# 6.3000: Signal Processing

#### **Continuous-Time Fourier Transform**

## **Synthesis Equation**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

## **Analysis Equation**

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Find the Fourier transforms of the following continuous-time signals.

- $x_1(t) = \begin{cases} e^{-t} & \text{if } t > 0\\ 0 & \text{otherwise} \end{cases}$
- $\bullet \quad x_2(t) = x_1(t t_0)$
- $x_3(t) = \text{Symmetric}\{x_1(t)\}$
- $x_4(t) = \text{Antisymmetric}\{x_1(t)\}$
- $x_5(t) = \frac{d}{dt} \text{Symmetric}\{x_1(t)\}$

Find the Fourier transform of the following signal

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t > 0\\ 0 & \text{otherwise} \end{cases}$$

$$X_1(\omega) = \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(1+j\omega)t}dt$$

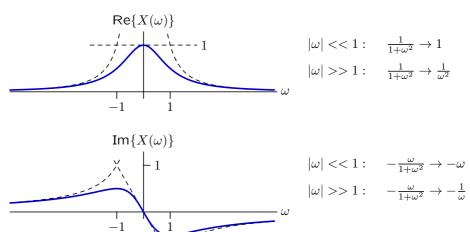
$$= -\frac{1}{1+j\omega}e^{-(1+j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{1+j\omega}$$

Find the real and imaginary parts of  $X(\omega)$ .

$$X(\omega) = \frac{1}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{1-j\omega}{1+\omega^2}$$

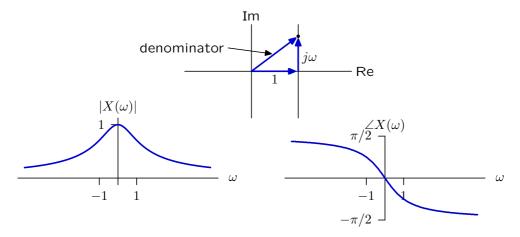
Plot these functions.



Alternative graphical method.

$$X(\omega) = \frac{1}{1 + j\omega}$$

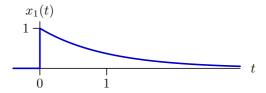
The denominator is the sum of two vectors.

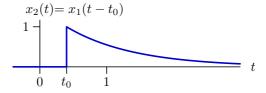


Magnitude is symmetric, angle is antisymmetric in  $\Omega$ .

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$





Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$

$$X_2(\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} x_1(t - t_0)e^{-j\omega t} dt$$

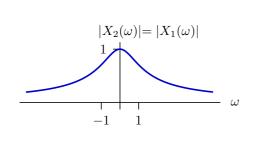
Let  $\tau = t - t_0$ .

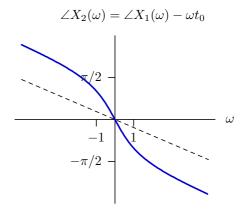
$$X_2(\omega) = \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega(\tau + t_0)} d\tau$$
$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega \tau} d\tau$$
$$= e^{-j\omega t_0} X_1(\omega)$$

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$

$$X_2(\omega) = e^{-j\omega t_0} X_1(\omega)$$



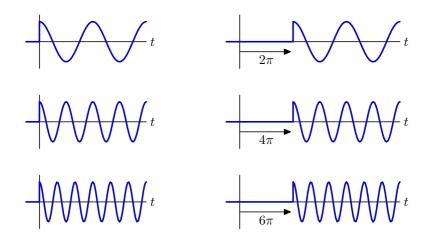


Magnitude is unchanged.

Angle offset by straight line through zero (still antisymmetric).

Why does time delay shift phase by angle proportional to frequency?

Why does time delay shift phase by angle proportional to frequency? Think about Fourier components of a signal that are each delayed by same time  $t_0$ .



The same amount of time corresponds to different amounts of phase.

Find the Fourier transform of the following signal.

$$x_3(t) = \operatorname{Symmetric}\{x_1(t)\}\$$

$$X_3(\omega) = \int_{-\infty}^{\infty} \text{Symmetric}\{x_1(t)\}e^{-j\omega t}dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (x_1(t) + x_1(-t))e^{-j\omega t}dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + \frac{1}{2} \int_{-\infty}^{\infty} x_1(-t)e^{-j\omega t}dt$$

$$= \frac{1}{2} X_1(\omega) + \frac{1}{2} \int_{-\infty}^{\infty} x_1(\tau)e^{j\omega \tau}d\tau$$

$$= \frac{1}{2} X_1(\omega) + \frac{1}{2} X_1(-\omega)$$

$$= \frac{1}{2} \left(\frac{1}{1+j\omega} + \frac{1}{1-j\omega}\right) = \frac{1}{1+\omega^2}$$

real and symmetric  $\stackrel{\text{CTFT}}{\Longrightarrow}$  real and symmetric

Find the Fourier transform of the following signal.

$$x_4(t) = \text{Antisymmetric}\{x_1(t)\}$$

$$X_4(\omega) = \int_{-\infty}^{\infty} \text{Antisymmetric}\{x_1(t)\}e^{-j\omega t}dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (x_1(t) - x_1(-t))e^{-j\omega t}dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt - \frac{1}{2} \int_{-\infty}^{\infty} x_1(-t)e^{-j\omega t}dt$$

$$= \frac{1}{2} X_1(\omega) - \frac{1}{2} \int_{-\infty}^{\infty} x_1(\tau)e^{j\omega \tau}d\tau$$

$$= \frac{1}{2} X_1(\omega) - \frac{1}{2} X_1(-\omega)$$

$$= \frac{1}{2} \left(\frac{1}{1+j\omega} - \frac{1}{1-j\omega}\right) = \frac{-j\omega}{1+\omega^2}$$

real and antisymmetric  $\stackrel{\text{CTFT}}{\Rightarrow}$  imaginary and antisymmetric

Find the Fourier transform of the following signal.

$$x_5(t) = \frac{d}{dt} \text{Symmetric}\{x_1(t)\}$$

$$X_{5}(\omega) = \int_{-\infty}^{\infty} \underbrace{\frac{dx_{3}(t)}{dt}}_{dv} \underbrace{e^{-j\omega t}}_{u} \underbrace{dt}_{dv}$$
$$= \underbrace{e^{-j\omega t}}_{u} \underbrace{x_{3}(t)}_{v} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x_{3}(t)}_{v} \underbrace{(-j\omega)e^{-j\omega t}}_{du} dt$$

The first term is zero since the CTFT of  $x_3(t)$  converged.

$$X_5(\omega) = j\omega \int_{-\infty}^{\infty} x_3(t) e^{-j\omega t} dt$$
$$= j\omega X_3(\omega) = \frac{j\omega}{1 + \omega^2}$$

real and antisymmetric  $\stackrel{\mathrm{CTFT}}{\Longrightarrow}$  imaginary and antisymmetric

Alternative method – start with synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) (j\omega) e^{j\omega t} d\omega$$

$$Y(\omega) = j\omega X(\omega)$$

Straightforward approach (start with analysis) not always simplest. This is a great example of how Fourier transforms simplify calculus. Differentiation in time corresponds to multiplication by  $j\omega$  in frequency.

#### **Inverse Continuous-Time Fourier Transform**

Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

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Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{\omega} e^{j\omega t} d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \frac{e^{(jt+1)\omega}}{jt+1} \Big|_{-\infty}^{0} + \frac{1}{2\pi} \frac{e^{(jt-1)\omega}}{jt-1} \Big|_{0}^{\infty}$$

$$= \frac{1}{2\pi} \frac{1}{jt+1} - \frac{1}{2\pi} \frac{1}{jt-1}$$

$$= \frac{1/\pi}{1+t^2}$$