

6.3000: Signal Processing

Discrete Cosine Transform

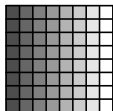
$$X_C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cos \left(\frac{\pi k}{N} \left(n + \frac{1}{2} \right) \right)$$

$$x[n] = X_C[0] + 2 \sum_{k=1}^{N-1} X_C[k] \cos \left(\frac{\pi k}{N} \left(n + \frac{1}{2} \right) \right)$$

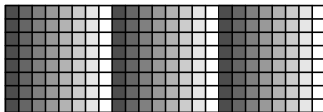
Motivation

The Discrete Fourier Transform (DFT) implicitly represents the frequencies that are contained in a **periodically extended** version of the input signal, and periodic extension can generate frequencies that are **not present** in the original signal.

Consider the following 8×8 example.

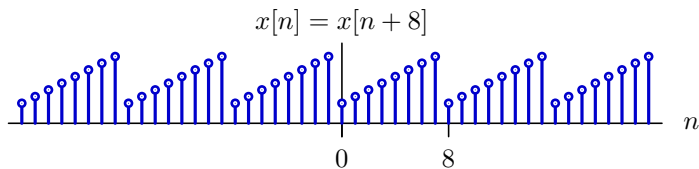


The brightnesses of left and right edges are different and will generate a sequence of large transitions when periodically extended.



Motivation: 1D

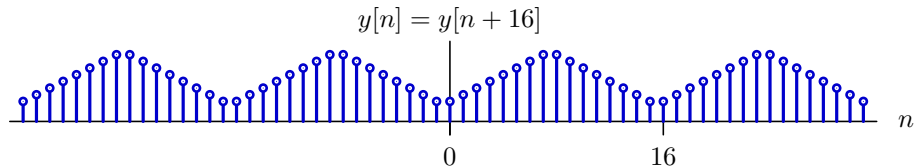
Consider a single row from the previous image. The DFT implicitly extends the signal (here a ramp) periodically.



Although the function is smooth from $n = 0$ to 7 , the periodic extension contains a series of steps.

Motivation: 1D

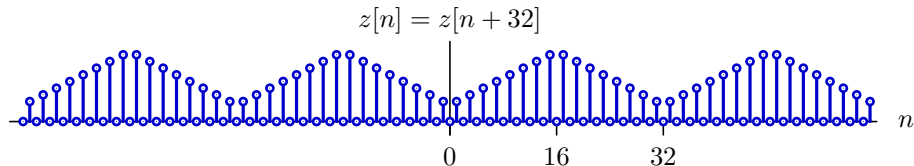
We can eliminate the step discontinuities by first replicating one period in reverse order and then extending the result periodically.



The resulting signal is continuous across the edges (however the slope is still discontinuous).

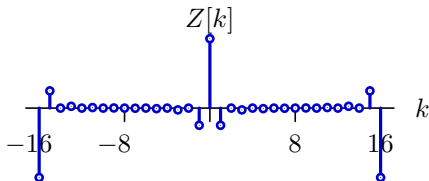
Motivation: 1D

Finally, insert zeros between successive samples, and shift result right by 1.



The resulting signal is real-valued, symmetric about $n = 0$, periodic in $4N$, and contains only odd numbered samples.

The DFT of this signal is real, symmetric about $k = 0$ and anti-periodic. It is completely characterized by N values: $Z[0]$ to $Z[7]$.



This process is captured in the **Discrete Cosine Transform**.

Discrete Cosine Transform

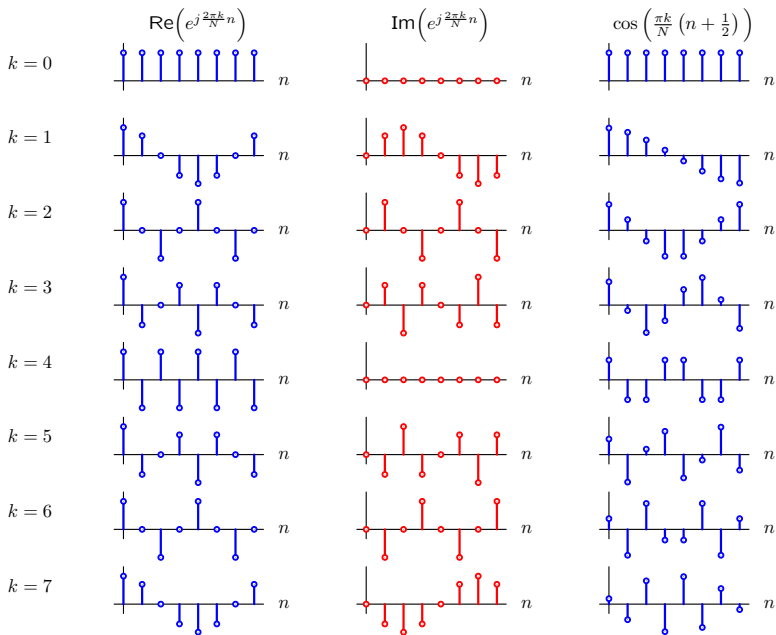
The Discrete Cosine Transform (DCT) is described by analysis and synthesis equations that are analogous to those of the DFT.

$$X_C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cos \left(\frac{\pi k}{N} \left(n + \frac{1}{2} \right) \right) \quad (\text{analysis})$$

$$x[n] = X_C[0] + 2 \sum_{k=1}^{N-1} X_C[k] \cos \left(\frac{\pi k}{N} \left(n + \frac{1}{2} \right) \right) \quad (\text{synthesis})$$

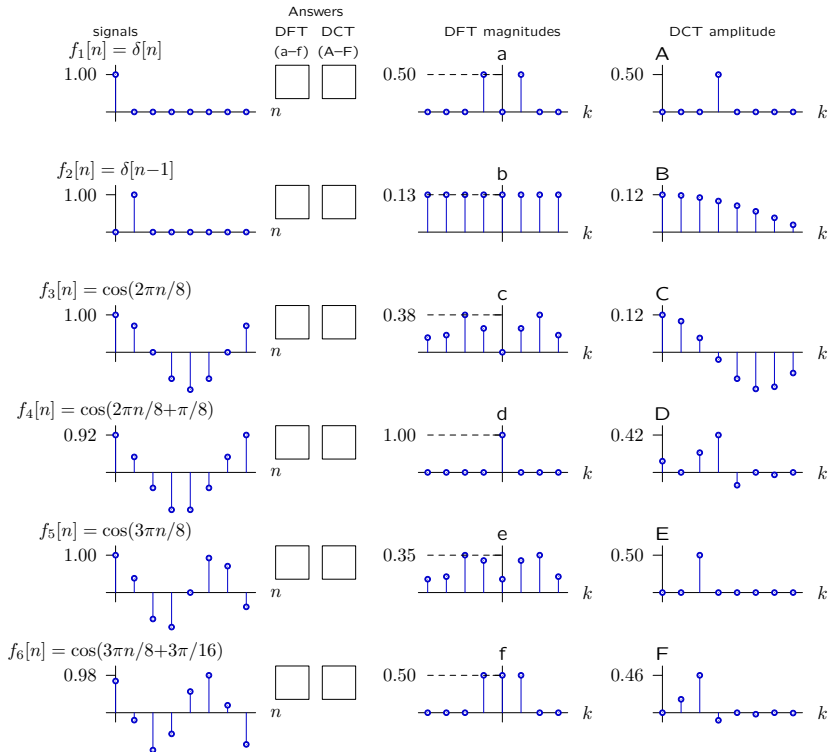
Comparison of DFT and DCT Basis Functions

DFT (real and imaginary parts) versus DCT.



DFT Magnitudes and DCT Amplitudes

For each of the DT signals shown in the left column on the following page, find the associated DFT magnitude from the center column and DCT amplitude from the right column, and enter the appropriate labels in the answer boxes.



DCT Basis Functions

Much of the utility of Fourier transforms in general and the DFT in particular results from properties of the Fourier basis functions:

$$\text{DTFT} \quad e^{j\Omega n}$$

$$\text{DTFS} \quad e^{j\frac{2\pi k}{N}n}$$

$$\text{DFT} \quad e^{j\frac{2\pi k}{N}n}$$

To better understand the DCT, we need to similarly understand its basis functions.

$$\text{DCT} \quad \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right)$$

DCT Basis Functions

The k^{th} DCT basis function of order N is given by

$$\phi_k[n] = \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right).$$

How many of the following symmetries are true?

- $\phi_k[n+2N] = \phi_k[n]$
- $\phi_k[n+N] = (-1)^k \phi_k[n]$
- $\phi_k[n-N] = (-1)^k \phi_k[n]$
- $\phi_k[(N-1)-n] = (-1)^k \phi_k[n]$

DCT Basis Functions

We can use the previous properties to calculate useful facts.

Show that

$$\sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right) = N\delta[k].$$

Orthogonality

Show that

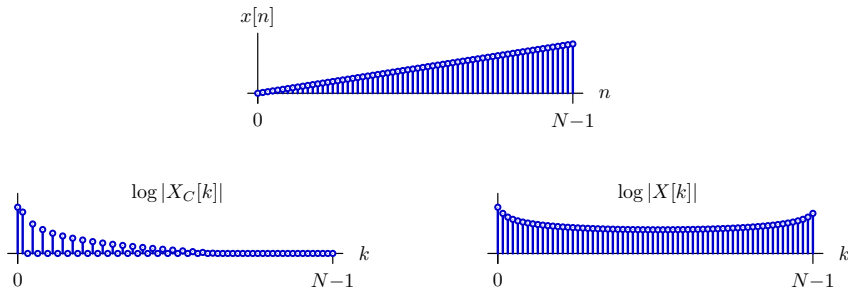
$$\sum_{n=0}^{N-1} \phi_k[n] \phi_l[n] = \begin{cases} N & \text{if } k = l = 0 \\ N/2 & \text{if } k = l \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

This orthogonality property is the basis of the analysis equation.

Compaction: Gradient

If a signal has predominately low-frequency content, then the higher order coefficients of the DCT tend to decrease faster than the corresponding coefficients of the DFT.

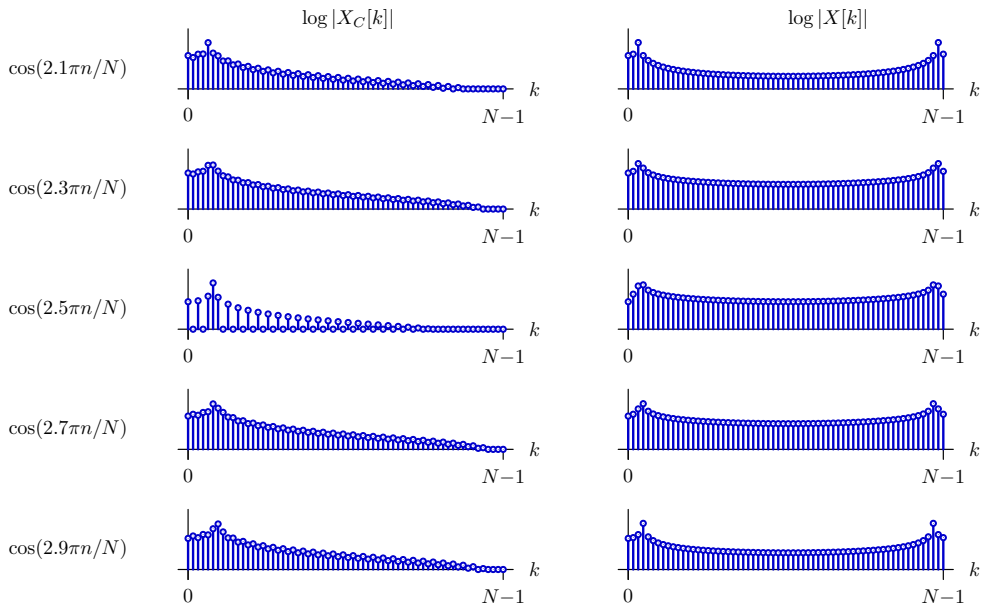
Here are results for a ramp.



Note that the same scales apply for X_C and X .

Compaction: Sinusoids

The same sort of compaction results for sinusoidal signals.



Same scales in each panel.