

6.3000: Signal Processing

Discrete-Time Fourier Transform

Synthesis Equation

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Analysis Equation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Discrete-Time Fourier Transform

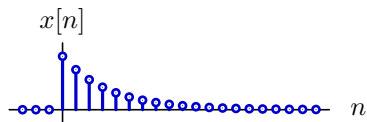
Find the Fourier transforms of the following discrete-time signals.

- $x_1[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- $x_2[n] = x_1[n-n_0]$
- $x_3[n] = \text{Symmetric}\{x_1[n]\}$
- $x_4[n] = \text{Antisymmetric}\{x_1[n]\}$
- $x_5[n] = nx_1[n]$

Discrete-Time Fourier Transform

Find the Fourier transform of

$$x_1[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{aligned} X_1(\Omega) &= \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n \\ &= \frac{1}{1 - ae^{-j\Omega}} \quad \text{provided } |a| < 1 \end{aligned}$$

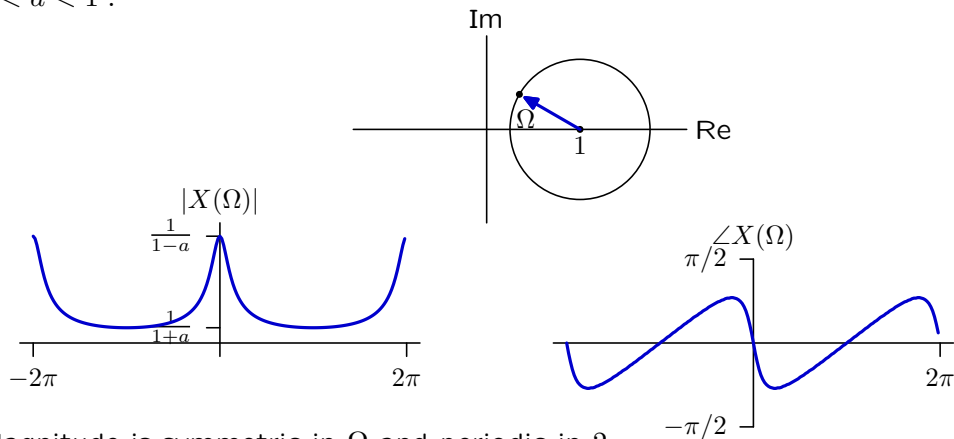
Discrete-Time Fourier Transform

Plot the transform.

$$x_1[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \xrightarrow{\text{DTFT}} X_1(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Note that denominator is sum of 2 complex numbers.

$0 < a < 1$:



Magnitude is symmetric in Ω and periodic in 2π .

Angle is antisymmetric and periodic in 2π .

How would these change if $-1 < a < 0$?

Discrete-Time Fourier Transform

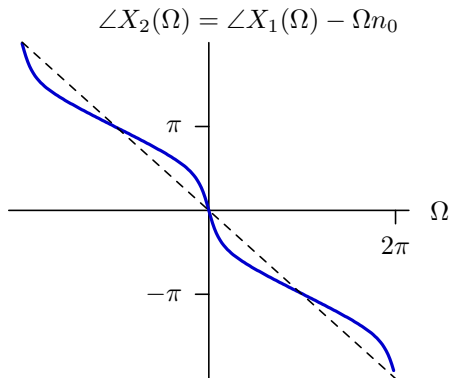
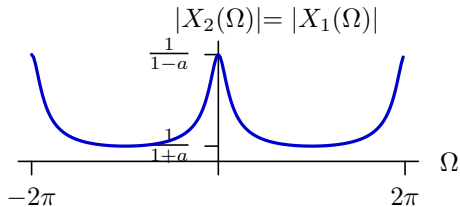
Find the Fourier transform of $x_2[n] = x_1[n - n_0]$.

$$\begin{aligned} X_2(\Omega) &= \sum_{n=-\infty}^{\infty} x_2[n] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x_1[n - n_0] e^{-j\Omega n} \\ &= \sum_{m=-\infty}^{\infty} x_1[m] e^{-j\Omega(m+n_0)} \\ &= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x_1[m] e^{-j\Omega m} \\ &= e^{-j\Omega n_0} X_1(\Omega) \end{aligned}$$

Discrete-Time Fourier Transform

Find the Fourier transform of $x_2[n] = x_1[n - n_0]$.

$$X_2(\Omega) = e^{-j\Omega n_0} X_1(\Omega)$$



Magnitude is unchanged.

Phase offset by $-\Omega n_0$.

- still antisymmetric?
- still periodic in 2π ?

Discrete-Time Fourier Transform

Find the Fourier transform of $x_3[n] = \text{Symmetric}\{x_1[n]\}$.

$$\begin{aligned}X_3(\Omega) &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(x_1[n] + x_1[-n])e^{-j\Omega n} \\&= \frac{1}{2} \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\Omega n} + \frac{1}{2} \sum_{n=-\infty}^{\infty} x_1[-n] e^{-j\Omega n} \\&= \frac{1}{2}X_1(\Omega) + \frac{1}{2} \sum_{m=-\infty}^{\infty} x_1[m] e^{j\Omega m} \\&= \frac{1}{2}X_1(\Omega) + \frac{1}{2}X_1(-\Omega) \\&= \frac{1}{2} \left(\frac{1}{1 - ae^{-j\Omega}} + \frac{1}{1 - ae^{j\Omega}} \right) = \frac{1}{2} \left(\frac{1 - ae^{-j\Omega} + 1 - ae^{j\Omega}}{1 - ae^{-j\Omega} - ae^{j\Omega} + a^2} \right) \\&= \frac{1 - a \cos \Omega}{1 - 2a \cos \Omega + a^2}\end{aligned}$$

real and symmetric $\xRightarrow{\text{DTFT}}$ real and symmetric

Discrete-Time Fourier Transform

Find the Fourier transform of $x_4[n] = \text{Antisymmetric}\{x_1[n]\}$.

$$\begin{aligned}X_4\Omega &= \sum_{n=-\infty}^{\infty} \frac{1}{2}(x_1[n] - x_1[-n])e^{-j\Omega n} \\&= \frac{1}{2} \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\Omega n} - \frac{1}{2} \sum_{n=-\infty}^{\infty} x_1[-n] e^{-j\Omega n} \\&= \frac{1}{2}X_1(\Omega) - \frac{1}{2} \sum_{m=-\infty}^{\infty} x_1[m] e^{j\Omega m} \\&= \frac{1}{2}X_1(\Omega) - \frac{1}{2}X_1(-\Omega) \\&= \frac{1}{2} \left(\frac{1}{1 - ae^{-j\Omega}} - \frac{1}{1 - ae^{j\Omega}} \right) = \frac{1}{2} \left(\frac{1 - ae^{j\Omega} - 1 + ae^{-j\Omega}}{1 - ae^{-j\Omega} - ae^{j\Omega} + a^2} \right) \\&= \frac{-ja \sin \Omega}{1 - 2a \cos \Omega + a^2}\end{aligned}$$

real and antisymmetric $\xrightarrow{\text{DTFT}}$ imaginary and antisymmetric

Discrete-Time Fourier Transform

Find the Fourier transform of $x_5[n] = nx_1[n]$.

$$X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\Omega n}$$

$$\frac{d}{d\Omega} X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n] (-jn) e^{-j\Omega n}$$

$$j \frac{d}{d\Omega} X_1(\Omega) = \sum_{n=-\infty}^{\infty} nx_1[n] e^{-j\Omega n}$$

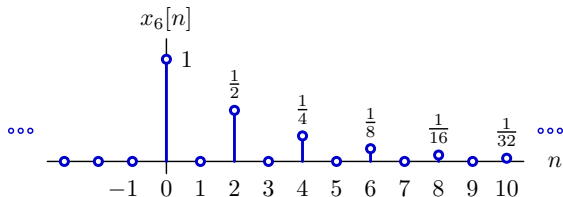
$$x_1[n] \xrightarrow{\text{DTFT}} \frac{1}{1 - ae^{-j\Omega}}$$

$$\begin{aligned} x_5[n] &\xrightarrow{\text{DTFT}} j \frac{d}{d\Omega} \left(\frac{1}{1 - ae^{-j\Omega}} \right) \\ &= -j \left(\frac{1}{1 - ae^{-j\Omega}} \right)^2 (-ae^{-j\Omega})(-j) \\ &= \frac{ae^{-j\Omega}}{(1 - ae^{-j\Omega})^2} \end{aligned}$$

Discrete-Time Fourier Transform

Find the Fourier transform of $x_6[n]$:

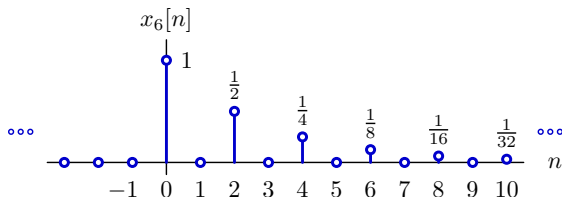
$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & n = 0, 2, 4, 6, 8, \dots, \infty \\ 0 & \text{otherwise} \end{cases}$$



Discrete-Time Fourier Transform

Find the Fourier transform of $x_6[n]$:

$$x_6[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & n = 0, 2, 4, 6, 8, \dots, \infty \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{aligned} X_6(\Omega) &= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{2}\right)^{n/2} e^{-j\Omega n} \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m e^{-j\Omega 2m} = \sum_{m=0}^{\infty} \left(\frac{1}{2} e^{-j2\Omega}\right)^m = \frac{1}{1 - \frac{1}{2} e^{-j2\Omega}} \end{aligned}$$

Stretching in time $\rightarrow$ compressing in frequency

Inverse Discrete-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\Omega) = e^{-j3\Omega}$$

Inverse Discrete-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\Omega) = e^{-j3\Omega}$$

$$\begin{aligned}x[n] &= \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \\&= \frac{1}{2\pi} \int_{2\pi} e^{-j3\Omega} e^{j\Omega n} d\Omega \\&= \frac{1}{2\pi} \int_{2\pi} e^{j\Omega(n-3)} d\Omega \\&= \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$