6.3000 Quiz 2 Spring 2025

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two 8.5×11 sheets of notes (four sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Short-Answer Questions (24 points)

Part a. The left column of the following table shows the input/output relations for four different systems, where x represents the input and y represents the output of the system. For each system, determine if that system is additive and/or homogeneous and/or time-invariant, and enter **Yes** or **No** in the boxes to the right.

	additive? (**Yes** or **No**)	homogeneous? (**Yes** or **No**)	time-invariant? (**Yes** or **No**)
$y[n]=\left(-\frac{1}{2}\right)^n\left(x[n]+1\right)$	**No**	**No**	**No**
$y[n]=\sin\left(x[n]\right)$	**No**	**No**	**Yes**
$y(t)=\int_{-\infty}^{t}x(\tau)d\tau$	**Yes**	**Yes**	**Yes**
$y(t)=tx(t)$	**Yes**	**Yes**	**No**

Part a1. Determine if the output of the sum of two inputs ($x_1[n]+x_2[n]$) is the sum of outputs ($y_1[n]$ and $y_2[n]$).

$$y_1[n]+y_2[n]=\left(-\frac{1}{2}\right)^n\Big(\left(x_1[n]+\mathbf{1}\right)+\left(x_2[n]+\mathbf{1}\right)\Big)\neq\left(-\frac{1}{2}\right)^n\left(x_1[n]+x_2[n]+\mathbf{1}\right)\neq y_1[n]+y_2[n]$$

Determine if scaling the input by α scales the output by α.

$$y_{scaled}[n]=\left(-\frac{1}{2}\right)^n\left(\alpha x[n]+\mathbf{1}\right)=\left(-\frac{1}{2}\right)^n\left(\alpha x[n]+1\right)\neq\left(-\frac{1}{2}\right)^n\left(\alpha x[n]+\alpha\right)$$

Determine if shifting the input shifts the output.

$$\left(-\frac{1}{2}\right)^n\left(x[n-m]+1\right)\neq\left(-\frac{1}{2}\right)^{n-\mathbf{m}}\left(x[n-m]+1\right)$$

Part a2.

$$\sin(x_1[n]+x_2[n])\neq\sin(x_1)+\sin(x_2)$$
$$\alpha\sin(x_1[n])\neq\sin(\alpha x_1[n])$$
$$\sin(x[n-m])=\sin(x[n-m])$$

Part a3.

The integral of a sum is the sum of the integrals.
Scaling the integrand scales the integral.

$$\int_{-\infty}^{t}x(\tau-t_0)d\tau=\int_{-\infty}^{t-t_0}x(\lambda)d\lambda=y(t-t_0)$$

Part a4.

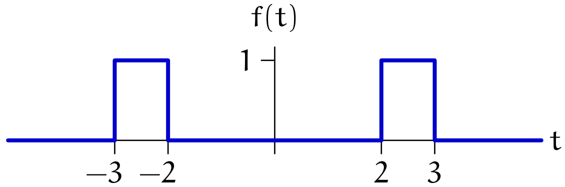
$tx_1(t) + tx_2(t) = t(x_1(t) + x_2(t))$

$t\alpha x(t) = \alpha t x(t)$

$t(x(t{-}t_0)) \neq (t{-}t_0)x(t{-}t_0)$

Part b. Determine the frequency response $F(\omega)$ of a linear, time-invariant system with the following impulse response:

$$f(t) = \begin{cases} 1 & \text{if } 2 < |t| < 3 \\ 0 & \text{otherwise} \end{cases}$$



Enter a closed form expression[1] for $F(\omega)$ in the box below.

$$F(\omega) = \boxed{2\frac{\sin 3\omega}{\omega} - 2\frac{\sin 2\omega}{\omega}}$$

The signal $f(t)$ can be written as the difference between a rectangular pulse that extends from -3 to 3 and a rectangular pulse that extends from -2 to 2. The Fourier transform of the former is

$$\int_{-3}^{3} e^{-j\omega t}\,dt = \frac{e^{-j\omega t}}{-j\omega}\bigg|_{-3}^{3} = 2\frac{\sin 3\omega}{\omega}$$

Similarly, the Fourier transform of the latter is $2\dfrac{\sin 2\omega}{\omega}$. Thus the total answer is

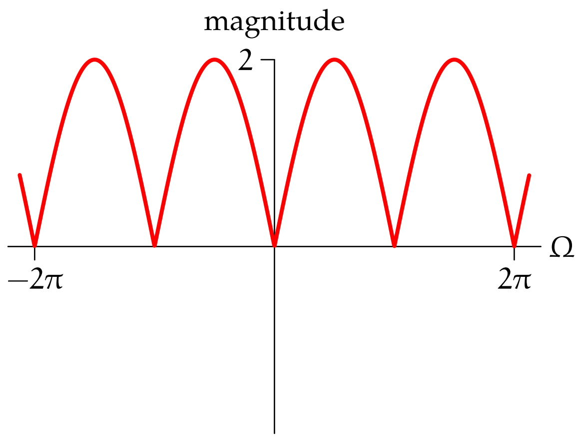
$$F(\omega) = 2\frac{\sin 3\omega}{\omega} - 2\frac{\sin 2\omega}{\omega}$$

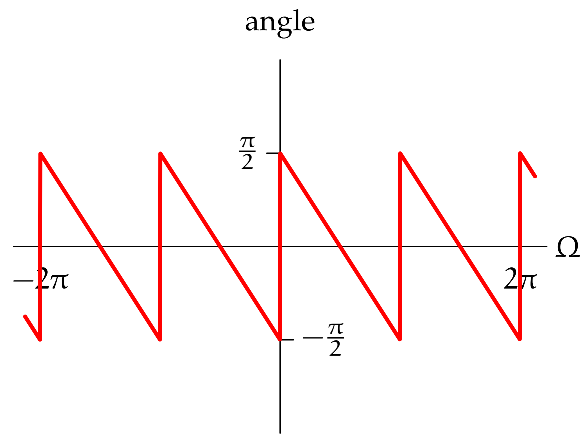
[1] Your expression should not include integrals or derivatives.

Part c. Determine the frequency response of a linear, time-invariant system with the following unit-sample response:

$$g[n] = \delta[n] - \delta[n-2]$$

Sketch the magnitude and angle of the frequency response on the axes below. Label the key points.





The frequency response is the Fourier transform of the unit-sample response:

$$G(\Omega) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\Omega n} = 1 - e^{-j2\Omega}$$

This frequency response can be simplified by realizing that $g[n]$ is a time-delayed version of $\delta[n+1]-\delta[n-1]$, which would correspond to a sinusoid in frequency. We can take advantage of this fact by factoring the time delay term out of the expression for $G(\Omega)$, as follows.

$$G(\Omega) = 1 - e^{-j2\Omega} = e^{-j\Omega}\left(e^{j\Omega} - e^{-j\Omega}\right) = j2\sin(\Omega)e^{-j\Omega}$$

The magnitude of the frequency response is the magnitude of $2\sin(\Omega)$.

The angle of the frequency response is determined by three factors. First, the j in $j2\sin(\Omega)e^{-j\Omega}$ contributes $+\pi/2$. Second, $\sin(\theta)$ is negative when $\pi < \theta < 2\pi$. Third, the phase term $e^{-j\Omega}$ contributes a linear term that decreases in proportion to Ω. When Ω is a small positive number, only the first factor contributes, so the angle of the frequency response is $\pi/2$. As Ω increases, the linear term decreases the angle of the frequency response until $\Omega = \pi/2$. At that point, the sign of $\sin(\Omega)$ is negative, so the phase jumps by π. Then the cycle repeats. Thus these three factors combine to give rise to the sawtooth function above.

Worksheet (intentionally blank)

2 Cascaded Systems (22 points)

Part a.

Let S_1 represent a linear, time-invariant (LTI) system whose unit-sample response $h_1[n]$ is a unit step:

$$x[n] \longrightarrow \boxed{S_1} \longrightarrow y_1[n]$$

$$h_1[n] = \begin{cases} 1 & \text{if } n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the response $y_1[n]$ of this system when the input $x[n]$ is the following geometric sequence:

$$x[n] = \begin{cases} (0.9)^n & \text{if } n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Enter the first 5 values of $y_1[n]$ in the boxes below.

$$y_1[0] = \boxed{1}$$

$$y_1[1] = \boxed{1 + 0.9}$$

$$y_1[2] = \boxed{1 + 0.9 + 0.9^2}$$

$$y_1[3] = \boxed{1 + 0.9 + 0.9^2 + 0.9^3}$$

$$y_1[4] = \boxed{1 + 0.9 + 0.9^2 + 0.9^3 + 0.9^4}$$

Enter a closed-form expression[2] for $y_1[n]$ in the box below.

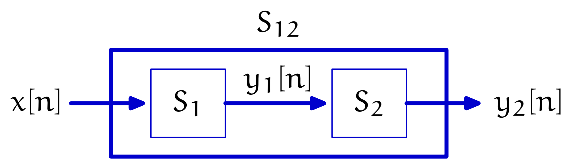
$$y_1[n] = \boxed{\frac{1-(0.9)^{n+1}}{1-0.9}\,u[n] \;=\; (10 - 9(0.9)^n)\,u[n]}$$

$$y_1[n] = \sum_{m=-\infty}^{\infty} h_1[m]\,u[n-m] = \sum_{m=-\infty}^{\infty} 0.9^m u[m]\,u[n-m] = \sum_{m=0}^{n} 0.9^m = \frac{1-0.9^{n+1}}{1-0.9} = 10 - 9\,(0.9)^n$$

[2] Your expression can contain additions, subtractions, multiplications, divisions, and exponentiations, but no other operators. Also, the number of operations required for each $y_1[n]$ should be bounded by a constant as $n \to \infty$.

Part b.

Let S_{12} represent the linear, time-invariant (LTI) system that results when two LTI systems (S_1 and S_2) are connected in cascade (so that the output of S_1 is the input to S_2) as shown in the figure to the right.



Determine the unit-sample response $h_{12}[n]$ of S_{12} when the unit-sample response $h_1[n]$ of system S_1 is a unit-step function $u[n]$ and the unit-sample response $h_2[n]$ of system S_2 is also a unit-step function $u[n]$:

$$h_1[n] = u[n]$$
$$h_2[n] = u[n]$$

Enter a closed-form expression[3] for $h_{12}[n]$ in the box below.

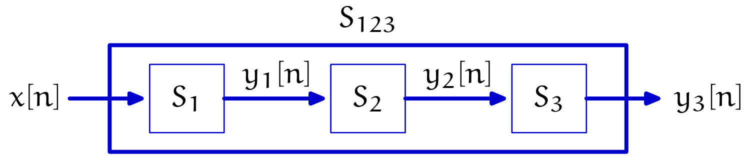
$$h_{12}[n] = \boxed{(n+1)\,u[n]}$$

$$\begin{aligned} h_{12}[n] &= (u * u)[n] \\ &= \sum_{m=-\infty}^{\infty} u[m]u[n-m] \\ &= \sum_{m=0}^{n} u[m]u[n-m] \\ &= (n+1)\,u[n] \end{aligned}$$

[3] Your expression can contain additions, subtractions, multiplications, divisions, and exponentiations, but no other operators. Also, the number of operations required for each $h_{12}[n]$ should be bounded by a constant as $n \to \infty$.

Part c.

Let S_{123} represent the LTI system that results when three LTI systems (S_1, S_2, and S_3) are connected in cascade, as shown in the figure to the right.



Determine the unit-sample response $h_{123}[n]$ of S_{123} when the unit-sample response $h_1[n]$ of system S_1 is a unit-step function $u[n]$, the unit-sample response $h_2[n]$ of system S_2 is a unit-step function $u[n]$, and the unit-sample response $h_3[n]$ of system S_3 is also a unit-step function $u[n]$:

$h_1[n] = u[n]$
$h_2[n] = u[n]$
$h_3[n] = u[n]$

Enter the first 5 values of $h_{123}[n]$ in the boxes below.

$h_{123}[0] =$	1
$h_{123}[1] =$	3
$h_{123}[2] =$	6
$h_{123}[3] =$	10
$h_{123}[4] =$	15

$$\begin{aligned}(u*u*u)[n] &= ((u*u)*u)[n] \\ &= \sum_{m=-\infty}^{\infty}(m+1)\,u[m]\,u[n-m] \\ &= \sum_{m=0}^{n}(m+1) \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

Worksheet (intentionally blank)

3 Pulsed Relations (27 points)

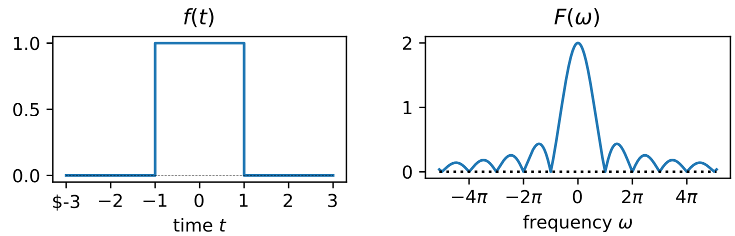
Let $f(t)$ represent the following signal

$$f(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $F(\omega)$ represent its Fourier transform

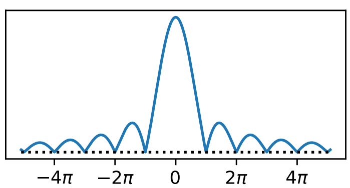
$$F(\omega) = 2\,\frac{\sin(\omega)}{\omega}$$

as shown on the right.

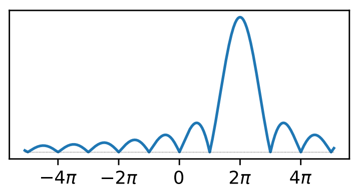


Each of the following plots shows the magnitude of the Fourier transform of a signal derived from $f(t)$. Note that the magnitude scales for these plots are not specified and may differ from one another.

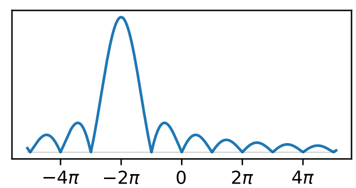
A



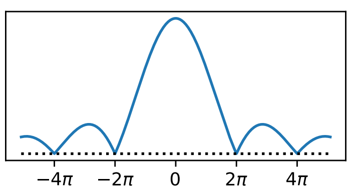
B



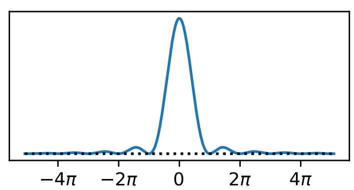
C



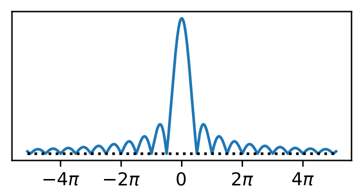
D



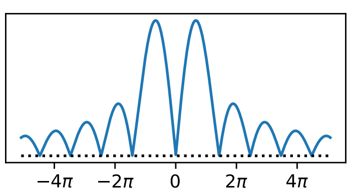
E



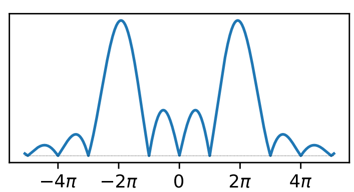
F



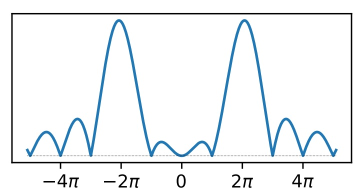
G



H



I



Identify which plot (A-I) shows the magnitude of each of the following derived signals.

$f(t/2)$: $\boxed{\text{F}}$

$(f * f)(t)$: $\boxed{\text{E}}$

$f(2t)$: $\boxed{\text{D}}$

$f(t-1)$: $\boxed{\text{A}}$

$tf(t)$: $\boxed{\text{G}}$

$f(t)\sin(2\pi t)$: $\boxed{\text{H}}$

$f(t)\cos(2\pi t)$: $\boxed{\text{I}}$

$f(t)e^{j2\pi t}$: $\boxed{\text{B}}$

$f(t)e^{-j2\pi t}$: $\boxed{\text{C}}$

Part 3a.

$$F_1(\omega) = \int_{-\infty}^{\infty} f(t/2)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega 2\tau}2d\tau = 2F(2\omega)$$

Answer: **F**

Part 3b.

$$F_2(\omega) = \int_{-\infty}^{\infty} (f * f)(t)e^{-j\omega t}dt = F^2(\omega)$$

Answer: **E**

Part 3c.

$$F_3(\omega) = \int_{-\infty}^{\infty} f(2t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau/2}d\tau/2 = F(\omega/2)/2$$

Answer: **D**

Part 3d.

$$F_4(\omega) = \int_{-\infty}^{\infty} f(t{-}1)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega(\tau+1)}dt = e^{-j\omega}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}dt = F(\omega)$$

Answer: **A**

Part 3e.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$\frac{dF(\omega)}{d\omega} = \int_{-\infty}^{\infty} -jtf(t)e^{-j\omega t}dt$$

$$F_5(\omega) = j\frac{dF(\omega)}{d\omega}$$

Answer: **G**

Part 3h.

$$F_8(\omega) = \int_{-\infty}^{\infty} f(t)e^{j2\pi t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{j2\pi t}e^{-j(\omega-2\pi)t}dt = F(\omega - 2\pi)$$

Answer: **B**

Part 3i.

$$F_9(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{j2\pi t}e^{-j(\omega+2\pi)t}dt = F(\omega + 2\pi)$$

Answer: **C**

Part 3g.

$$F_7(\omega) = \int_{-\infty}^{\infty} f(t)\cos(2\pi t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)\frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right)e^{-j\omega t}dt = \frac{1}{2}\Big(F(\omega{-}2\pi) + F(\omega{+}2\pi)\Big)$$

Answer: **I**

Part 3f.

$$F_6(\omega) = \int_{-\infty}^{\infty} f(t)\sin(2\pi t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)\frac{1}{j2}\left(e^{j2\pi t} - e^{-j2\pi t}\right)e^{-j\omega t}dt = \frac{1}{j2}\Big(F(\omega{-}2\pi) - F(\omega{+}2\pi)\Big)$$

Answer: **H**

Modulation by the cosine function produces two main lobes with the same peak amplitudes (one at $\omega{=}2\pi$ and one at $\omega{=}-2\pi$) as well as a series of sidelobes.

The distinquishing feature of panels H and I are that the side lobes have different amplitudes.

The cosine can be written as a sum of $e^{j\omega t}$ and $e^{-j\omega t}$. Both of these terms produce two sidelobes near $\omega{=}0$.

For the $e^{-j\omega t}$ term, the main lobe is at $\omega{=}2\pi$, the sidelobe to the right of $\omega{=}0$ is negative, and the sidelobe to the left of $\omega{=}0$ is positive.

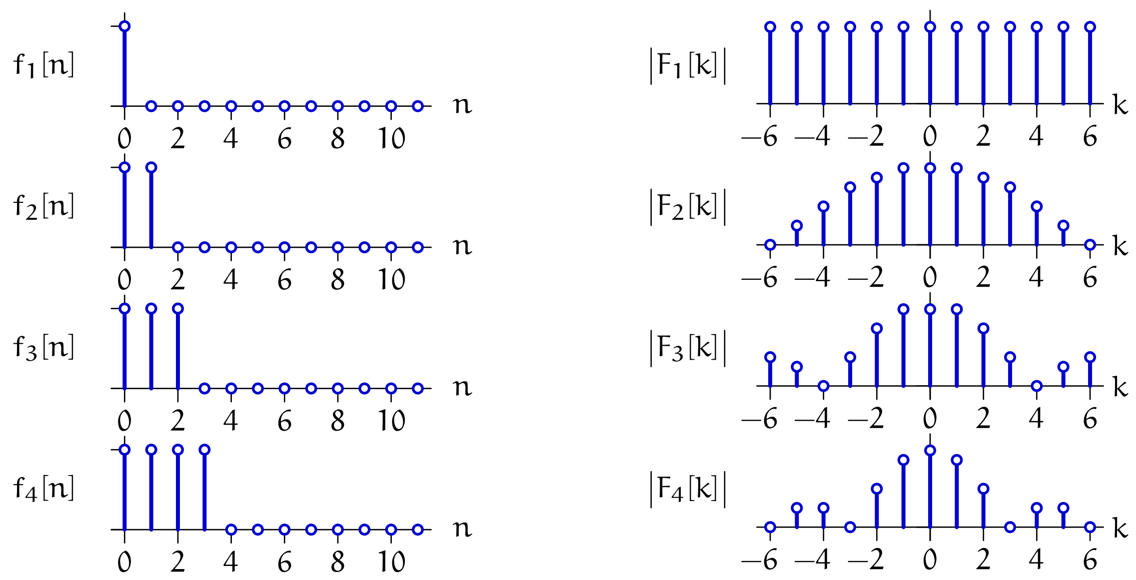
For the $e^{j\omega t}$ term, the main lobe is at $\omega{=}-2\pi$, the sidelobe to the left of $\omega{=}0$ is negative, and the sidelobe to the right of $\omega{=}0$ is positive.

Because the contributions to the sidelobes to the right of $\omega{=}0$ are opposite in sign, they tend to cancel. The same is true for the sidelobes to the left of ω.

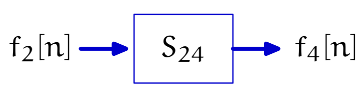
For the sine terms, the main lobes are opposite in sign, and the two sidelobes to the right of $\omega{=}0$ have the same sign.

4 Pulses In Pulses Out (27 points)

The first 12 samples of four **periodic** signals that are each periodic in N=12 are shown in the left column below, and the magnitudes of their Fourier series coefficients are shown in the right column.



Part a. Consider a system S_{24} that produces $f_4[n]$ as output when $f_2[n]$ is its input.

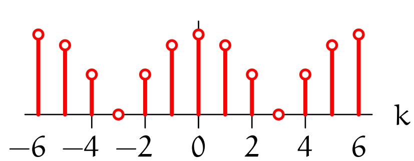


Determine the unit-sample response $h_{24}[n]$ and frequency response $H_{24}[k]$ of system S_{24} and enter expressions for $h_{24}[n]$ and $H_{24}[k]$ in the boxes below.

$h_{24}[n] =$ $\boxed{\delta[n] + \delta[n-2]}$

$H_{24}[k] =$ $\boxed{\frac{1}{12}\left(1 + e^{-j\pi k/3}\right)}$

Sketch the magnitude of the frequency response $H_{24}[k]$ on the axes below.



Part b. Consider 16 possible systems that are each defined by its input signal $f_i[n]$ and output signal $f_o[n]$:

$$f_i[n] \longrightarrow \boxed{S_{io}} \longrightarrow f_o[n]$$

Some of these systems could be linear and time invariant (LTI); others cannot.

Determine which of the 16 systems cannot possibly be LTI, and enter **X** in the corresponding box below.

i \ o	1	2	3	4
1				
2	X		X	
3	X	X		X
4	X	X	X	

If a system is LTI, then its frequency response $H_{ij}[k]$ is the ratio of the Fourier series coefficients of its output $F_j[k]$ divided by its input $F_i[k]$:

$$H_{ij}[k] = \frac{F_j[k]}{F_i[k]}$$

Notice that this ratio is infinite if $F_i[k] = 0$ when $F_j[k] \neq 0$. This relation follows from the filtering property of LTI systems: if $F_i[k] = 0$ then there is no possible value of $H_{ij}[k]$ for which $F_j[k]$ could be nonzero, and the system cannot be LTI.

Row 1: The Fourier series coefficients $F_1[k]$ are all nonzero, so we can define $H_{1j}[k] = F_1[k]/F_j[k]$ for all k. All of the entries in the first row correspond to possible LTI systems.

Row 3: The fourth harmonic of $f_3[n]$ is missing (i.e., its amplitude is zero). Therefore if $f_3[n]$ is the input signal, then none of the other signals other than $f_3[n]$ could result if the corresponding system is LTI. Therefore S_{31}, S_{32}, and S_{34} must be nonlinear. Of course S_{33} could be LTI (with $h33[k] = \delta[k]$).

Row 2: The sixth harmonic is missing from $f_2[n]$ but not from $f_1[n]$ or $f_3[n]$. therefore S_{21} and S_{23} must be nonlinear. The sixth harmonic is missing from both $f_2[n]$ and $f_4[n]$. Therefore the ratio of $F_4[n]$ to $F_2[n]$ is indeterminant. It follows that H_{24} could be any number and S_{24} could be LTI.

Worksheet (intentionally blank)

Worksheet (intentionally blank)

Worksheet (*intentionally blank*)