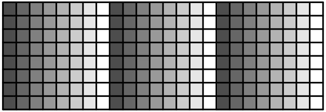
Motivation

The Discrete Fourier Transform (DFT) implicitly represents the frequencies that are contained in a **periodically extended** version of the input signal, and periodic extension can generate frequencies that are **not present** in the original signal.

Consider the following 8×8 example.

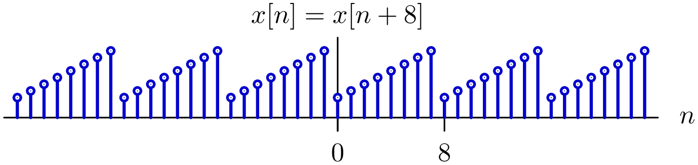


The brightnesses of left and right edges are different and will generate a sequence of large transitions when periodically extended.



Motivation: 1D

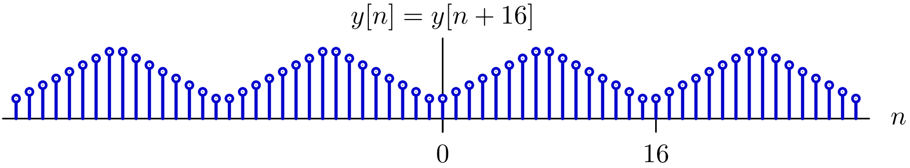
Consider a single row from the previous image. The DFT implicitly extends the signal (here a ramp) periodically.



Although the function is smooth from $n = 0$ to 7, the periodic extention contains a series of steps.

Motivation: 1D

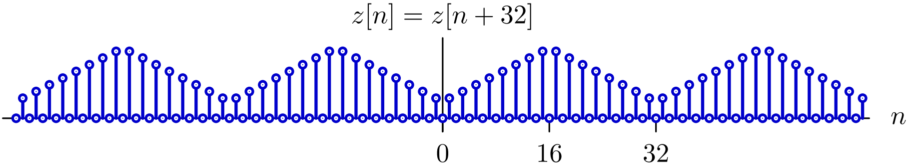
We can eliminate the step discontinuities by first replicating one period in reverse order and then extending the result periodically.



The resulting signal is continuous across the edges (however the slope is still discontinuous).

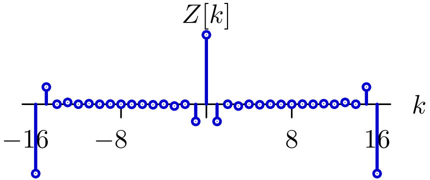
Motivation: 1D

Finally, insert zeros between successive samples, and shift result right by 1.



The resulting signal is real-valued, symmetric about $n = 0$, periodic in $4N$, and contains only odd numbered samples.

The DFT of this signal is real, symmetric about $k = 0$ and anti-periodic. It is completely characterized by N values: $Z[0]$ to $Z[7]$.



This process is captured in the **Discrete Cosine Transform**.

Discrete Cosine Transform

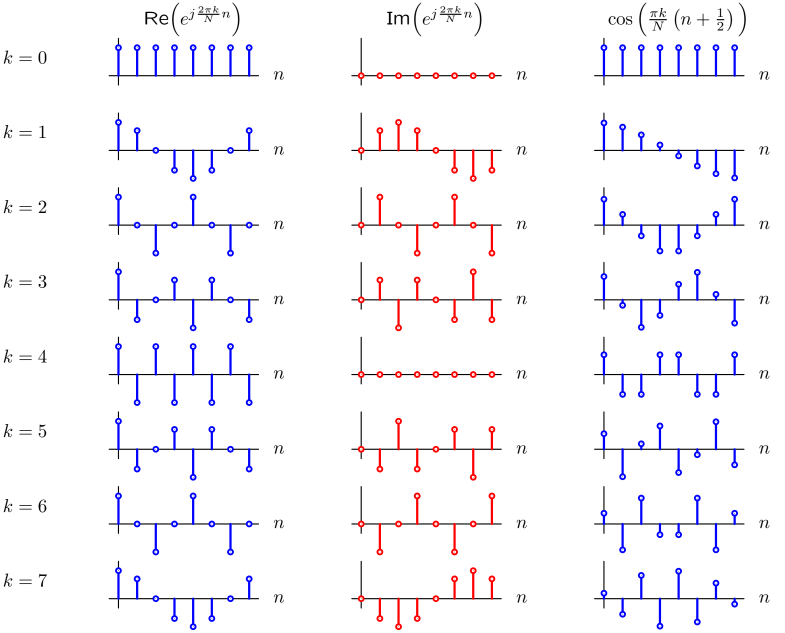
The Discrete Cosine Transform (DCT) is described by analysis and synthesis equations that are analogous to those of the DFT.

$$X_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \qquad \text{(analysis)}$$

$$x[n] = X_C[0] + 2\sum_{k=1}^{N-1} X_C[k]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \qquad \text{(synthesis)}$$

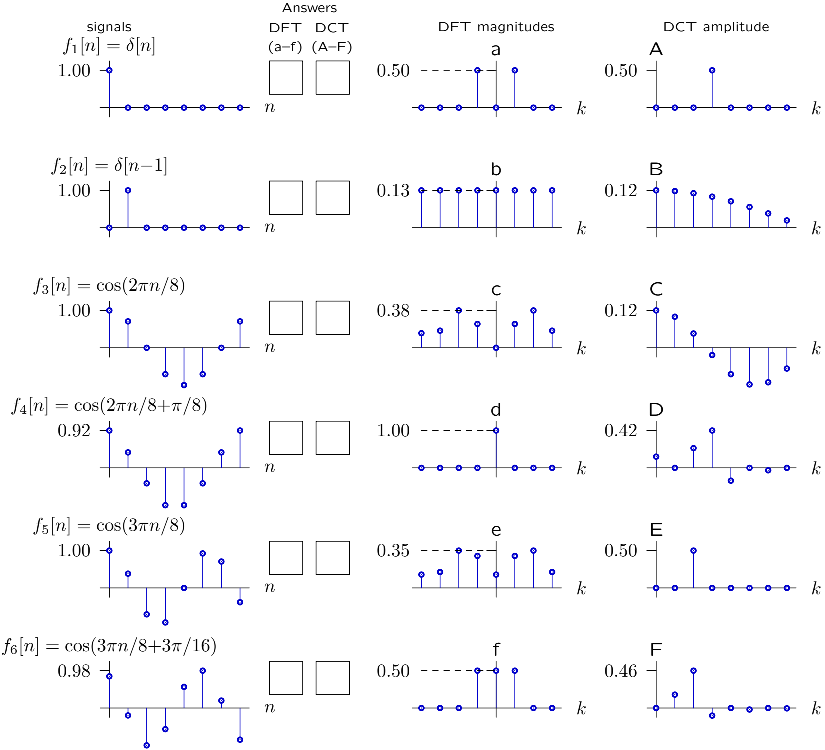
Comparison of DFT and DCT Basis Functions

DFT (real and imaginary parts) versus DCT.



DFT Magnitudes and DCT Amplitudes

For each of the DT signals shown in the left column on the following page, find the associated DFT magnitude from the center column and DCT amplitude from the right column, and enter the appropriate labels in the answer boxes.



Answers
signals
DFT
DCT
DFT magnitudes
DCT amplitude
(a–f)
(A–F)
f1[n] = δ[n]
f2[n] = δ[n−1]
f3[n] = cos(2πn/8)
f4[n] = cos(2πn/8+π/8)
f5[n] = cos(3πn/8)
f6[n] = cos(3πn/8+3π/16)
1.00
1.00
1.00
0.92
1.00
0.98
n
a
b
c
d
e
f
0.50
0.13
0.38
1.00
0.35
0.50
A
B
C
D
E
F
0.50
0.12
0.12
0.42
0.50
0.46
k

Part 1. The DFT of $\delta[n]$ is $1/N$. Therefore $F_1[k] = 1/N \rightarrow$ panel b.

$$F_{C1}[k] = \frac{1}{N}\sum_{n=0}^{N-1} f_1[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \frac{1}{N}\sum_{n=0}^{N-1} \delta[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \frac{1}{N}\cos\left(\frac{\pi k}{2N}\right)$$

$\rightarrow$ panel B.

The DCT of a unit-sample signal is NOT the same as the DFT of a unit-sample signal!

Part 2. The DFT of $\delta[n-1]$ is $\frac{1}{8}e^{-j\frac{2\pi k}{N}}$. Therefore $|F_2[k]| = 1/N \rightarrow$ panel b.

$$F_{C2}[k] = \frac{1}{N}\sum_{n=0}^{N-1} \delta[n-1]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \frac{1}{N}\cos\left(\frac{\pi k}{N}\frac{3}{2}\right)$$

$\rightarrow$ panel C.

Time shifts are not as simple with the DCT as they were with the DFT.

Part 3. $f_3[n]$ is the fundamental frequency for $N = 8$. The DFT has non-zero contributions only at $k = \pm 1$. Therefore $|F_3[k]| \rightarrow$ panel a.

DFT basis functions are integer multiples of the fundamental frequency. By contrast, DCT basis functions are half-integer multiples. Therefore we expect a non-zero component at $k = 2$. However, the last sample of all DCT basis functions is either equal to the first sample or equal to the negative of the first sample. Therefore $f_3[n]$ is NOT a basis function for the DCT.

$\rightarrow$ panel F.

Part 4. $f_4[n]$ is also at the fundamental frequency for $N = 8$. The DFT has non-zero contributions only at $k = \pm 1$. Therefore $|F_4[k]| \rightarrow$ panel a.

The frequency of $f_4[n]$ is the same as that of $f_3[n]$ but the phase is different. $f_4[7] = f_4[0]$. Therefore $f_4[n]$ IS a basis function.

$\rightarrow$ panel E.

Notice that both $f_3[n]$ and $f_4[n]$ have simple representations as DFTs but not as DCTs.

Part 5. The frequency of $f_5[n]$ is $\frac{3}{2}$ times the fundamental frequency. The discontinuity created by periodic extension of $f_5[n]$ generates components at all frequencies, although the peak is near $k = 3/2$. Thus the answer could be c or e. The DC component of $f_5[n]$ is non-zero (4 positive numbers and 3 negative).

$\rightarrow$ panel e.

Since the frequency of $f_5[n]$ is a half-integer multiple of the fundamental, it could be a DCT basis function at $k = 3$. However, the final value is not equal to the initial value or its negative.

$\rightarrow$ panel D.

Part 6. $f_6[n]$ is a phase shifted version of $f_5[n]$. Now the DC value is zero.

$\rightarrow$ panel c.

$f_6[n]$ is a basis function of the DCT.

$\rightarrow$ panel A.

Notice the $f_6[n]$ has a simple representation as a DCT but not as a DFT.

DCT Basis Functions

Much of the utility of Fourier transforms in general and the DFT in particular results from properties of the Fourier basis functions:

DTFT	$e^{j\Omega n}$
DTFS	$e^{j\frac{2\pi k}{N}n}$
DFT	$e^{j\frac{2\pi k}{N}n}$

To better understand the DCT, we need to similarly understand its basis functions.

DCT $\quad \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right)$

DCT Basis Functions

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right) .$$

How many of the following symmetries are true?

- $\phi_k[n+2N] = \phi_k[n]$
- $\phi_k[n+N] = (-1)^k \phi_k[n]$
- $\phi_k[n-N] = (-1)^k \phi_k[n]$
- $\phi_k[(N-1)-n] = (-1)^k \phi_k[n]$

DCT Basis Functions

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right) .$$

The first property

$$\phi_k[n+2N] = \phi_k[n]$$

follows from the periodicity of the cosine function, as follows.

$$\begin{aligned}\phi_k[n+2N] &= \cos\left(\frac{\pi k}{N}\left(2N+n+\frac{1}{2}\right)\right) \\ &= \cos\left(\frac{\pi k}{N}2N+\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\ &= \cos\left(2\pi k+\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\ &= \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \phi_k[n]\end{aligned}$$

Notice that $\phi_k[n]$ is not periodic in N!

DCT Basis Functions

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) .$$

The second property

$$\phi_k[n+N] = (-1)^k \phi_k[n]$$

addresses symmetry in N.

$$\begin{aligned}
\phi_k[n+N] &= \cos\left(\frac{\pi k}{N}\left(N+n+\frac{1}{2}\right)\right) \\
&= \cos\left(\frac{\pi k}{N}N+\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= \cos\left(\pi k+\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= (-1)^k \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = (-1)^k \phi_k[n]
\end{aligned}$$

DCT Basis Functions

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right) .$$

The third property

$$\phi_k[n-N] = (-1)^k \phi_k[n]$$

follows from the first two.

$$\phi_k[n-N] = \phi_k[n+2N-N] = \phi_k[n+N] = (-1)^k \phi_k[n]$$

DCT Basis Functions

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right) .$$

The fourth property

$$\phi_k[(N-1)-n] = (-1)^k \phi_k[n]$$

describes symmetry about the point $n = (N-1)/2$.

$$\begin{aligned}
\phi_k[(N-1)-n] &= \cos\left(\frac{\pi k}{N}\left(N-1-n+\frac{1}{2}\right)\right) \\
&= \cos\left(\frac{\pi k}{N}\left(N-\left(n+\frac{1}{2}\right)\right)\right) \\
&= \cos(\pi k)\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) + \sin(\pi k)\sin\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \\
&= (-1)^k \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = (-1)^k \phi_k[n]
\end{aligned}$$

DCT Basis Functions

We can use the previous properties to calculate useful facts.

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right) = N\delta[k]\,.$$

DCT Basis Functions

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = N\delta[k]\,.$$

If k is odd, then by property 4, the sum $\sum_{n=0}^{N-1} \phi_k[n]$ zero.

If k is even, then we can replace k by $2l$ where l is an integer:

$$\sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi l}{N}\left(n+\frac{1}{2}\right)\right)$$

Rewrite the cosine terms as the real parts of complex exponentials:

$$\sum_{n=0}^{N-1} \phi_k[n] = \mathrm{Re}\left(\sum_{n=0}^{N-1} e^{j\frac{2\pi l}{N}\left(n+\frac{1}{2}\right)}\right) = \mathrm{Re}\left(e^{j\frac{\pi l}{N}} \sum_{n=0}^{N-1} e^{j\frac{2\pi l}{N}n}\right)$$

(continued on next page)

continued

Let

$$S = \sum_{n=0}^{N-1} e^{j\frac{2\pi l}{N}n}$$

$$= 1 + e^{j\frac{2\pi l}{N}} + e^{j\frac{2\pi l}{N}2} + \cdots + e^{j\frac{2\pi l}{N}(N-1)}$$

If $l = 0$, then $S = N$.

Otherwise

$$S - Se^{j\frac{2\pi l}{N}} = S(1 - e^{j\frac{2\pi l}{N}}) = 1 - e^{j\frac{2\pi l}{N}N} = 1 - e^{j2\pi l} = 0$$

and $S = 0$.

Thus the sum

$$\sum_{n=0}^{N-1} \phi_k[n] = N\delta[k]\,.$$

Orthogonality

Show that

$$\sum_{n=0}^{N-1} \phi_k[n]\phi_l[n] = \begin{cases} N & \text{if } k = l = 0 \\ N/2 & \text{if } k = l \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

This orthogonality property is the basis of the analysis equation.

Orthogonality

Show that

$$\sum_{n=0}^{N-1} \phi_k[n]\phi_l[n] = \begin{cases} N & \text{if } k = l = 0 \\ N/2 & \text{if } k = l \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=0}^{N-1} \cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \cos\left(\frac{\pi l}{N}\left(n+\frac{1}{2}\right)\right)$$

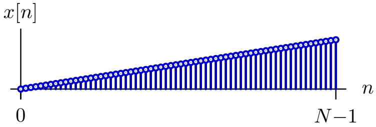
$$= \frac{1}{2}\sum_{n=0}^{N-1} \cos\left(\frac{\pi(k-l)}{N}\left(n+\frac{1}{2}\right)\right) + \frac{1}{2}\sum_{n=0}^{N-1} \cos\left(\frac{\pi(k+l)}{N}\left(n+\frac{1}{2}\right)\right)$$

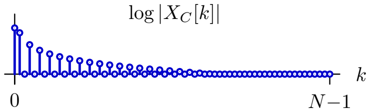
Now we can use the previous result to show that the first sum is equal to $\frac{N}{2}\delta[k-l]$ and the second sum is equal to $\frac{N}{2}\delta[k+l]$.
Since both k and l must be between 0 and $N-1$, the first term is $\frac{N}{2}$ if $k = l$ and the second term is $\frac{N}{2}$ if $k = l = 0$.

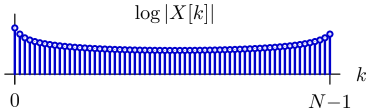
Compaction: Gradient

If a signal has predominately low-frequency content, then the higher order coefficients of the DCT tend to decrease faster than the corresponding coefficients of the DFT.

Here are results for a ramp.



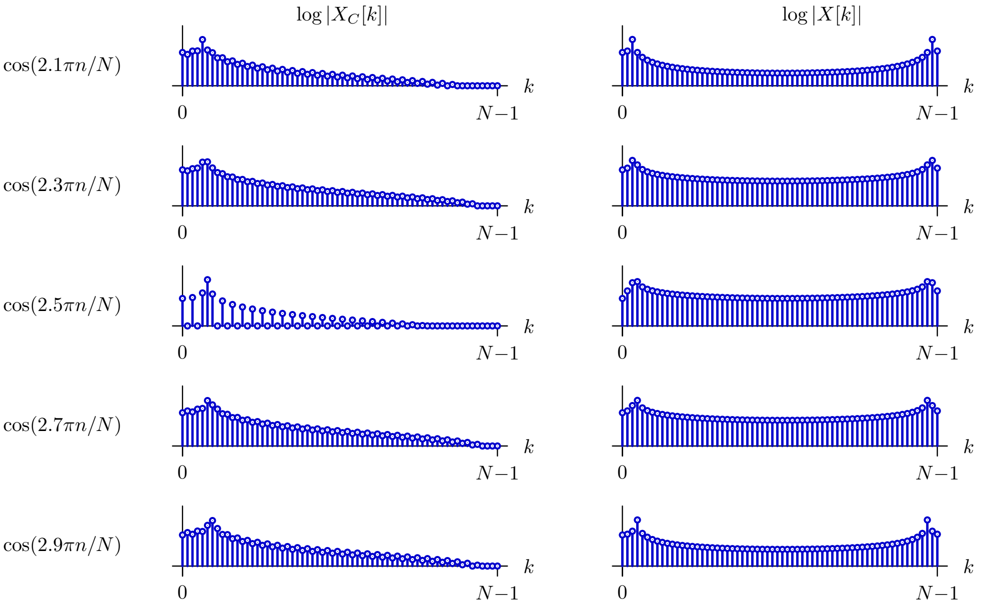




Note that the same scales apply for X_C and X.

Compaction: Sinusoids

The same sort of compaction results for sinusoidal signals.



Same scales in each panel.