6.3000 Final Exam Spring 2025

Name: Solutions

Kerberos (Athena) Username:

Please WAIT until we tell you to begin.

This exam is closed book, but you may use three 8.5×11 sheets of notes (six sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

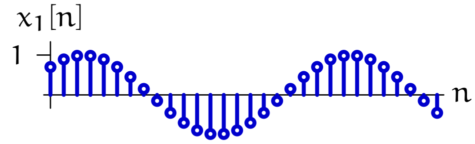
We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

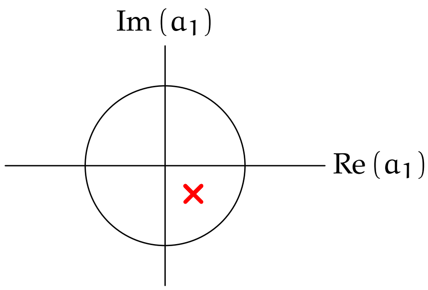
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

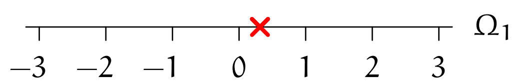
1 Describing Sinusoids (11 points)

Part a. Let $x_1[n] = a_1 e^{j\Omega_1 n} + a_1^* e^{-j\Omega_1 n}$ as shown in the following figure.



Estimate a_1 and Ω_1, where Ω_1 is real-valued and a_1 may be complex. Place an "×" on the complex plane shown below (where the circle has a radius of 1) to indicate the value of a_1. Also, place an "×" on the number line shown below to indicate the value of Ω_1.





This signal can be written as

$$x_1[n] = a_1 e^{j\Omega_1 n} + a_1^* e^{-j\Omega_1 n} = a_1 e^{j\Omega_1 n} + (a_1 e^{j\Omega_1 n})^* = 2\,\mathrm{Re}\left(a_1 e^{j\Omega_1 n}\right) = \mathrm{Re}\left(2a_1 e^{j\Omega_1 n}\right) .$$

Thus this signal is the real part of a vector $2a_1$ in the complex plane whose angle increases by Ω_1 radians per sample n. Since the period of $x_1[n]$ is 20,

$$e^{j\Omega_1 n} = e^{j\Omega_1 (n+20)} = e^{j\Omega_1 n} e^{j20\Omega_1}$$

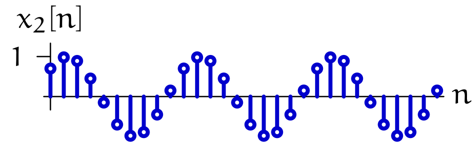
and the period of the complex exponential is 2π, it follows that $20\Omega_1 = 2\pi$, and

$$\Omega_1 = \frac{2\pi}{20} \approx 0.314 .$$

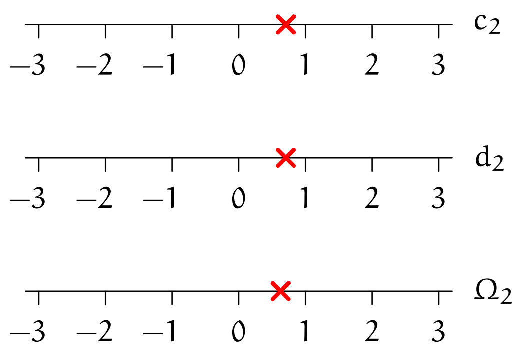
The peak amplitude is approximately 1, so $|2a_1| = 1$, and $|a_1| = 0.5$. The first peak of $x_1[n]$ occurs about 2.5 samples after $n = 0$, which corresponds to approximately $2.5/20 = 1/8$ of a cycle. So the angle of a_1 must start at approximately $-\pi/4$ and therefore $\angle a_1 \approx -\frac{\pi}{4}$. Thus

$$a_1 \approx \frac{1}{2} e^{-j\pi/4} .$$

Part b. Let $x_2[n] = c_2 \cos(\Omega_2 n) + d_2 \sin(\Omega_2 n)$ as shown in the following figure.



Estimate the real-valued constants c_2, d_2, and Ω_2. Place an "$\times$" on each of the number lines shown below to indicate these values.



We can express $x_2[n] = c_2 \cos \Omega_2 n + d_2 \sin \Omega_2 n$ as

$$\begin{aligned} x_2[n] &= c_2 \left(\frac{e^{j\Omega_2 n} + e^{-j\Omega_2 n}}{2} \right) + d_2 \left(\frac{e^{j\Omega_2 n} - e^{-j\Omega_2 n}}{2j} \right) \\ &= \frac{1}{2}(c_2 - jd_2)e^{j\Omega_2 n} + \frac{1}{2}(c_2 + jd_2)e^{-j\Omega_2 n} \\ &= \mathrm{Re}\left((c_2 - jd_2)e^{j\Omega_2 n} \right) \end{aligned}$$

Since the period of $x_2[n]$ is 10,

$$e^{j\Omega_2 n} = e^{j\Omega_2 (n+10)} = e^{j\Omega_2 n} e^{j10\Omega_2}$$

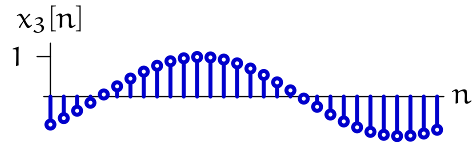
and the period of the complex exponential is 2π, it follows that $10\Omega_2 = 2\pi$, and

$$\Omega_2 = \frac{2\pi}{10} \approx 0.628\,.$$

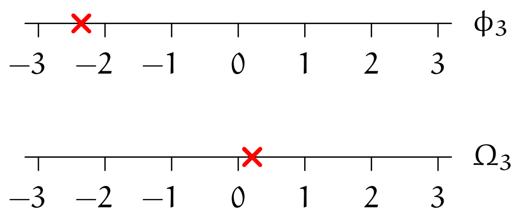
The peak amplitude of $x_2[n]$ is approximately 1, so $|c_2 - jd_2| = \sqrt{c_2^2 + d_2^2} = 1$. The first peak of $x_2[n]$ occurs at about 1/8 of a cycle. So the angle of $c_2 - jd_2$ must be approximately $-\pi/4$. Therefore

$$c_2 \approx d_2 \approx \frac{1}{\sqrt{2}} \approx 0.7\,.$$

Part c. Let $x_3[n] = \cos(\Omega_3 n + \phi_3)$ as shown in the following figure.



Estimate the real-valued constants ϕ_3 and Ω_3. Place an "$\times$" on each of the number lines shown below to indicate these values.



The period of $x_3[n]$ is approximately 30, so

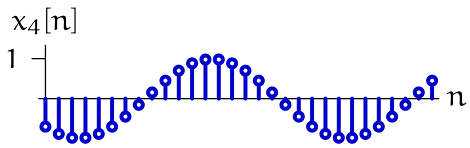
$$\Omega_3 \approx \frac{2\pi}{30} \approx 0.2\,.$$

The signal peaks at $n = 11$, so

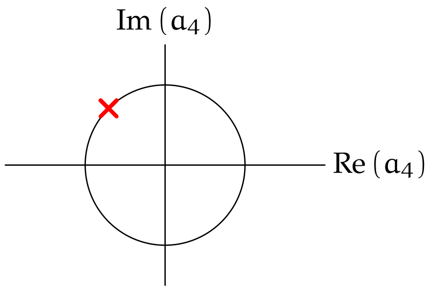
$$x_3[n_0] = \cos(\Omega_3 n_0 + \phi_3) \approx 1$$

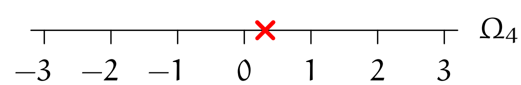
so $\phi_3 \approx -11\Omega_3 = 22\pi/30 \approx -2.3$.

Part d. Let $x_4[n] = \text{Re}\left(a_4 e^{j\Omega_4 n}\right)$ as shown in the following figure.



Estimate a_4 and Ω_4, where Ω_4 is real-valued and a_4 may be complex. Place an "$\times$" on the complex plane shown below (where the circle has a radius of 1) to indicate the value of a_4. Also, place an "$\times$" on the number line shown below to indicate the value of Ω_4.





The period of $x_4[n]$ is 20, so

$$\Omega_4 = \frac{2\pi}{20} \approx 0.314\,.$$

The peak amplitude is approximately 1, so $|a_4| \approx 1$. The first peak of $x_4[n]$ occurs near $n = 12.5$, which corresponds to approximately $12.5/20 = 5/8$ of a cycle, so that the angle of a_4 must be approximately $\angle a_4 \approx -\frac{5}{4}\pi$.

2 Periodic Extension (20 points)

Let $f[n]$ represent a real-valued, discrete-time signal, and let $g[n]$ represent a periodically extended version of $f[n]$, as follows:

$$g[n] = \begin{cases} f[n] & \text{if } 0 \leqslant n < N \\ f[n-N] & \text{if } N \leqslant n < 2N \\ f[n-2N] & \text{if } 2N \leqslant n < 3N \\ f[n-mN] & \text{if } mN \leqslant n < (m+1)N \end{cases}$$

where m represents an integer that is greater than or equal to 3.

Let $F[k]$ represent the DFT of the first N samples of $f[n]$.

Let $G[k]$ represent the DFT of the first $2N$ samples of $g[n]$.

Part a. Determine an expression for $G[0]$ in terms of the DFT coefficients $F[k]$ as well as familiar constants such as π and e. Simplify your expression as much as possible, and enter your result in the box below.

$$G[0] = \boxed{F[0]}$$

The first half of G is the same as F. Therefore the DC value of the first half of G is equal to the DC value of F.

The second half of G is the same as F. Therefore the DC value of the second half of G is equal to the DC value of F.

The DC value of G is the average of the DC values of its first and second halves. Therefore the DC value of G is equal to the DC value of F.

Part b. Determine an expression for $G[1]$ in terms of the DFT coefficients $F[k]$ as well as familiar constants such as π and e. Simplify your expression as much as possible, and enter your result in the box below.

$$G[1] = \boxed{0}$$

$$\begin{aligned} G[1] &= \sum_{n=0}^{2N-1} g[n]e^{\frac{-j2\pi kn}{2N}} \\ &= \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} + \sum_{n=N}^{2N-1} f[n-N]e^{\frac{-j2\pi kn}{2N}} \\ &= \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} + \sum_{m=0}^{N-1} f[m]e^{\frac{-j2\pi k(m+N)}{2N}} \\ &= \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} + \sum_{m=0}^{N-1} f[m]e^{\frac{-j2\pi km}{2N}} e^{\frac{-j2\pi kN}{2N}} \\ &= (1+e^{-j\pi k}) \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi kn}{2N}} = (1+e^{-j\pi}) \sum_{n=0}^{N-1} f[n]e^{\frac{-j2\pi n}{2N}} \\ &= 0 \end{aligned}$$

Part c. Determine an expression for $G[k]$ in terms of the DFT coefficients $F[k]$ as well as familiar constants such as π and e. Simplify your expression as much as possible, and enter your result in the box below.

$$G[k] = \boxed{\begin{cases} F[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}}$$

$$\begin{aligned}
G[k] &= \frac{1}{2N}\sum_{n=0}^{2N-1} g[n]e^{-\frac{j2\pi kn}{2N}} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} + \frac{1}{2N}\sum_{n=N}^{2N-1} f[n-N]e^{-\frac{j2\pi kn}{2N}} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} + \frac{1}{2N}\sum_{m=0}^{N-1} f[m]e^{-\frac{j2\pi k(m+N)}{2N}} \\
&= \frac{1}{2N}\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} + \frac{1}{2N}\sum_{m=0}^{N-1} f[m]e^{-\frac{j2\pi km}{2N}}e^{-\frac{j2\pi kN}{2N}} \\
&= \frac{1}{2N}\left(1+e^{-\frac{j2\pi kN}{2N}}\right)\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} \\
&= \frac{1}{2N}\left(1+(-1)^k\right)\sum_{n=0}^{N-1} f[n]e^{-\frac{j2\pi kn}{2N}} \\
&= \left(\frac{1+(-1)^n}{2}\right)F[k/2] \\
&= \begin{cases} F[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

3 Impulsive Images (15 points)

Pixel values for three images ($f_1[r,c]$, $f_2[r,c]$, and $f_3[r,c]$) are shown below for $|r| \leqslant 2$ and $|c| \leqslant 2$. Pixel values outside the indicated regions are zero.

$f_1[r,c]$

r \ c	−2	−1	0	1	2
−2	0	0	0	1	0
−1	0	0	0	0	0
0	0	0	0	0	0
1	0	0	0	0	0
2	0	1	0	0	0

$\left|F_1[k_r,k_c]\right| = $ **G**

$f_2[r,c]$

r \ c	−2	−1	0	1	2
−2	0	0	0	0	0
−1	0	−1	0	1	0
0	0	−2	0	2	0
1	0	−1	0	1	0
2	0	0	0	0	0

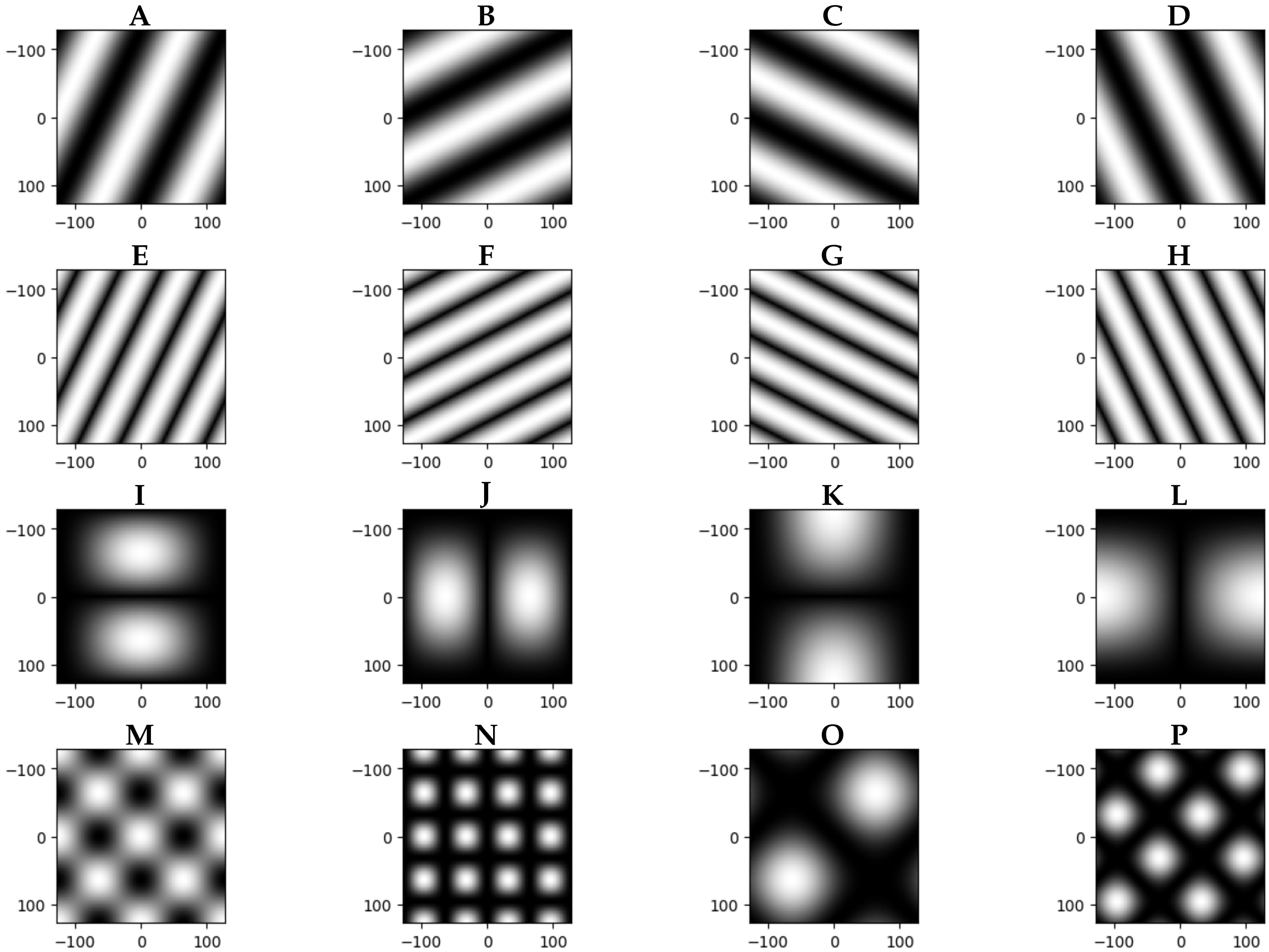
$\left|F_2[k_r,k_c]\right| = $ **J**

$f_3[r,c]$

r \ c	−2	−1	0	1	2
−2	0	0	1	0	0
−1	0	0	0	0	0
0	−1	0	4	0	1
1	0	0	0	0	0
2	0	0	−1	0	0

$\left|F_3[k_r,k_c]\right| = $ **P**

Each of the panels below show the magnitude of a 2D DFT calculated for $-128 \leqslant r, c < 128$ and displayed with black and white representing the minimum and maximum magnitude in each image. Determine which of the panels below shows the magnitude of the 2D DFT for each of the images shown above, and enter its label (A-P) in the corresponding box above.



Let $R = C = 128 = N$.

$$f_1[r,c] = \delta[r{+}2, c{-}1] + \delta[r{-}2, c{+}1]$$

$$F_1[k_r,k_c] = \frac{1}{N^2}\sum_r\sum_c f_1[r,c]e^{-\frac{j2\pi}{N}\left(rk_r+ck_c\right)}$$

$$= \frac{1}{N^2}\left(e^{-\frac{j2\pi}{N}\left(-2k_r+1k_c\right)} + e^{-\frac{j2\pi}{N}\left(+2k_r-1k_c\right)}\right)$$

$$= \frac{2}{N^2}\cos\left(\frac{2\pi}{N}(2k_r{-}k_c)\right)$$

$$\Big|F_1[k_r,k_c]\Big| = \Big|\frac{2}{N^2}\cos\left(\frac{2\pi}{N}(2k_r{-}k_c)\right)\Big|$$

Notice that all points along the line

$$k_r = k_c/2$$

have the same value of $|F_1[k_r,k_c]|$, and that similar statements hold for lines that are parallel to $k_r = k_c/2$. Thus, pixels along downward sloping lines in the k_r-k_c plane have the same brightness.

Rows of $F_1[k_r,k_c]$ go through one period of brightness for $-C/2 \leqslant c < C/2$. However, rows of $|F_1[k_r,k_c]|$ go through **TWO** periods of brightness for $-C/2 \leqslant c < C/2$ because negative peaks of F_1 are positive peaks of $|F_1|$.

Similarly, columns of $F_1[k_r,k_c]$ go through two periods of brightness for $-R/2 \leqslant r < R/2$. And columns of $|F_1[k_r,k_c]|$ go through **FOUR** periods of brightness for $-R/2 \leqslant r < R/2$.

Therefore the answer is **G**.

$$f_2[r,c] = -\delta[r{+}1,c{+}1] + -2\delta[r,c{+}1] + -\delta[r{-}1,c{+}1] + \delta[r{+}1,c{-}1] + 2\delta[r,c{-}1] + \delta[r{-}1,c{-}1]$$

$$F_2[k_r,k_c] = \frac{1}{N^2}\sum_r\sum_c f_1[r,c]e^{-\frac{j2\pi}{N}\left(rk_r+ck_c\right)}$$

$$= \frac{1}{N^2}\Big(-e^{-\frac{j2\pi}{N}\left(-1k_r-1k_c\right)} - 2e^{-\frac{j2\pi}{N}\left(+0k_r-1k_c\right)} - e^{-\frac{j2\pi}{N}\left(+1k_r-1k_c\right)}$$
$$+ e^{-\frac{j2\pi}{N}\left(-1k_r+1k_c\right)} + 2e^{-\frac{j2\pi}{N}\left(+0k_r+1k_c\right)} + e^{-\frac{j2\pi}{N}\left(+1k_r+1k_c\right)}\Big)$$

$$= j\frac{4}{N^2}\Big(1+\cos(2\pi k_r/R)\Big)\sin(2\pi k_c/C)$$

$$\Big|F_2[k_r,k_c]\Big| = \frac{4}{N^2}\Big(1+\cos(2\pi k_r/R)\Big)\Big|\sin(2\pi k_c/C)\Big|$$

The cosine term is always positive. It reaches a peak at $k_r = k_c = 0$ and its closest zeros are at the top and bottom of the image.

The sine term is zero at $k_r = k_c = 0$, it has a peak at $k_c = C/4$ and a negative peak at $c_c = -C/4$. After taking the absolute value, the negative peak becomes a positive peak.

Therefore the answer is **J**.

$$f_3[r,c] = \delta[r+2,c] + -\delta[r-2,c] + -\delta[r,c+2] + \delta[r,c-2] + +4\delta[r,c]$$

$$\begin{aligned}
F_3[k_r,k_c] &= \frac{1}{N^2}\sum_r\sum_c f_1[r,c]e^{-\frac{j2\pi}{N}\left(rk_r+ck_c\right)} \\
&= \frac{1}{N^2}\Big(+1e^{-\frac{j2\pi}{N}\left(-2k_r+0k_c\right)} - 1e^{-\frac{j2\pi}{N}\left(+2k_r+0k_c\right)} - 1e^{-\frac{j2\pi}{N}\left(+0k_r-2k_c\right)} \\
&\qquad + 1e^{-\frac{j2\pi}{N}\left(+0k_r+2k_c\right)} + 4e^{-\frac{j2\pi}{N}\left(+0k_r+0k_c\right)}\Big) \\
&= \frac{1}{N^2}\Big(4 + j2\sin(4\pi k_r/N) - j2\sin(4\pi k_c/N)\Big)
\end{aligned}$$

If we break the DFT into 16 equal-sized segments (on a 4×4 grid), then $\sin(4\pi k_r/N)$ is positive in the top row and the third row of segments (left figure below) and $-\sin(4\pi k_c/N)$ is positive in the second and fourth columns of segments (center figure below). The sum is greatest in the diagonal pattern shown in the right figure below.

+	+	+	+
−	−	−	−
+	+	+	+
−	−	−	−

−	+	−	+
−	+	−	+
−	+	−	+
−	+	−	+

	+		+
−		−	
	+		+
−		−	

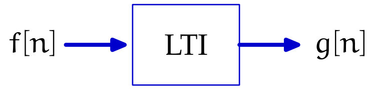
The magnitudes of the peak negative values of F_3 and the magnitudes of the peak negative values of F_3 are equal. Therefore the bright spots in $\left|F_3[k_r,k_c]\right|$ occur in the quadrants shown below.

	+		+
+		+	
	+		+
+		+	

Therefore the answer is **P**.

4 Recurring Delay (22 points)

A discrete-time, linear, time-invariant system has input $f[n]$ and output $g[n]$



and these signals are related by the following difference equation:

$$g[n] = f[n] - \frac{1}{2}g[n-3]$$

Part a. Assume that the output $g[n] = 0$ for $n < 0$, and find the output of the system when the input $f[n] = \delta[n]$. Enter an expression for your result in the following box.

$$g[n] \ = \ \boxed{\begin{cases} \left(-\frac{1}{2}\right)^{n/3} & \text{if } n = 0, 3, 6, 9, \ldots \\ 0 & \text{otherwise} \end{cases}}$$

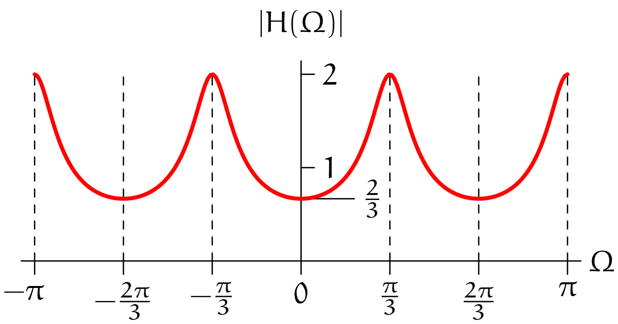
If $g[n] = 0$ for $n \leqslant 0$ and $f[n] = \delta[n]$ then

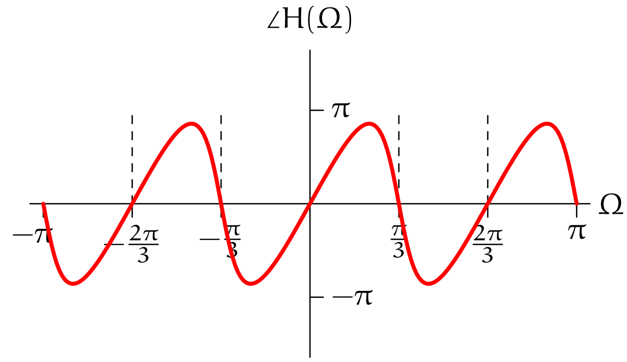
$$\begin{aligned}
g[0] &= f[0] - \tfrac{1}{2}g[-3] = 1 - 0 = 1 \\
g[1] &= f[1] - \tfrac{1}{2}g[-2] = 0 - 0 = 0 \\
g[2] &= f[2] - \tfrac{1}{2}g[-1] = 0 - 0 = 0 \\
g[3] &= f[3] - \tfrac{1}{2}g[0] \quad = 0 - \tfrac{1}{2} = -\tfrac{1}{2} \\
g[4] &= f[4] - \tfrac{1}{2}g[1] \quad = 0 - 0 = 0 \\
g[5] &= f[5] - \tfrac{1}{2}g[2] \quad = 0 - 0 = 0 \\
g[6] &= f[6] - \tfrac{1}{2}g[3] \quad = 0 + \tfrac{1}{4} = \tfrac{1}{4} \\
g[7] &= f[7] - \tfrac{1}{2}g[4] \quad = 0 - 0 = 0 \\
g[8] &= f[8] - \tfrac{1}{2}g[5] \quad = 0 - 0 = 0 \\
g[9] &= f[9] - \tfrac{1}{2}g[6] \quad = 0 + \tfrac{1}{8} = -\tfrac{1}{8} \\
&\ldots
\end{aligned}$$

In general,

$$g[n] = \begin{cases} \left(-\frac{1}{2}\right)^{n/3} & \text{if } n = 0, 3, 6, 9, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Part b. Sketch the frequency response of this system on the axes below, with magnitude on the left and phase on the right. Label all of the important points on your sketches.

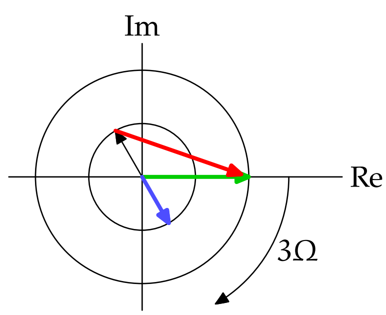




$$G(\Omega) = F(\Omega) - \frac{1}{2}e^{-j3\Omega}G(\Omega)$$

$$H(\Omega) = \frac{G(\Omega)}{F(\Omega)} = \frac{1}{1+\frac{1}{2}e^{-j3\Omega}} = \frac{1+\frac{1}{2}e^{j3\Omega}}{(1+\frac{1}{2}e^{-j3\Omega})(1+\frac{1}{2}e^{j3\Omega})} = \frac{1+\frac{1}{2}\cos(3\Omega)+\frac{1}{2}j\sin(3\Omega)}{\frac{5}{4}+\cos(3\Omega)}$$

$$H(\Omega) = \frac{1}{1+\frac{1}{2}e^{-j3\Omega}} = \frac{1}{1-\frac{1}{2}e^{j\pi}e^{-j3\Omega}} = \frac{1}{1-\frac{1}{2}e^{j(-3\Omega-\pi)}}$$



green arrow: 1

blue arrow: $\frac{1}{2}e^{-j3\Omega}$

black arrow: $\frac{1}{2}e^{-j(3\Omega-\pi)}$

red arrow: $1-\frac{1}{2}e^{-j(3\Omega-\pi)}$

5 Tetra (16 points)

Let $f[r,c]$ represent the following image as an array of pixel brightnesses in the range $[-\frac{1}{2},\frac{1}{2}]$ where black corresponds to $-1/2$, white corresponds to $+1/2$, and the background grey corresponds to 0. The pixels are indexed by their row number r (with $-\frac{1}{2}R \leqslant r < \frac{1}{2}R$) and column number c (with $-\frac{1}{2}C \leqslant c < \frac{1}{2}C$), and $R = C = 200$.

Note: High-quality images of this figure as well as those on the following page have been provided on a separate sheet. Use the high-quality images to answer the questions below, but record your answers on this page (which has QR codes).

Also consider four additional images, two of which are defined in the spatial domain:

$$h_1[r,c] = \sin\left(\frac{40\pi r}{R}\right)$$

$$h_2[r,c] = \begin{cases} 1 & \text{if } r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

and two defined by their two-dimensional Fourier transforms:

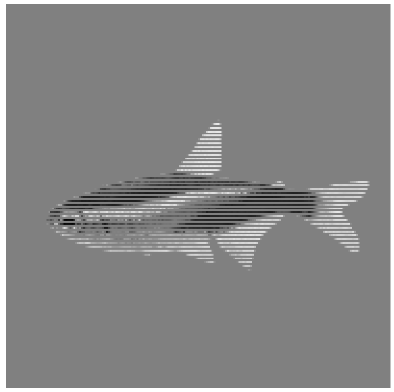
$$H_3[k_r,k_c] = j\sin\left(\frac{40\pi k_r}{R}\right)$$

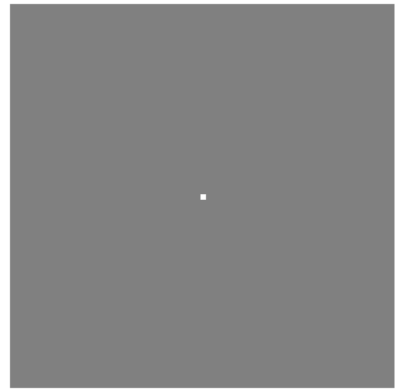
$$H_4[k_r,k_c] = \begin{cases} 1 & \text{if } k_r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

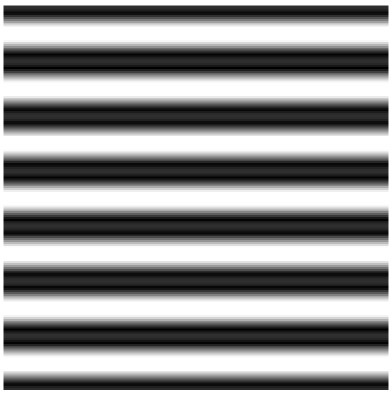
Determine which of the images on the next page corresponds to each of the following image combinations, and enter the appropriate label (A-L) in each of the boxes below.

$(f \times h1)[r,c]$	E	$(f \circledast h1)[r,c]$	A
$(f \times h2)[r,c]$	B	$(f \circledast h2)[r,c]$	F
$(f \times h3)[r,c]$	G	$(f \circledast h3)[r,c]$	I
$(f \times h4)[r,c]$	C	$(f \circledast h4)[r,c]$	K

A: 

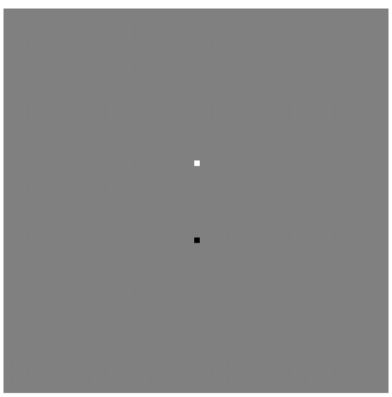
B: 

C: 

D: 

E: 

F: 

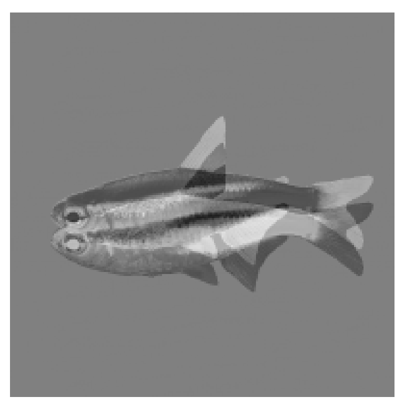
G: 

H: 

I: 

J: 

K: 

L: 

$$h_1[r,c] = \sin\left(\frac{40\pi r}{R}\right)$$

This image consists of 20 horizontal stripes with brightnesses changing sinusoidally along the vertical dimension. Multiplying the fish image by this signal puts horizontal stripes on the fish but does not affect the background since its brightness is zero.

Thus the answer is **E**.

Convolving the fish image with h_1 is equivalent to multiplying their 2D DFTs. The 2D DFT of h_1 is a positive dot at $r = -20, c = 0$ and a negative dot at $r = 20, c = 0$. Multiplying these dots times the 2D DFT of the fish selects only the two basis functions at the dots location.

The result is a striped image shown in **A**.

$$h_2[r,c] = \begin{cases} 1 & \text{if } r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

This image consists of 100 alternating horizontal stripes of white and black. Multiplying the fish image by this signal puts finely spaced horizontal stripes on the fish but does not affect the background since its brightness is zero.

Thus the answer is **B**.

Convolving the image with h_2 is equivalent to multiplying their 2D DFTs. The 2D DFT of h_1 is a positive dot at $r = -1, c = 0$ and a negative dot at $r = 1, c = 0$. Multiplying these dots times the 2D DFT of the fish selects only the two basis functions, corresponding to the dots' locations. However, the amplitudes of these components in the fish image are small. Thus the resulting stripes are not visible at the output.

The result is shown in **F**.

$$H_3[k_r,k_c] = j\sin\left(\frac{40\pi k_r}{R}\right)$$

This 2D DFT consists of 20 horizontal stripes with amplitudes changing sinusoidally along the vertical dimension. The inverse DFT of this pattern is two dots, much like image G. Multiplying the fish image by this signal creates an image that has the background gray color (which represents 0) everywhere except for the two dots in G.

Thus the answer is **G**.

Convolving the fish image with the two dots that comprise h_3 produces a positive image of the fish that is shifted up 20 pixels plus a negative image of the fish that is shifted down 20 pixels.

The result is a striped image shown in **I**.

$$H_4[k_r,k_c] = \begin{cases} 1 & \text{if } k_r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

This 2D DFT consists of 100 horizontal stripes that alternate between white and black along the vertical dimension. The inverse DFT of this pattern is two dots: one at the origin and one in the center of the top row of the image. Multiplying the fish image times this signal sets the top dot to zero, making it blend in with the background. Only the bottom dot remains.

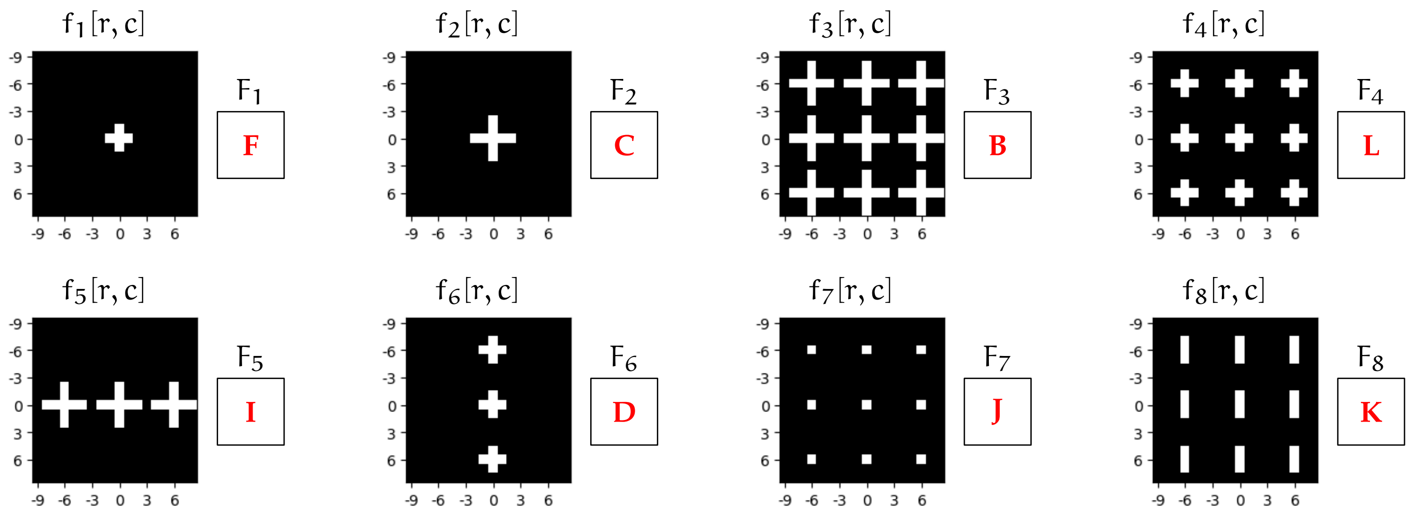
Thus the answer is **C**.

Convolving the fish image with the two dots that comprise h_4 produces two images of the fish. The first is at the original location of the fish. The second is shifted up by half the frame, with the upper part of this shifted fish circularly shifted into the bottom part of the new image.

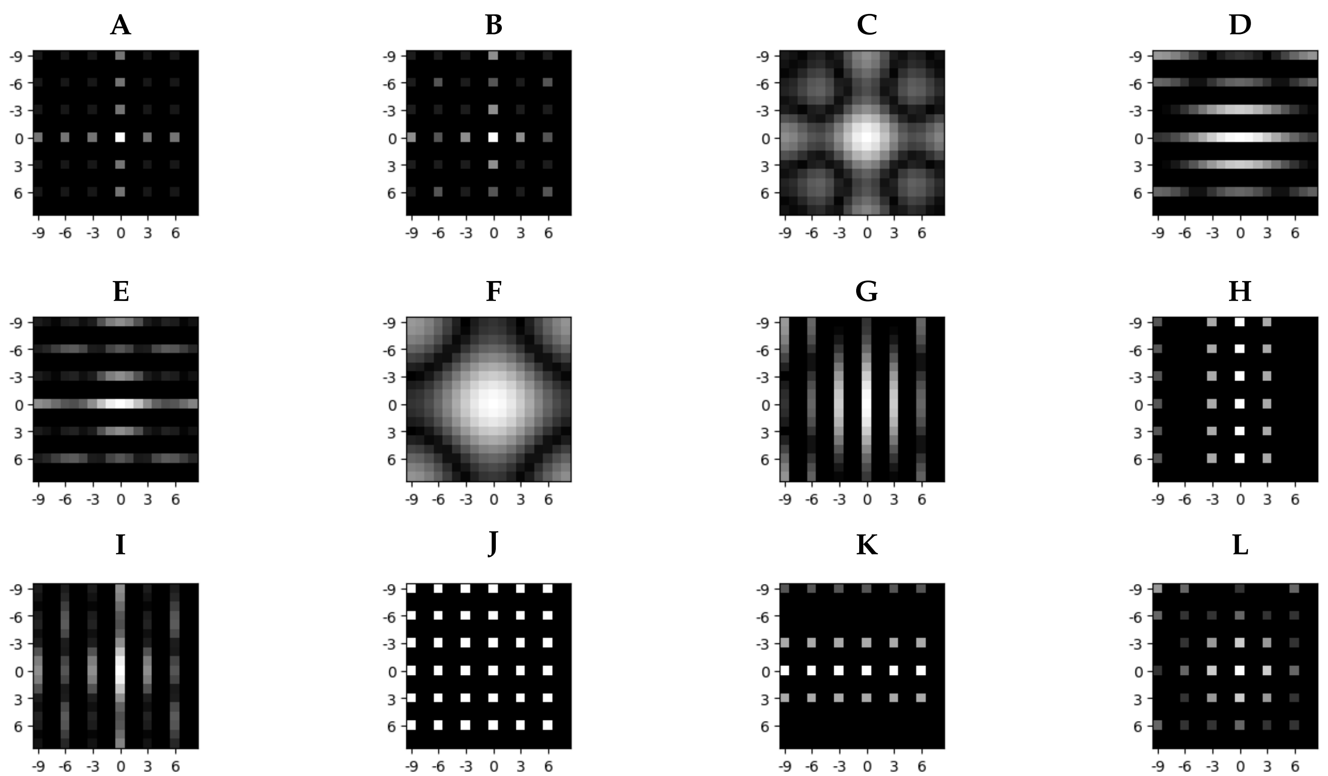
The result is **K**.

6 Pluses and Minuses (16 points)

The panels below show eight images ($f_1[r, c]$ to $f_8[r, c]$ with black (0) or white (1) pixels that are indexed by row number r and column number c where $-9 \leqslant r < 9$ and $-9 \leqslant c < 9$.



Each of the following panels shows the magnitude of a 2D DFT. For each panel, black represents a value of 0 and white represents the largest magnitude in that panel (which may be different for each panel). Determine which of the panels shows the magnitude of the 2D DFT of each of the images above, and enter the corresponding letter (A-L) in the appropriate box.



$f_1[r, c]$

This signal contains a DC component (due to the white square at $r = c = 0$, as well as a horizontal fundamental component (due to white squares at $c = \pm 1$, and a vertical fundamental component (due to white squares at $r = \pm 1$. The periods of the fundamental components are equal to the image width $C = 18$ and height $R = 18$. Thus the 2D DFT is panel **F**.

$f_2[r, c]$

This signal includes not only fundamental components (as in f_1 but also second harmonic terms (due to the which squares at $c = \pm 2$ and $r = \pm 2$). Thus the 2D DFT is panel **C**.

$f_7[r, c]$

This image is periodic in r and c, so its 2D DFT will be composed of isolated dots. Since there are 3 dots across the field in both the r and c directions, the lowesst, non-zero harmonic in each direction will be the third. Thus the 2D DFT is panel **J**.

$f_3[r, c]$

The f_3 image is the convolution of f_2 with f_7. Therefore F_3 will be the product of F_2 (C) with F_7 (J). Thus the 2D DFT is panel **B**.

$f_4[r, c]$

The f_4 image is the convolution of f_1 with f_7. Therefore F_4 will be the product of F_1 (F) with F_7 (J). Thus the 2D DFT is panel **L**.

$f_8[r, c]$

The f_8 image is the convolution of a large, rotated minus sign with f_7. The rotated minus sign is skinny and tall. Therefore, its 2D DFT would be short and fat. The only pattern of isolated dots that fits this description is panel **K**.

$f_5[r, c]$

This image is periodic in c but not in r. Therefore its 2D DFT is dots in k_c continuous in k_r. This description could fit with I or G. The big plus signs produces the shorter vertical bars in panel **I**.

$f_6[r, c]$

This image is a rotated version of f_5 with small plus signs, which produces the longer horizontal bars in panel **D**.

Worksheet (intentionally blank)

Worksheet (intentionally blank)

Worksheet (*intentionally blank*)

Worksheet (intentionally blank)

Worksheet (intentionally blank)