6.300: Signal Processing

Quiz #1 **Solutions** (Fall 2025)

Name:			
Kerberos:			
Recitation:	4-237 (with Mark)	4-370 (with Titus)	N/A

Instructions

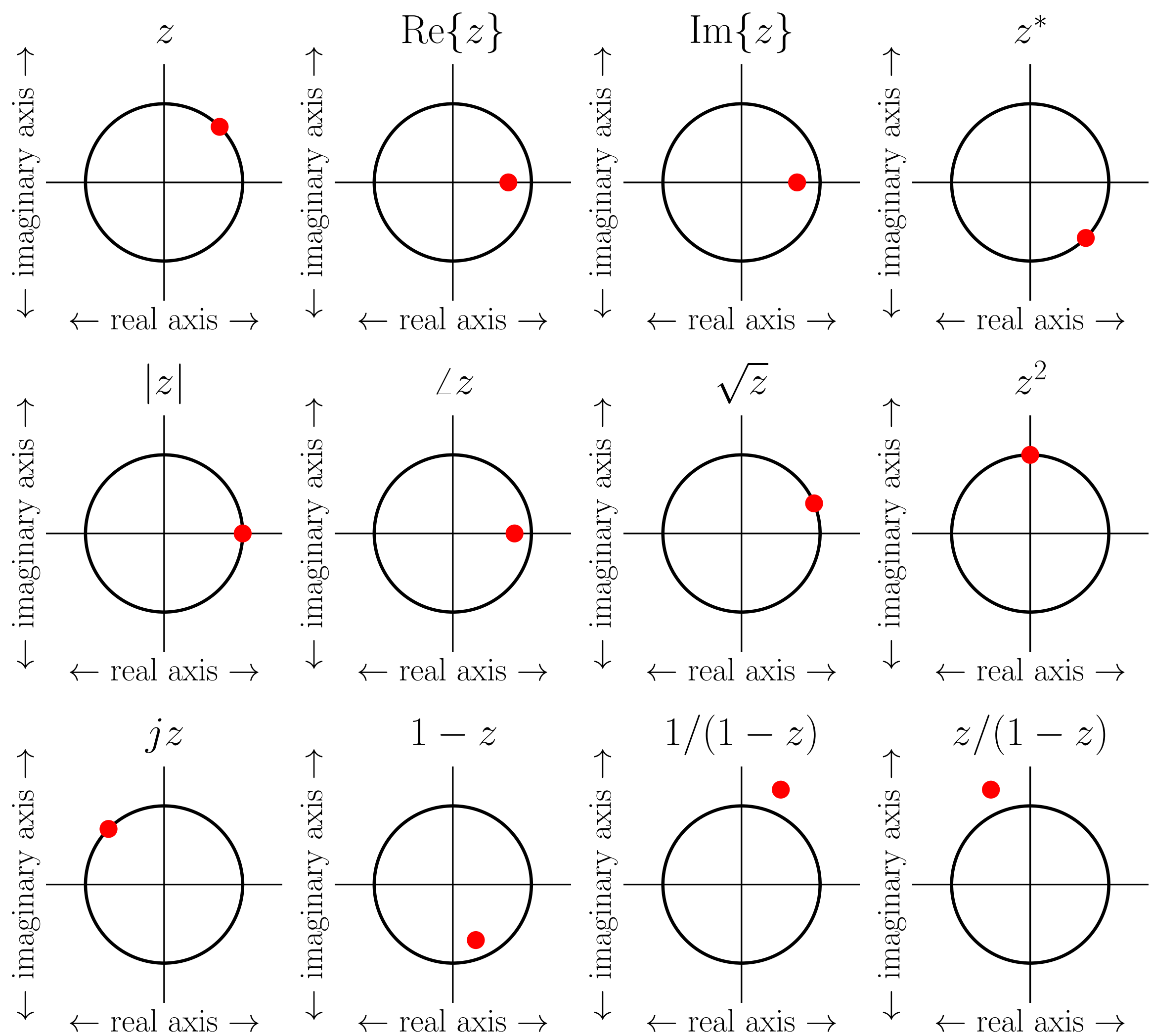
1. Do not write on the QR codes.
2. You may use one 8.5″ × 11″ double-sided sheet of handwritten notes.
   (No phones, laptops, tablets, music players, digital calculators, slide rules, abaci, etc.)
3. If you have a question, come to the front of the room to ask.
4. If you finish the quiz in the last 10 minutes, quietly remain at your desk and wait to hand in your quiz until we call time.
5. Write neatly. We cannot award credit for answers we cannot read.
6. You may not discuss the quiz with anyone other than course instructors until quiz scores have been released.
7. We will report any violations of academic integrity to the MIT Committee on Discipline.

1 Complex Numbers (25%)

Consider the complex number

$$z = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}.$$

On each plot below, clearly sketch the indicated transformation of z in the complex plane. The unit circle is shown for reference.



It will be helpful to express z in polar form.

$$z = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = e^{j\pi/4}$$

So, z has unit magnitude (i.e., $|z| = 1$) and angle $\angle z = \pi/4$ radians $= 45°$.

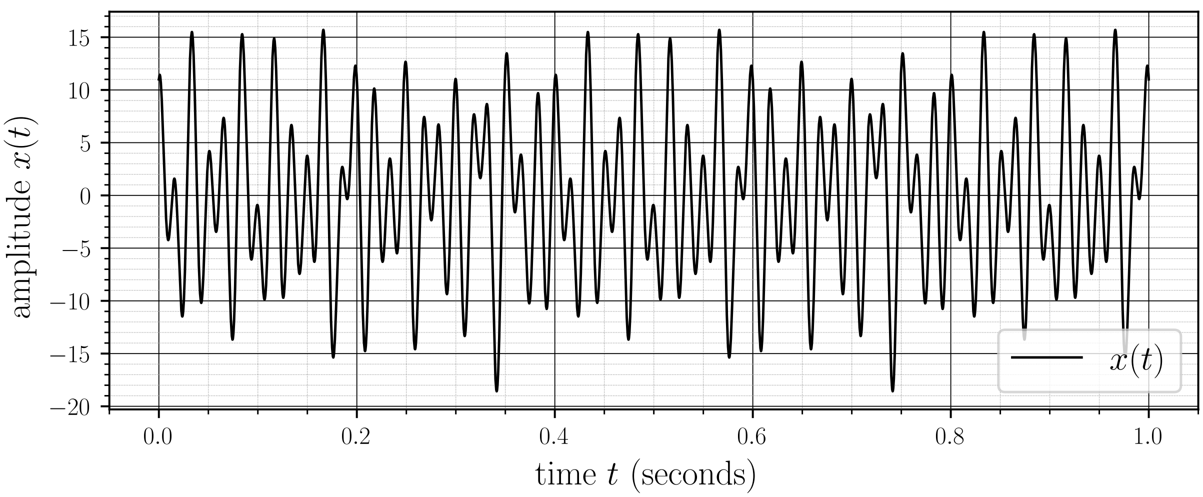
1. $z = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = e^{j\pi/4}$ — Unit magnitude. Angle: $\frac{\pi}{4}$ radians (45°)
2. $\text{Re}\{z\} = \frac{1}{\sqrt{2}} \approx 0.707$ — Must be purely real.
3. $\text{Im}\{z\} = \frac{1}{\sqrt{2}} \approx 0.707$ — Must be purely real.
4. $z^* = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = e^{-j\pi/4}$ — Unit magnitude. Angle: $-\frac{\pi}{4}$ radians or $-45°$.
5. $|z| = |e^{j\pi/4}| = 1$ — Must be purely real.
6. $\angle z = \angle(e^{j\pi/4}) = \pi/4 \approx 0.785$ — Must be purely real.
7. $\sqrt{z} = (e^{j\pi/4})^{1/2} = e^{j\pi/8}$ — Unit magnitude. Angle: $\frac{\pi}{8}$ radians or 22.5°.
8. $z^2 = (e^{j\pi/4})^2 = e^{j\pi/2} = j$ — Unit magnitude. Angle: $\frac{\pi}{2}$ radians or 90°.
9. $jz = e^{j\pi/2}e^{j\pi/4} = e^{j3\pi/4}$ — Unit magnitude. Angle: $\frac{3\pi}{4}$ radians or 135°.
10. $1 - z$ — Vector addition: $1 + (-z)$.
11. $1/(1 - z)$ — Magnitude: Reciprocal of $|1 - z|$. Angle: $-\angle(1 - z)$.
12. $z/(1 - z)$ — Rotate $1/(1 - z)$ counterclockwise by $\frac{\pi}{4}$ radians or 45°.

2 Continuous-Time Fourier Series (25%)

Consider the continuous-time (CT) signal

$$x(t) = x(t+T) = 3\cos(50\pi t) - 2\sin(60\pi t) + 6\sin(75\pi t) + 8\cos(120\pi t)$$

which is periodic with fundamental period T. Suppose that T is measured in seconds. A plot of $x(t)$ over $0 \le t \le 1$ is shown below. You should not necessarily assume that $T = 1$ second.



2.1

Determine the fundamental angular frequency ω_0 (radians per second) of $x(t)$. Specify an exact numerical answer, not an approximation based solely on the plot shown above. You may express your answer in terms of familiar constants such as π and e.

$$\boldsymbol{\omega_0} = 5\pi$$

(An explanation follows on the next page.)

The sum of periodic signals is periodic. Express $x(t)$ in the form

$$x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$$

where $x_i(t)$ has fundamental angular frequency ω_i.

The fundamental angular frequency of $x(t)$ will be

$$\begin{aligned} \omega_0 &= \text{GCF}\{\omega_1, \omega_2, \omega_3, \omega_4\} \\ &= \text{GCF}\{50\pi, 60\pi, 75\pi, 120\pi\} \\ &= 5\pi. \end{aligned}$$

To check that $\omega_0 = 5\pi$ is indeed the fundamental (i.e., smallest) angular frequency, we can express $x(t)$ in the form

$$x(t) = 3\cos(10\omega_0 t) - 2\sin(12\omega_0 t) + 6\sin(15\omega_0 t) + 8\cos(24\omega_0 t)$$

and check that the harmonic numbers (i.e., k's) are co-prime: $\text{GCF}\{10, 12, 15, 24\} = 1$.

Yet another way to verify this answer is to look over the plot of $x(t)$. From inspection, we see that the fundamental period is $T = 0.4$ seconds. Consequently, the fundamental angular frequency is $\omega_0 = 2\pi/0.4 = 5\pi$ radians per second.

2.2

Recall that $x(t)$ is given by

$$x(t) = x(t+T) = 3\cos(50\pi t) - 2\sin(60\pi t) + 6\sin(75\pi t) + 8\cos(120\pi t).$$

Determine the coefficients $\{c_k, d_k\}_{k=0}^{\infty}$ for a Fourier series expansion of the form

$$x(t) = \sum_{k=0}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=0}^{\infty} d_k \sin(k\omega_0 t)$$

where, as before, ω_0 denotes the fundamental angular frequency (radians per second) of $x(t)$.

1. Specify c_k and d_k for five values of k for which either $c_k \neq 0$ or $d_k \neq 0$.
2. If there are fewer than five such values of k, enter an **X** in the remaining boxes.
3. If there are more than five such values of k, enter any five.

Specify numerical values. You may express your answer in terms of familiar constants such as π and e. This problem can be answered with relatively few calculations.

k	c_k	d_k
10	3	0
12	0	-2
15	0	6
24	8	0
X	**X**	**X**

Solution: Earlier, we determined that $\omega_0 = 5\pi$. We can express $x(t)$ in the form

$$x(t) = 3\cos(10\omega_0 t) - 2\sin(12\omega_0 t) + 6\sin(15\omega_0 t) + 8\cos(24\omega_0 t)$$

and read off the coefficients $\{c_k, d_k\}_{k=0}^{\infty}$.

$$c_k = \begin{cases} 3 & k = 10 \\ 8 & k = 24 \\ 0 & \text{otherwise} \end{cases} \qquad d_k = \begin{cases} -2 & k = 12 \\ 6 & k = 15 \\ 0 & \text{otherwise} \end{cases}$$

Reading off the coefficients is far simpler than chugging through the analysis formulae.

2.3

Recall that $x(t)$ is given by

$$x(t) = x(t+T) = 3\cos(50\pi t) - 2\sin(60\pi t) + 6\sin(75\pi t) + 8\cos(120\pi t).$$

Determine the coefficients $\{a_k\}_{k=-\infty}^{\infty}$ for a Fourier series expansion of the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where, as before, ω_0 denotes the fundamental angular frequency (radians per second) of $x(t)$.

1. Specify a_k for five values of k for which $a_k \neq 0$ and $a_{-k} \neq 0$.
2. If there are fewer than five such values of k, enter an **X** in the remaining boxes.
3. If there are more than five such values of k, enter any five.

Specify numerical values. You may express your answer in terms of familiar constants such as j, π, and e. This problem can be answered with relatively few calculations.

$\boldsymbol{k}$	$\boldsymbol{a_k}$	$\boldsymbol{a_{-k}}$
10	$3/2$	$3/2$
12	j	$-j$
15	$-3j$	$3j$
24	4	4
X	**X**	**X**

Solution: Apply Euler's formula.

$$\cos(\theta) = \tfrac{1}{2}e^{j\theta} + \tfrac{1}{2}e^{-j\theta}$$
$$\sin(\theta) = \tfrac{1}{2j}e^{j\theta} - \tfrac{1}{2j}e^{-j\theta}$$

Expand the $\cos(\cdot)$ and $\sin(\cdot)$ terms into $e^{j(\cdot)}$ terms and read off the $\{a_k\}_{k=-\infty}^{\infty}$ coefficients.

$$3\cos(10\omega_0 t) = \tfrac{3}{2}e^{j10\omega_0 t} + \tfrac{3}{2}e^{-j10\omega_0 t} \implies a_{10} = a_{-10} = \tfrac{3}{2}$$
$$-2\sin(12\omega_0 t) = -\frac{1}{j}e^{j12\omega_0 t} + \frac{1}{j}e^{-j12\omega_0 t} \implies a_{12} = -\frac{1}{j} = j \text{ and } a_{-12} = \frac{1}{j} = -j$$
$$6\sin(15\omega_0 t) = \frac{3}{j}e^{j15\omega_0 t} - \frac{3}{j}e^{-j12\omega_0 t} \implies a_{15} = \frac{3}{j} = -3j \text{ and } a_{-15} = -\frac{3}{j} = 3j$$
$$8\cos(24\omega_0 t) = 4e^{j24\omega_0 t} + 4e^{-j24\omega_0 t} \implies a_{24} = a_{-24} = 4$$

Summarizing, we have

$$a_k = \begin{cases} \frac{3}{2} & k = \pm 10 \\ \pm j & k = \pm 12 \\ \mp 3j & k = \pm 15 \\ 4 & k = \pm 24 \\ 0 & \text{otherwise} \end{cases}$$

as the Fourier series coefficients. No trigonometry or calculus required here — just skillful use of Euler's formula. Observe that a_k is conjugate-symmetric (i.e., $a_k = a^*_{-k}$) as one would expect from the Fourier series coefficients of a real-valued signal.

2.4

Recall that $x(t)$ is given by

$$x(t) = x(t+T) = 3\cos(50\pi t) - 2\sin(60\pi t) + 6\sin(75\pi t) + 8\cos(120\pi t).$$

Determine the coefficients $\{m_k, \phi_k\}_{k=0}^{\infty}$ for a Fourier series expansion of the form

$$x(t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_0 t + \phi_k)$$

where, as before, ω_0 denotes the fundamental angular frequency (radians per second) of $x(t)$, and where $m_k \geq 0$ and $-\pi \leq \phi_k < \pi$ for all k.

1. Specify m_k and ϕ_k for five values of k for which either $m_k \neq 0$ or $\phi_k \neq 0$.
2. If there are fewer than five such values of k, enter an **X** in the remaining boxes.
3. If there are more than five such values of k, enter any five.

Specify numerical values. You may express your answer in terms of familiar constants such as π and e. This problem can be answered with relatively few calculations.

k	m_k	ϕ_k
10	3	0
12	2	$\pi/2$
15	6	$-\pi/2$
24	8	0
X	**X**	**X**

Solution: As before, we seek to match $x(t)$ to the form of the desired Fourier series expansion. We did not introduce analysis formulae to determine the $\{m_k, \phi_k\}_{k=0}^{\infty}$ terms in class — nor did we intend to — so pattern-matching is the most direct way to go about this problem. Still, there are multiple valid approaches to determine these $\{m_k, \phi_k\}_{k=0}^{\infty}$ terms. Perhaps the easiest way to approach this problem is to recognize that, for the $\cos(\cdot)$ terms, we have ought to have

$$m_k = |c_k|$$
$$\phi_k = \begin{cases} 0 & c_k > 0 \\ -\pi & c_k < 0 \\ \text{indeterminate} & c_k = 0 \end{cases}$$

while for the $\sin(\cdot)$ terms, we ought to have

$$m_k = |d_k|$$
$$\phi_k = \begin{cases} -\frac{\pi}{2} & d_k > 0 \\ \frac{\pi}{2} & d_k < 0 \\ \text{indeterminate} & d_k = 0 \end{cases}$$

simply from matching $m_k \cos(k\omega_0 t + \phi_k)$ to the form of $c_k \cos(k\omega_0 t)$ or $d_k \sin(k\omega_0 t)$.

Alternatively, one could note that

$$\begin{aligned} m_k \cos(k\omega_0 t + \phi_k) &= m_k \cos(\phi_k)\cos(k\omega_0 t) - m_k \sin(\phi_k)\sin(k\omega_0 t) \\ &= c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \end{aligned}$$

and determine that $c_k = m_k \cos(\phi_k)$ and $d_k = -m_k \sin(\phi_k)$. From here, one may solve for m_k and ϕ_k in terms of c_k and d_k as

$$m_k = \sqrt{c_k^2 + d_k^2}$$
$$\tan(\phi_k) = -\frac{d_k}{c_k}$$

using elementary trigonometry.

Yet another approach is to recognize that

$$\begin{aligned}
m_k\cos(k\omega_0 t+\phi_k) &= \text{Re}\{m_k e^{j(k\omega_0 t+\phi_k)}\} \\
&= \text{Re}\{m_k e^{j\phi_k} e^{jk\omega_0 t}\} \\
&= \text{Re}\{a_k e^{jk\omega_0 t}\} \\
&= \tfrac{1}{2}a_k e^{jk\omega_0 t} + \tfrac{1}{2}a_k^* e^{-jk\omega_0 t} \\
&= \tfrac{1}{2}a_k\Big(\cos(k\omega_0 t) + j\sin(k\omega_0 t)\Big) + \tfrac{1}{2}a_k^*\Big(\cos(k\omega_0 t) - j\sin(k\omega_0 t)\Big) \\
&= \left(\frac{a_k+a_k^*}{2}\right)\cos(k\omega_0 t) + j\left(\frac{a_k-a_k^*}{2}\right)\sin(k\omega_0 t) \\
&= \text{Re}\{a_k\}\cos(k\omega_0 t) - \text{Im}\{a_k\}\sin(k\omega_0 t)
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
m_k &= \sqrt{\text{Re}\{a_k\}^2 + \text{Im}\{a_k\}^2} \\
\tan(\phi_k) &= -\frac{\text{Im}\{a_k\}}{\text{Re}\{a_k\}}
\end{aligned}$$

using properties of complex numbers.

All three approaches ought to yield the same answer.

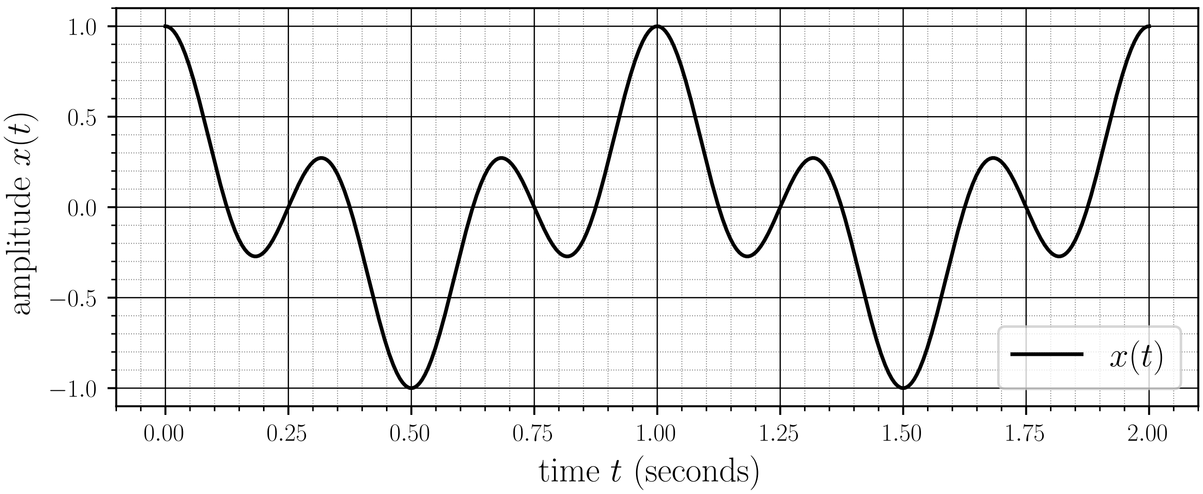
$$m_k = \begin{cases} 3 & k=10 \\ 2 & k=12 \\ 6 & k=15 \\ 8 & k=24 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_k = \begin{cases} \frac{\pi}{2} & k=12 \\ -\frac{\pi}{2} & k=15 \\ 0 & \text{otherwise} \end{cases}$$

The point of this problem was not to, say, derive analysis formulae for the m_k and ϕ_k terms; rather, the point was to emphasize that the very same pattern-matching tricks that work for our fairly familiar cosine-sine $\{c_k, d_k\}_{k=0}^{\infty}$ and complex-exponential $\{a_k\}_{k=-\infty}^{\infty}$ forms of Fourier series also apply to this less familiar magnitude-phase $\{m_k, \phi_k\}_{k=0}^{\infty}$ form of a Fourier series.

3 Sampling (25%)

A periodic continuous-time (CT) signal, $x(t)$, is shown in the plot below. While $x(t)$ is plotted over $0 \le t \le 2$, you should not necessarily assume that the fundamental period is $T = 2$.



3.1

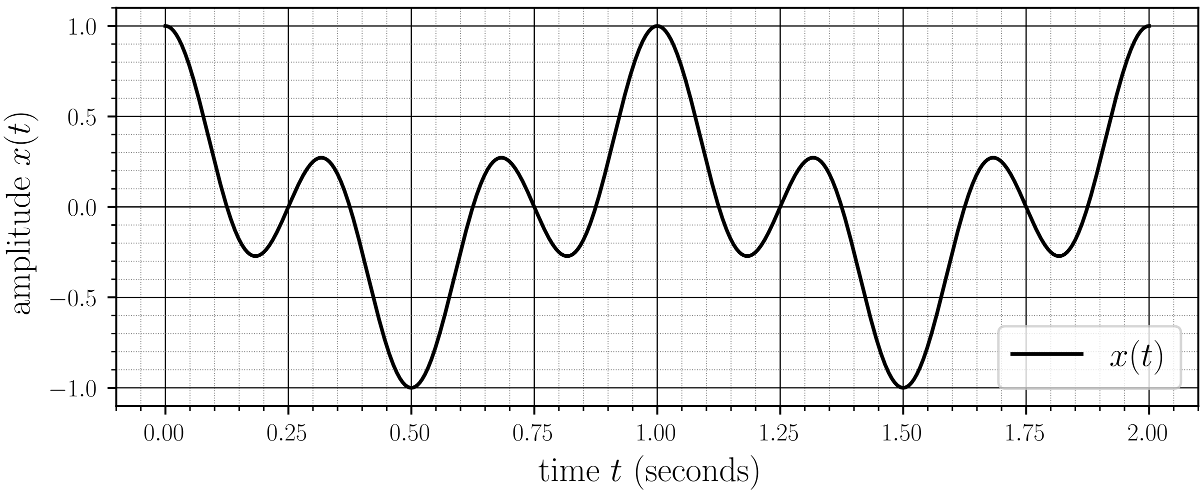
Based on the plot of $x(t)$ shown above, answer the following questions. Specify numerical values. You may express these numerical values in terms of constants such as π and e.

1. Determine the fundamental period T (seconds) of $x(t)$.
2. Determine the fundamental cyclical frequency f_0 (cycles per second, i.e., hertz) of $x(t)$.
3. Determine the fundamental angular frequency ω_0 (radians per second) of $x(t)$.

T	f_0	ω_0
1 second	**1 cycle per second**	**2π radians per second**

3.2

For convenience, the plot of $x(t)$ is reproduced below.

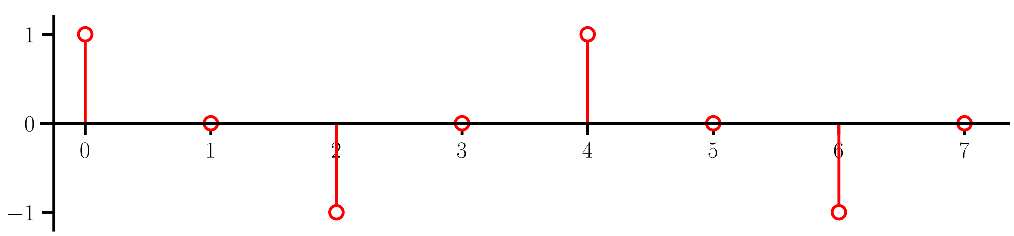


Suppose we sample $x(t)$ every $\Delta_1 = 0.25$ seconds — or, equivalently, with a sampling rate of $f_1 = 4$ cycles per second (i.e., hertz) — to produce a discrete-time (DT) signal $x_1[n]$.

1. Specify N_1, the fundamental period (in samples) of $x_1[n]$. In addition, specify Ω_1 (where $0 \le \Omega_1 \le \pi$), the fundamental angular frequency (in radians per sample) of $x_1[n]$. If $x_1[n]$ does not have a fundamental period, write **None**.
2. Sketch $x_1[n]$ over $0 \le n \le 7$.

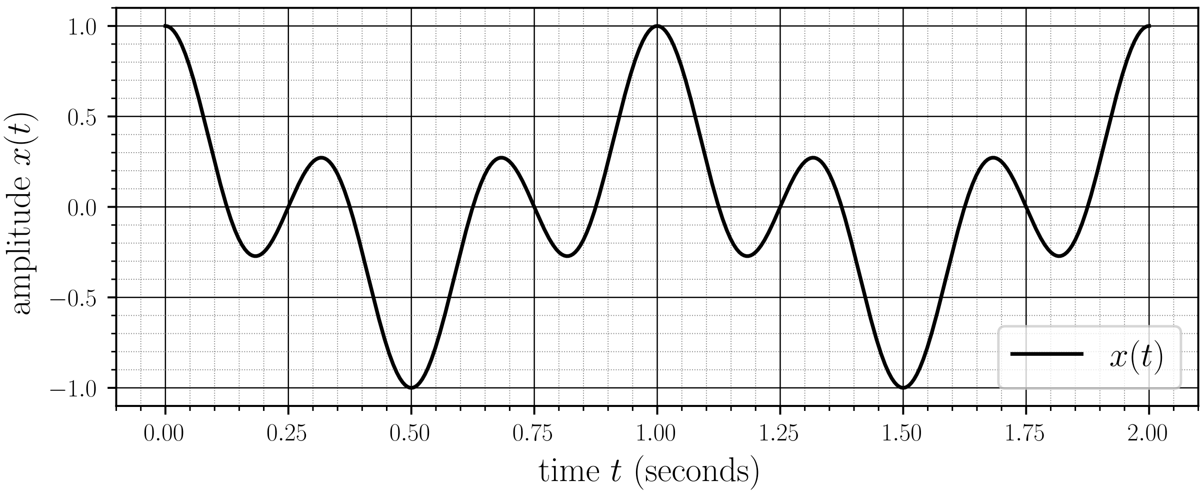
You may express numerical values in terms of constants such as π and e.

$\boldsymbol{N_1}$	$\boldsymbol{\Omega_1}$
4 samples	$\pi/2$ **radians per sample**



3.3

For convenience, the plot of $x(t)$ is reproduced below.



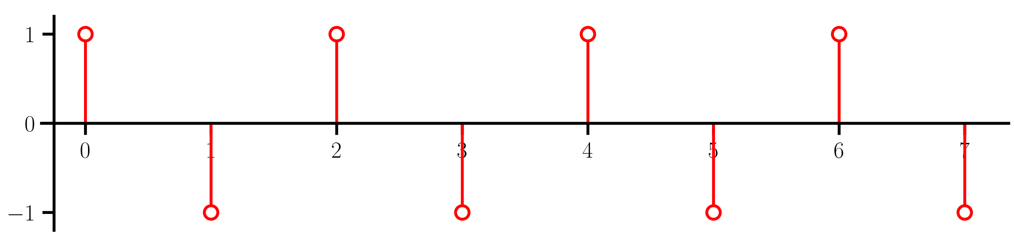
Suppose we sample $x(t)$ every $\Delta_2 = 0.5$ seconds — or, equivalently, with a sampling rate of $f_2 = 2$ cycles per second (i.e., hertz) — to produce a discrete-time (DT) signal $x_2[n]$.

1. Specify N_2, the fundamental period (in samples) of $x_2[n]$. In addition, specify Ω_2 (where $0 \le \Omega_2 \le \pi$), the fundamental angular frequency (in radians per sample) of $x_2[n]$. If $x_2[n]$ does not have a fundamental period, write **None**.

2. Sketch $x_2[n]$ over $0 \le n \le 7$.

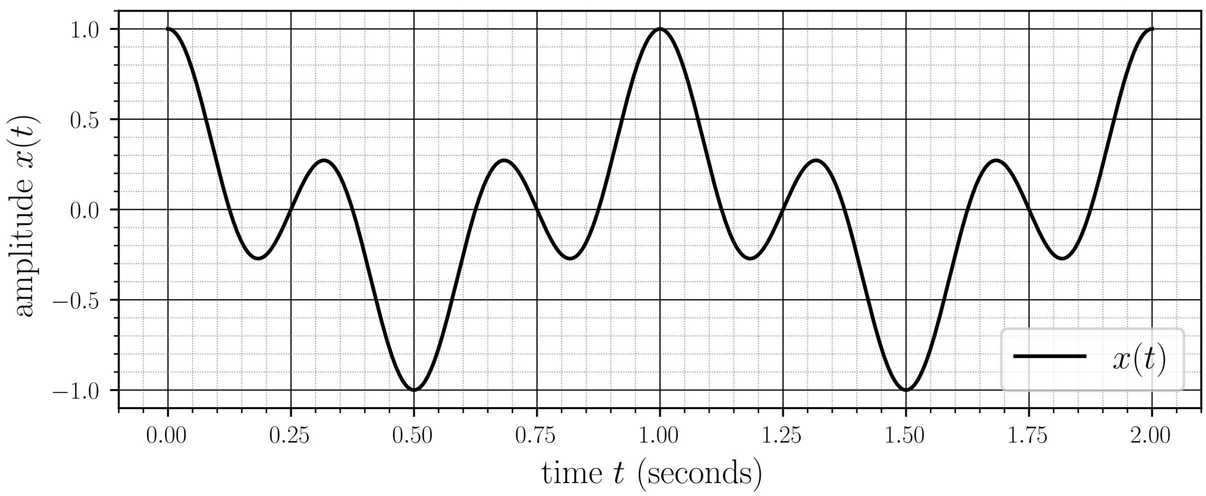
You may express numerical values in terms of constants such as π and e.

$\boldsymbol{N_2}$	$\boldsymbol{\Omega_2}$
2 samples	π **radians per sample**



3.4

For convenience, the plot of $x(t)$ is reproduced below.

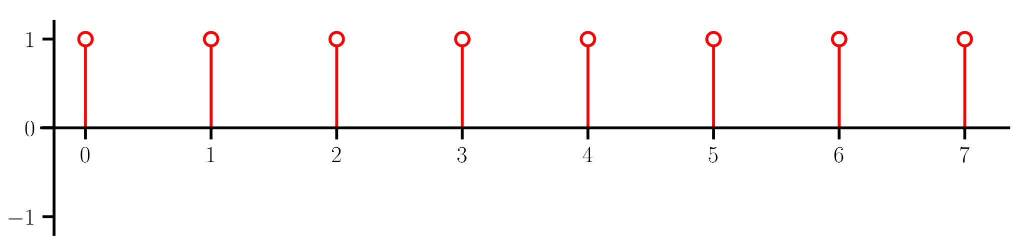


Suppose we sample $x(t)$ every $\Delta_3 = 1$ second — or, equivalently, with a sampling rate of $f_3 = 1$ cycle per second (i.e., hertz) — to produce a discrete-time (DT) signal $x_3[n]$.

1. Specify N_3, the fundamental period (in samples) of $x_3[n]$. In addition, specify Ω_3 (where $0 \le \Omega_3 \le \pi$), the fundamental angular frequency (in radians per sample) of $x_3[n]$. If $x_3[n]$ does not have a fundamental period, write **None**.
2. Sketch $x_3[n]$ over $0 \le n \le 7$.

You may express numerical values in terms of constants such as π and e.

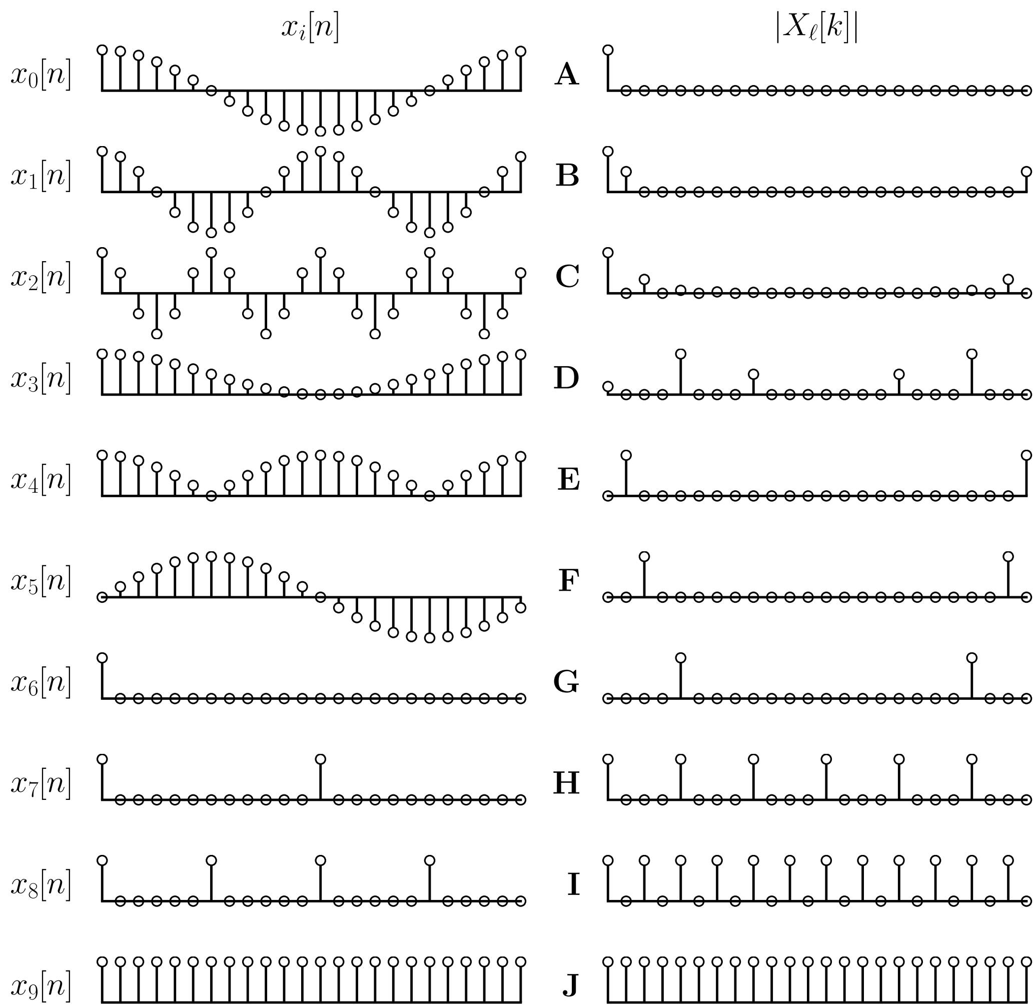
$\boldsymbol{N_3}$	$\boldsymbol{\Omega_3}$
1 or None	**0 radians per sample**



4 Discrete-Time Fourier Series (25%)

Match each periodic discrete-time sequence $x_i[n]$ shown on the left to the corresponding plot on the right (**A** – **J**) that shows the magnitude of its Fourier series coefficients $|X_i[k]|$ computed with a period of $N = 24$ samples. Multiple $x_i[n]$'s may match to the same plot on the right.

Left	$x_0[n]$	$x_1[n]$	$x_2[n]$	$x_3[n]$	$x_4[n]$	$x_5[n]$	$x_6[n]$	$x_7[n]$	$x_8[n]$	$x_9[n]$
Right	**E**	**F**	**G**	**B**	**C**	**E**	**J**	**I**	**H**	**A**



Let $\Omega_0 = 2\pi/24$ denote the fundamental angular frequency in radians per sample.

$$x_0[n] \propto \cos(\Omega_0 n) = \tfrac{1}{2}e^{j\Omega_0 n} + \tfrac{1}{2}e^{-j\Omega_0 n}$$
$$X_0[k] \propto \tfrac{1}{2}\delta[k-1] + \tfrac{1}{2}\delta[k-1] \implies \textbf{E}$$
$$x_1[n] \propto \cos(2\Omega_0 n) = \tfrac{1}{2}e^{j(2)\Omega_0 n} + \tfrac{1}{2}e^{-j(2)\Omega_0 n}$$
$$X_1[k] \propto \tfrac{1}{2}\delta[k-2] + \tfrac{1}{2}\delta[k+2] \implies \textbf{F}$$
$$x_2[n] \propto \cos(4\Omega_0 n) = \tfrac{1}{2}e^{j(4)\Omega_0 n} + \tfrac{1}{2}e^{-j(4)\Omega_0 n}$$
$$X_2[k] \propto \tfrac{1}{2}\delta[k-4] + \tfrac{1}{2}\delta[k+4] \implies \textbf{G}$$
$$x_3[n] \propto 1 + \cos(\Omega_0 n) = 1 + e^{j\Omega_0 n} + e^{-j\Omega_0 n}$$
$$X_3[k] \propto \delta[k] + \tfrac{1}{2}\delta[k-1] + \tfrac{1}{2}\delta[k+1] \implies \textbf{B}$$
$$x_4[n] \propto |\cos(\Omega_0 n)| \rightsquigarrow \text{non-zero DC term, completes 2 cycles (so } k = \pm 2) \implies \textbf{C}$$
$$x_5[n] \propto x_0[n-6]$$
$$X_5[k] \propto e^{-jk\Omega_0(6)}X_0[k] \implies |X_5[k]| = |X_0[k]| \implies \textbf{E}$$

If $x[n]$ completes M periods in N samples, then $X[k]$ completes a period in M samples.

$$x_6[n] \propto \delta[n]$$
$$X_6[k] \propto 1 \text{ (for all } k) \implies \textbf{J}$$
$$x_7[n] \propto \delta[n] + \delta[n-12]$$
$$X_7[k] \propto \delta[k] + \delta[k-2] + \delta[k+2] + \cdots + \delta[k-10] + \delta[k+10] + \delta[k-12] \implies \textbf{I}$$
$$x_8[n] \propto \delta[n] + \delta[n-6] + \delta[n-12] + \delta[n-18]$$
$$X_8[k] \propto \delta[k] + \delta[k-4] + \delta[k+4] + \delta[k-8] + \delta[k+8] + \delta[k-12] \implies \textbf{H}$$
$$x_9[n] \propto 1 \text{ (for all } n)$$
$$X_9[k] \propto \delta[k] \implies \textbf{A}$$