6.300: Signal Processing

Quiz #2 Review (Lecture)

- Quiz #2 is in Walker (50-340) on Tuesday, 11/04 at 2:00 p.m.
- Bring two 8.5″ × 11.0″ pages (four sides) of handwritten notes.

November 2025

Calculus Analogy

You wouldn't want to walk into a calculus quiz without knowing

$$\frac{d\sin(\theta)}{d\theta} = \cos(\theta)$$

by heart, right? Going back to the derivation wastes precious time!

$$\lim_{\phi \to 0} \frac{\sin(\theta + \phi) - \sin(\theta)}{\phi}$$

$$\lim_{\phi \to 0} \frac{\sin(\theta)\cos(\phi) + \sin(\phi)\cos(\theta) - \sin(\theta)}{\phi}$$

$$\lim_{\phi \to 0} \underbrace{\left[\frac{\sin(\phi)}{\phi}\right]}_{1 \text{ as } \phi \to 0} \cos(\theta) + \lim_{\phi \to 0} \underbrace{\left[\frac{\cos(\phi) - 1}{\phi}\right]}_{0 \text{ as } \phi \to 0} \sin(\theta)$$

Calculus Analogy

To do well on a calculus quiz,
you probably need to know at least a few things by heart.

common functions and their derivatives

$$\frac{d(t^n)}{dt} = nt^{n-1} \qquad \frac{d\left(e^{\lambda t}\right)}{dt} = \lambda e^{\lambda t} \qquad \frac{d\sin(t)}{dt} = \cos(t)$$

differentiation rules

$$\frac{d\left[c_1 f(t) + c_2 g(t)\right]}{dt} = c_1 \frac{df}{dt} + c_2 \frac{dg}{dt} \qquad \frac{dg\left(f(t)\right)}{dt} = \frac{dg}{df} \cdot \frac{df}{dt}$$

Calculus Analogy

To do well on a signal processing quiz,
you probably need to know at least a few things by heart.

common signals and their Fourier transforms

$$\delta[n-n_0] \iff e^{-j\Omega n_0} \qquad e^{j\Omega_0 n} \iff 2\pi\delta\big((\Omega-\Omega_0) \bmod 2\pi\big)$$

Fourier properties

$$\begin{aligned} c_1x_1[n]+c_2x_2[n] &\iff c_1X_1(\Omega)+c_2X_2(\Omega) \\ x[n-n_0] &\iff e^{-j\Omega n_0}X(\Omega) \\ e^{j\Omega_0 n}x[n] &\iff X(\Omega-\Omega_0) \end{aligned}$$

Fourier Transforms (CT)

$$\begin{aligned}
\textbf{time domain} \quad &\Longleftrightarrow \quad \textbf{frequency domain} \\
\delta(t) \quad &\Longleftrightarrow \quad 1 \\
\delta(t - t_0) \quad &\Longleftrightarrow \quad e^{-j\omega t_0} \\
1 \quad &\Longleftrightarrow \quad 2\pi\delta(\omega) \\
e^{j\omega_0 t} \quad &\Longleftrightarrow \quad 2\pi\delta(\omega - \omega_0)
\end{aligned}$$

Duality: Notice the common trend in transform pairs.

$$\begin{aligned}
x(t) \quad &\Longleftrightarrow \quad X(\omega) \\
X(t) \quad &\Longleftrightarrow \quad 2\pi x(-\omega)
\end{aligned}$$

Fourier Transforms (DT)

All discrete-time Fourier transforms are 2π-periodic.

$$\begin{aligned}
\textbf{time domain} &\iff \textbf{frequency domain} \\
\delta[n] &\iff 1 \\
\delta[n-n_0] &\iff e^{-j\Omega n_0} \\
1 &\iff 2\pi\delta(\Omega \bmod 2\pi) \\
e^{j\Omega_0 n} &\iff 2\pi\delta\big((\Omega-\Omega_0) \bmod 2\pi\big)
\end{aligned}$$

Duality: Not so easy with discrete-time Fourier transforms.
$x[n]$ is discrete in time, but $X(\Omega)$ is continuous in frequency!

Some Fourier Properties

$$
\begin{array}{rcl}
\textbf{time domain} & \Longleftrightarrow & \textbf{frequency domain} \\
c_1 x_1[n] + c_2 x_2[n] & \Longleftrightarrow & c_1 X_1(\Omega) + c_2 X_2(\Omega) \\
(x_1 * x_2)[n] & \Longleftrightarrow & X_1(\Omega) X_2(\Omega) \\
x_1[n] x_2[n] & \Longleftrightarrow & \frac{1}{2\pi}\big(X_1 * X_2\big)(\Omega) \\
x[-n] & \Longleftrightarrow & X(-\Omega) \\
x[nM] & \Longleftrightarrow & X(\frac{\Omega}{M}) \\
x[n - n_0] & \Longleftrightarrow & e^{-j\Omega n_0} X(\Omega) \\
e^{j\Omega_0 n} x[n] & \Longleftrightarrow & X(\Omega - \Omega_0) \\
n\,x[n] & \Longleftrightarrow & j\frac{d}{d\Omega} X(\Omega) \\
\frac{d}{dt} x(t) & \Longleftrightarrow & j\omega X(\omega)
\end{array}
$$

Linearity and Time-Invariance

Linearity

$$x_1[n] \rightarrow \boxed{\text{linear system}} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{\text{linear system}} \rightarrow y_2[n]$$

$$c_1x_1[n] + c_2x_2[n] \rightarrow \boxed{\text{linear system}} \rightarrow c_1y_1[n] + c_2y_2[n]$$

Time-Invariance

$$x[n] \rightarrow \boxed{\text{time-invariant system}} \rightarrow y[n]$$

$$x[n - n_0] \rightarrow \boxed{\text{time-invariant system}} \rightarrow y[n - n_0]$$

Check Yourself

Are the following systems **linear** and **time-invariant**?
(Recall: Together, additivity and homogeneity imply linearity.)

$$y[n] = \tfrac{1}{3}x[n-1] + \tfrac{1}{3}x[n] + \tfrac{1}{3}x[n+1]$$

$$y[n] = Mx[n] + B \quad \text{for constants } M \text{ and } B$$

$$y(t) = \int_0^t x(\tau)d\tau$$

Check Yourself

$$y[n] = \frac{1}{3}x[n-1] + \frac{1}{3}x[n] + \frac{1}{3}x[n+1]$$

Linear? By inspection, yes!

$$x[n] = c_1x_1[n] + c_2x_2[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = c_1y_1[n] + c_2y_2[n]$$

Time-invariant? By inspection, yes!

$$x[n-n_0] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n-n_0]$$

Check Yourself

$$y[n] = Mx[n] + B \quad \text{for constants } M \text{ and } B$$

Linear? If $M \neq 0$ and $B = 0$, yes.

$$x[n] = c_1x_1[n] + c_2x_2[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = c_1y_1[n] + c_2y_2[n]$$

Time-invariant? Yes.

$$x[n-n_0] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n-n_0]$$

Check Yourself

$$y(t) = \int_0^t x(\tau) d\tau$$

Linear? Yes. Integration is a linear operation.

$$\int_0^t c_1 x_1(\tau) + c_2 x_2(\tau) d\tau = c_1 \int_0^t x_1(\tau) d\tau + c_2 \int_0^t x_2(\tau) d\tau$$

Time-invariant? No.

$$y(t-t_0) = \int_0^{t-t_0} x(\tau) d\tau \neq \int_{-t_0}^{t-t_0} x(\tau') d\tau' \text{ for } \tau' = \tau - t_0$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

LTI Systems

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- frequency response

Difference Equations and Differential Equations

Impose time-domain constraints on the input and output.

$$y[n] = \tfrac{1}{2}y[n-1] + x[n]$$

$$\frac{dy(t)}{dt} = x(t) - \tfrac{1}{2}y(t)$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

Unit-Sample Response

Characterize a system by a single time-domain signal.

$$\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$$

$$x[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow \sum_k h[k]x[n-k]$$

Convolution

Convolving $x[n]$ with $\delta[n-n_0]$ time-shifts $x[n]$.

$$h[n] = \delta[n-n_0] \implies (x*h)[n] = x[n-n_0]$$

Convolving $x[n]$ with a sum of scaled and time-shifted δ signals produces a sum of scaled and time-shifted $x[n]$.

$$h[n] = \sum_k h[k]\delta[n-k]$$

$$(x*h)[n] = \sum_k \underbrace{h[k]}_{\text{scale}} \underbrace{x[n-k]}_{\text{time-shift}}$$

Convolution

$(x * h)[n]$ is a superposition of scaled and time-shifted $x[n]$.

$$\vdots$$

$$\begin{aligned}(x * h)[n] = {} & h[0]\, x[n] + \\ & h[1]\, x[n-1] + \\ & h[2]\, x[n-2] + \end{aligned}$$

$$\vdots$$

Convolution is Commutative

$(x * h)[n]$ is a superposition of scaled and time-shifted $h[n]$.

$$
\begin{aligned}
& \vdots \\
(x * h)[n] = {} & x[0]\,h[n] + \\
& x[1]\,h[n-1] + \\
& x[2]\,h[n-2] + \\
& \vdots
\end{aligned}
$$

Convolution

n	=	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{2}}$	$\boxed{\mathbf{3}}$	$\boxed{\mathbf{4}}$	$\boxed{\mathbf{5}}$	$\boxed{\mathbf{6}}$	$\boxed{\mathbf{7}}$
$x[n]$	=	1	1	1	1	0	0	0	0
$h[n]$	=	1	2	3	0	0	0	0	0

n	=	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{2}}$	$\boxed{\mathbf{3}}$	$\boxed{\mathbf{4}}$	$\boxed{\mathbf{5}}$	$\boxed{\mathbf{6}}$	$\boxed{\mathbf{7}}$
$h[0]\,x[n-0]$	=	1	1	1	1	0	0	0	0
$h[1]\,x[n-1]$	=	0	2	2	2	2	0	0	0
$h[2]\,x[n-2]$	=	0	0	3	3	3	3	0	0
$(x * h)[n]$	=	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{3}}$	$\boxed{\mathbf{6}}$	$\boxed{\mathbf{6}}$	$\boxed{\mathbf{5}}$	$\boxed{\mathbf{3}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$

Check Yourself

Consider an LTI system with unit-sample response $h[n]$.

$$\boxed{h[n] = \delta[n] + \delta[n-1] + \delta[n-2]}$$

Suppose that the input to the system is $x[n]$.

$$\boxed{x[n] = \cos\left(\frac{2\pi}{3}n\right)}$$

Determine a closed-form expression for the output $y[n]$.

Check Yourself

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$y[n] = (x * h)[n] = x[n] + x[n-1] + x[n-2]$$

$x[n] = \cos\left(\frac{2\pi}{3}n\right)$ is periodic in $N = 3$ samples.

$$x[0] = 1 \qquad x[1] = -\tfrac{1}{2} \qquad x[2] = -\tfrac{1}{2}$$

$$x[n] + x[n-1] + x[n-2] = 0 \text{ for all } n$$

Check Yourself

Alternatively, think of the frequency response.

$$\begin{aligned} H(\Omega) &= 1 + e^{-j\Omega} + e^{-j2\Omega} \\ &= e^{j\Omega}\left(e^{-j\Omega} + 1 + e^{-j\Omega}\right) \\ &= e^{j\Omega}\left(1 + 2\cos(\Omega)\right) \end{aligned}$$

$$x[n] = \cos\left(\tfrac{2\pi}{3}\right) \iff X(\Omega) = \tfrac{1}{2}e^{j\frac{2\pi}{3}n} + \tfrac{1}{2}e^{-j\frac{2\pi}{3}n}$$

$$H\left(\tfrac{2\pi}{3}\right) = 0 \implies Y(\Omega) = 0 \iff y[n] = 0$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
- unit-sample response (DT) or impulse response (CT)
- frequency response

Frequency Response

Complex exponentials are eigenfunctions of LTI systems!

Characterize a system by how it shapes a signal's spectrum.

$$e^{j\Omega n} \rightarrow \boxed{\textbf{LTI}} \rightarrow H(\Omega)e^{j\Omega n}$$

$$X(\Omega) \rightarrow \boxed{\textbf{LTI}} \rightarrow H(\Omega)X(\Omega)$$

Eigenfunctions (if you're interested)

An eigenvalue-eigenvector pair $(\lambda, \boldsymbol{v})$ satisfy the eigenequation.

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

Likewise, eigenvalue-eigenfunction pairs satisfy eigenequations.

$$\frac{d}{dt}\{e^{\lambda t}\} = \lambda e^{\lambda t} \qquad \underbrace{\mathcal{R}\{\lambda^n\}}_{\text{right shift}} = \lambda^{-1}\lambda^n$$

Exponential functions $e^{\lambda t}$ are eigenfunctions of the d/dt operator.
Set $\lambda = j\omega \implies$ Eigenfunctions are CTFT basis functions!

Geometric sequences λ^n are eigenfunctions of the $\mathcal{R}$ operator.
Set $\lambda = e^{j\Omega} \implies$ Eigenfunctions are DTFT basis functions!

Eigenfunctions (if you're interested)

Let $P(\boldsymbol{A})$ denote a polynomial in $\boldsymbol{A}$. $P(\boldsymbol{A})$ has the same eigenvectors $\boldsymbol{v}_k$, but the corresponding eigenvalues are $P(\lambda_k)$.

$$P(\boldsymbol{A})\boldsymbol{v} = P(\lambda)\boldsymbol{v}$$

Likewise …

$$P\left(\frac{d}{dt}\right)e^{\lambda t} = P(\lambda)e^{\lambda t}$$

$$P(\mathcal{R})\lambda^n = P(\lambda^{-1})\lambda^n$$

Expressing a signal in a basis of eigenfunctions facilitates analysis.

(e.g., The homogeneous solution to a linear differential equation with constant coefficients is a linear combination of eigenfunctions that lie in the null space of the polynomial differential operator.)

Eigenfunctions (if you're interested)

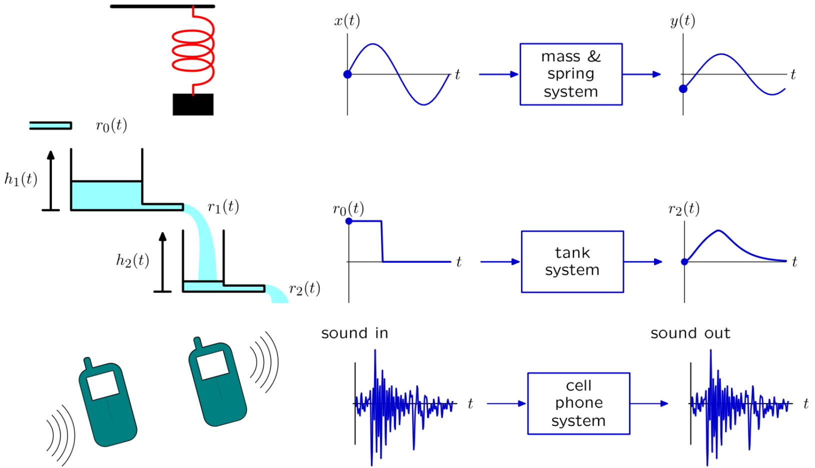
How do we interpret $\boldsymbol{Ax} = \boldsymbol{b}$?

- express $\boldsymbol{x} = \sum_k c_k \boldsymbol{v}_k$ in basis spanned by eigenvectors of $\boldsymbol{A}$
- scale each eigenvector $\boldsymbol{v}_k$ by the eigenvalue λ_k
- $\boldsymbol{b} = \sum_k c_k \lambda_k \boldsymbol{v}_k$

How do we interpret $x[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n]$?

- express $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$ in eigenfunction basis
- scale each eigenfunction $e^{j\Omega n}$ by the eigenvalue $H(\Omega)$
- $y[n] = \frac{1}{2\pi} \int_{2\pi} Y(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$

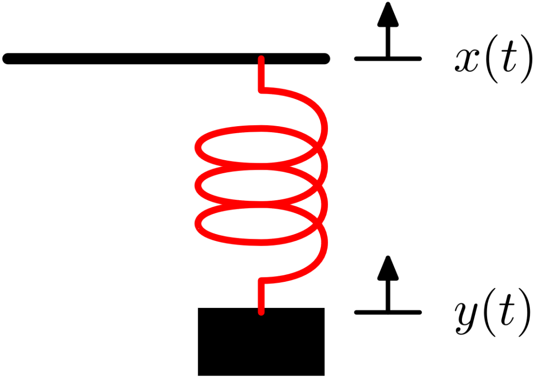
Signals and Systems



(Graphic: Denny Freeman)

Signals and Systems

Example: Mass on a Spring



(Graphic: Denny Freeman)

Signals and Systems

Example: Mass on a Spring

- **signals:** position $x(t)$ and position $y(t)$
- **parameters:** mass M and spring constant K

$$M\frac{d^2y(t)}{dt^2} = K\big(x(t) - y(t)\big)$$

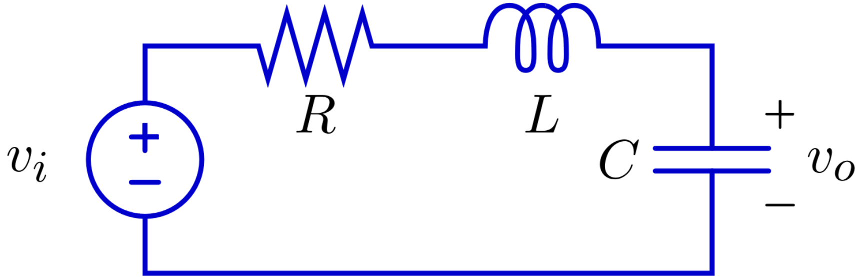
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\omega_0^2}{\omega_0^2 - \omega^2} \qquad \omega_0 = \sqrt{\frac{K}{M}}$$

$$\cos(\omega t) \to \boxed{\textbf{LTI}} \to |H(\omega)| \cos\big(\omega t + \angle H(\omega)\big)$$

very responsive to sinusoidal oscillations at $\omega \approx \omega_0$

Signals and Systems

Example: Series RLC Circuit



(Graphic: Denny Freeman)

Signals and Systems

Example: Series RLC Circuit

- **signals:** input voltage $v_i(t)$ and output voltage $v_o(t)$
- **parameters:** resistance R, inductance L, and capacitance C

$$C\frac{d^2v_o(t)}{dt^2} = \frac{1}{L}\Big(v_i(t) - RC\frac{dv_o(t)}{dt} - v_o(t)\Big)$$

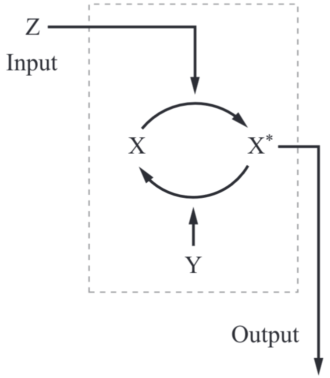
$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{\omega_0^2}{\omega_0^2 + \frac{1}{\tau}j\omega - \omega^2} \qquad \omega_0 = \sqrt{\frac{1}{LC}} \qquad \tau = \frac{L}{R}$$

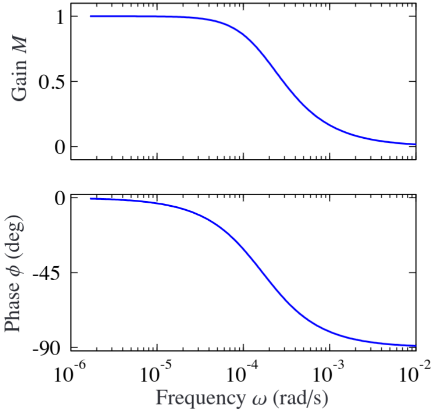
$$\cos(\omega t) \rightarrow \boxed{\textbf{LTI}} \rightarrow |H(\omega)|\cos\big(\omega t + \angle H(\omega)\big)$$

damped harmonic oscillator

Signals and Systems

Example: Phosphorylation Cycle





(*Biomolecular Feedback Systems*, D. Del Vecchio and R. M. Murray)

Signals and Systems

Example: Phosphorylation Cycle

- **signals:** kinase $x(t)$ and phosphorylated substrate $y(t)$
- **parameters:** production rate β and decay rate γ

$$\frac{dy(t)}{dt} = \beta x(t) - \gamma y(t) \iff j\omega Y(\omega) = \beta X(\omega) - \gamma Y(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\beta}{\gamma + j\omega}$$

$$\left|H(\omega)\right| = \frac{\beta}{\sqrt{\gamma^2 + \omega^2}} \qquad \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{\gamma}\right)$$

low-pass filter: unresponsive to rapidly-varying stimuli

Check Yourself

Difference Equation $\rightarrow$ Unit-Sample Response

Determine the unit-sample response $h[n]$ for the following linear constant-coefficient difference equation. Assume that the system is initially at rest: For $n < 0$, $x[n] = y[n] = 0$.

$$\boxed{y[n] = \tfrac{1}{2}y[n-1] + x[n]}$$

Check Yourself

We could set $x[n] = \delta[n]$ and notice that the response $y[n] = h[n]$ is a decaying geometric sequence. Alternatively, we could determine the frequency response $H(\Omega)$ by computing the DTFT of the difference equation. The unit-sample response $h[n]$ is the inverse DTFT of the frequency response $H(\Omega)$.

$$Y(\Omega) = \tfrac{1}{2}e^{-j\Omega}Y(\Omega) + X(\Omega) \iff H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} \frac{e^{j\Omega n}}{1 - \frac{1}{2}e^{-j\Omega}}d\Omega = \left(\tfrac{1}{2}\right)^n u[n]$$

Check Yourself

Frequency Response → Differential Equation

Determine a linear ordinary differential equation with constant coefficients with frequency response $H(\omega)$.

$$\boxed{H(\omega) = \frac{1 - j\omega}{1 - 4\,\omega^2}}$$

Check Yourself

Multiplication by $j\omega$ in the frequency domain corresponds to differentiation with respect to t in the time domain.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - j\omega}{1 + 4(j\omega)^2}$$

$$y(t) + 4\,\frac{d^2y(t)}{dt^2} = x(t) - \frac{dx(t)}{dt}$$

LTI Systems

Three representations for LTI systems:

- difference equation (DT) or differential equation (CT)
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- frequency response

Communications Systems

Amplitude Modulation

$$x(t) \rightarrow \boxed{\textbf{AM}} \rightarrow y(t) = x(t)\cos(\omega_c t)$$

Is an amplitude modulator a linear system?
Is an amplitude modulator a time-invariant system?

Communications Systems

Amplitude Modulation

$$x(t) \rightarrow \boxed{\textbf{AM}} \rightarrow y(t) = x(t)\cos(\omega_c t)$$

Is an amplitude modulator a linear system?
Is an amplitude modulator a time-invariant system?

Linear? Yes.

$$\big(c_1 x_1(t) + c_2 x_2(t)\big)\cos(\omega_c t) = c_1 x_1(t)\cos(\omega_c t) + c_2 x_2(t)\cos(\omega_c t)$$

Time-invariant? No! The carrier $\cos(\omega_c t)$ is time-varying.
The system generates new non-zero frequencies in the output!

Communications Systems

Amplitude Modulation

Transmission: Multiply $x(t)$ by sinusoidal carrier signal $c(t)$ (modulation) and transmit the modulated signal $y(t) = x(t)c(t)$.

Reception: Recover $x(t)$ from the amplitude-modulated signal $y(t)$ through demodulation and low-pass filtering.

$$c(t) = \cos(\omega_c t) = \tfrac{1}{2}e^{j\omega_c t} + \tfrac{1}{2}e^{-j\omega_c t}$$

$$y(t) = x(t)c(t) \iff Y(\omega) = \tfrac{1}{2\pi}\big(X * C\big)(\omega)$$

$$Y(\omega) = \underbrace{\tfrac{1}{2}X(\omega - \omega_c) + \tfrac{1}{2}X(\omega + \omega_c)}_{\text{copies of } X(\omega) \text{ shifted outward by } \omega_c}$$

More Modulation

We examined amplitude modulation in class. Perhaps you've heard of frequency modulation (FM) or phase modulation (PM) — but you don't need to know these for the quiz, per se.

Sinusoidal Modulation

$$y(t) = A\cos(\omega t + \phi)$$

- amplitude (AM) — time-varying amplitude $A = A(t)$
- frequency (FM) — time-varying frequency $\omega = \omega(t)$
- phase (PM) — time-varying phase $\phi = \phi(t)$

DT Fourier Representations

The **DTFS** is for periodic signals. No real-world periodic signals!

- finite summation over n (infinite-length periodic signals)
- frequency variable k of discrete domain

The **DTFT** may only be computed in theory.

- infinite summation over n (infinite-length aperiodic signals)
- frequency variable Ω of continuous domain

The **DFT** can be computed in practice.

- finite summation over n (finite-length aperiodic signals)
- frequency variable k of discrete domain

The **FFT** refers to a family of algorithms for computing the DFT.

The **STFT** is a "moving-window Fourier transform."

- For practical computation, use the DFT.

Discrete Fourier Transform

The DFT is a discrete-time, discrete-frequency Fourier transform.

- finite-length signals $x_w[n] = x[n]w[n]$
- discrete in time (n) N time-samples
- discrete in frequency (k) N frequency-samples

Discrete Fourier Transform

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad \textbf{analysis}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n} \qquad \textbf{synthesis}$$

Discrete Fourier Transform

DFT vs. Discrete-Time Fourier Series (DTFS)

The length-N DFT is equivalent to the discrete-time Fourier series of an N-periodic extension of windowed signal $x_w[n] = x[n]w[n]$.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_w[n \bmod N] e^{-jk\frac{2\pi}{N}n}$$

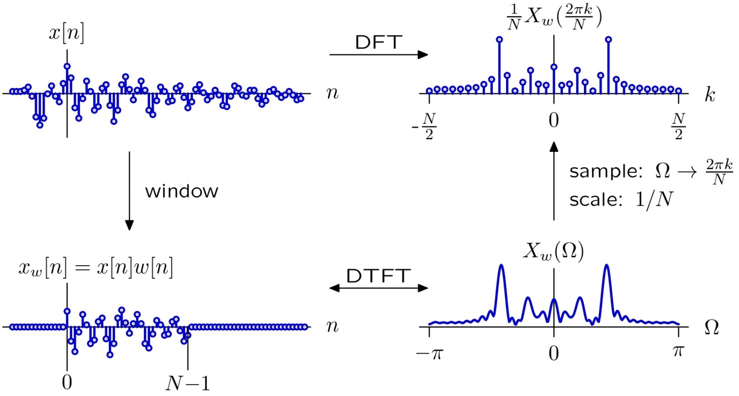
DFT vs. Discrete-Time Fourier Transform (DTFT)

$$X[k] = \tfrac{1}{N} X_w\left(\tfrac{2\pi}{N} k\right)$$

DFT frequency resolution: $\frac{f_s}{N}$ hertz or $\frac{2\pi}{N}$ radians

Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



While sampling and scaling are important, it is the **windowing** that most affects frequency content.

(Graphic: Denny Freeman)

Window Functions

Multiplying $x[n]$ by the window function $w[n]$ corresponds to convolving the DTFT of $x[n]$ with the DTFT of $w[n]$.

Windowing

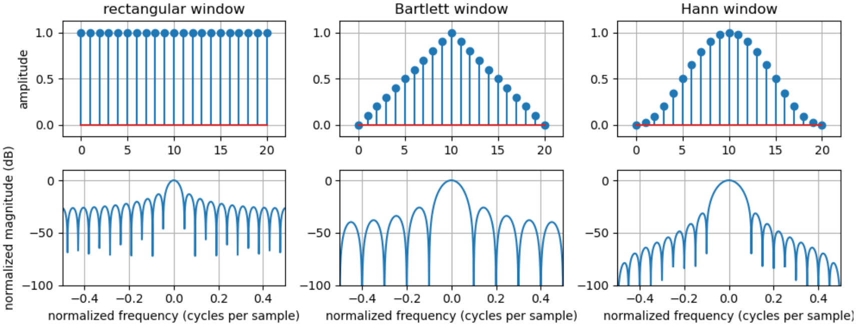
$$x_w[n] = x[n]w[n] \iff X_w(\Omega) = \tfrac{1}{2\pi}\big(X * W\big)(\Omega)$$

long $w[n] \iff$ narrow $W(\Omega)$

There are many window functions.

SciPy: Bartlett, Bartlett-Hann, Blackman, Blackman-Harris, Bohman, box-car, cosine, discrete prolate spheroidal sequences, Dolph-Chebyshev, exponential, flat-top, Gaussian, generalized Hamming, Hamming, Hann, Kaiser, Kaiser-Bessel, Lanczos, Nutall, Parzen, Taylor, triangular, Tukey, …

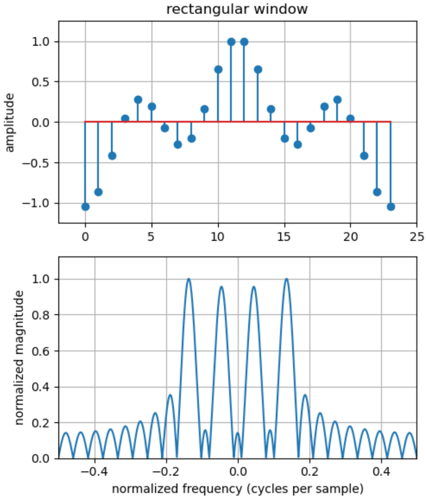
Window Functions



The window to use depends on the task at hand.

- What's most important? Narrow mainlobe? Low sidelobes?

Window Functions



DFT: Circular Convolution

Multiplication of N-point DFTs in the frequency domain corresponds to circular convolution in the time domain.

$$\begin{aligned}(x \circledast h)[n] &= N\,\mathrm{DFT}_N^{-1}\{X_N[k]H_N[k]\} \\ &= \sum_{m=0}^{N-1} x[m]h\Big[(n-m) \bmod N\Big]\end{aligned}$$

Circular convolution seems complicated, but it is really simple. You do need to know how to do regular convolution, though.

Circular Convolution

- Compute the regular (non-circular) convolution.
- Wrap the result into a length-N interval.
- Periodically extend this length-N interval.

Circular Convolution

n	=	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$
$x[n]$	=	1	1	1	1	0	0	0	0
$h[n]$	=	1	2	3	0	0	0	0	0

n	=	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$
$(x * h)[n]$	=	1	3	6	6	5	3	0	0
$(x \circledast h)_6[n]$	=	1	3	6	6	5	3	1	3
$(x \circledast h)_5[n]$	=	4	3	6	6	5	4	3	6
$(x \circledast h)_4[n]$	=	6	6	6	6	6	6	6	6

Check Yourself

Suppose that $x[n]=0$ and $h[n]=0$ for $n \notin \{0,1,2,3,\ldots,9\}$.

$$y[n] = \underbrace{\text{DTFT}^{-1}\left\{X(\Omega)H(\Omega)\right\}}_{(x*h)[n]} \qquad z[n] = \underbrace{\text{DFT}_5^{-1}\left\{X\left(\tfrac{2\pi}{5}k\right)H\left(\tfrac{2\pi}{5}k\right)\right\}}_{(x \circledast h)[n]}$$

n	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{2}}$	$\boxed{\mathbf{3}}$	$\boxed{\mathbf{4}}$	$\boxed{\mathbf{5}}$	$\boxed{\mathbf{6}}$	$\boxed{\mathbf{7}}$	$\boxed{\mathbf{8}}$	$\boxed{\mathbf{9}}$
$y[n]$	4	3	7	7	0	A	B	C	D	E
$z[n]$	4	3	14	13	1	4	3	14	13	1

Determine appropriate values for the constants A, B, C, D, and E. Give a few choices of $x[n]$ and $h[n]$ that produce $y[n]$.

Check Yourself

Suppose that $x[n] = 0$ and $h[n] = 0$ for $n \notin \{0, 1, 2, 3, \ldots, 9\}$.

$$y[n] = \underbrace{\mathrm{DTFT}^{-1}\left\{X(\Omega)H(\Omega)\right\}}_{(x*h)[n]} \qquad z[n] = \underbrace{\mathrm{DFT}_5^{-1}\left\{X\left(\tfrac{2\pi}{5}k\right)H\left(\tfrac{2\pi}{5}k\right)\right\}}_{(x \circledast h)[n]}$$

n	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	$\boxed{9}$
$y[n]$	4	3	7	7	0	A	B	C	D	E
$z[n]$	4	3	14	13	1	4	3	14	13	1

Determine appropriate values for the constants A, B, C, D, and E. Give a few choices of $x[n]$ and $h[n]$ that produce $y[n]$.

$A = 0 \qquad B = 0 \qquad C = 7 \qquad D = 6 \qquad E = 1$

Short-Time Fourier Transforms

Think of short-time Fourier transforms as "moving-window Fourier transforms."

Any Fourier transform can be a short-time Fourier transform.

Short-Time CTFT: $X(\omega, \tau) = \int_{-\infty}^{\infty} x(t) \underbrace{w(t-\tau)}_{\text{window}} e^{-j\omega t} dt$

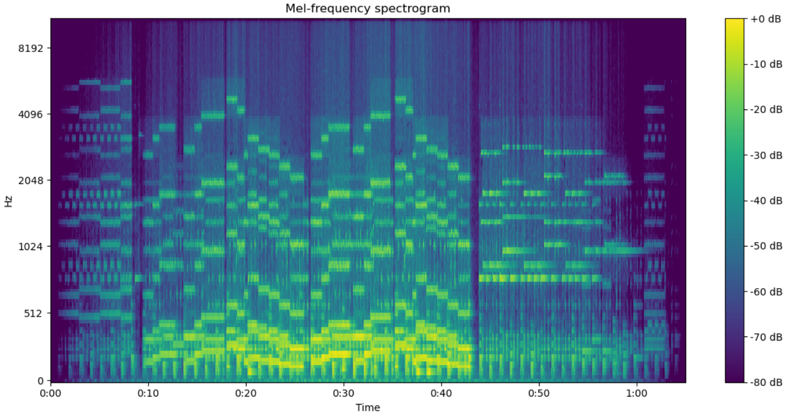
Short-Time DTFT: $X(\Omega, m] = \sum_{n=-\infty}^{\infty} x[n] \underbrace{w[n-m]}_{\text{window}} e^{-j\Omega n}$

Window Functions

$$x_w[n] = x[n]w[n] \iff X_w(\Omega) = \frac{1}{2\pi}\big(X * W\big)(\Omega)$$

Spectrograms

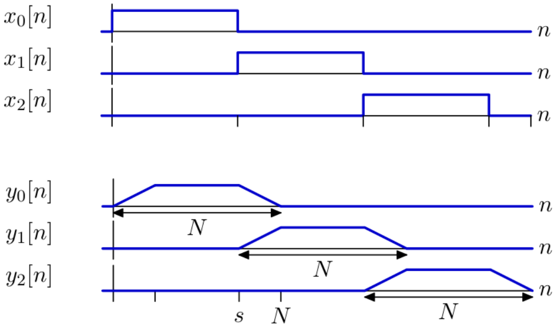
Examine the power of a signal's time-varying spectrum.



(spectrogram of "Les Patineurs" performed on Hammond organ)

Overlap-Add Method

How can we process long signals block-by-block? Divide the input $x[n]$ into blocks — each of length s. Convolve each block with $h[n]$.



The output is $y[n] = y_0[n] + y_1[n] + y_2[n] + \cdots$ Hence *overlap-add*.

(Graphic: Denny Freeman)

Fast Fourier Transform (FFT)

Gauss, circa 1805: "…truly, that method greatly reduces the tediousness of mechanical calculations …"

Radix-2 Decimation-in-Time Algorithm

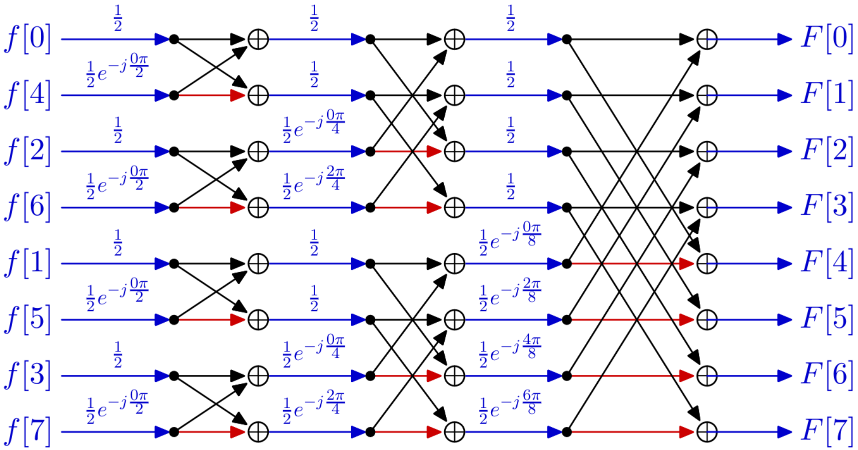
- Split a length-N DFT into a sum of two length-$(N/2)$ DFTs.

$$X_N[k] = \tfrac{1}{2}\left(X_{N/2}^{\text{even}}[k] + W_N^k X_{N/2}^{\text{odd}}[k]\right)$$

$$W_N = e^{-j\frac{2\pi}{N}} \quad (N^{\text{th}} \text{ root of unity, or "twiddle factor"})$$

- Repeat ($\uparrow$) until $N/2 = 1$, when we can't divide by 2 anymore.
- The DFT of a length-1 signal is the signal itself: $X[0] = x[0]$.

FFT: Decimation in Time



(Graphic: Denny Freeman)

Summary

- Fourier transform pairs and properties
- linearity and time-invariance
- difference equations (DT) and differential equations (CT)
- unit-sample response (DT) and impulse response (CT)
- frequency response
- convolution and filtering
- modulation and communications systems
- discrete Fourier transform (DFT)
- window functions
- circular convolution
- short-time Fourier transforms
- fast Fourier transform (FFT)

“Signals and Systems” Subjects

Signal Processing

6.300	Signal Processing	fall, spring
6.301	Signals, Systems, and Inference	spring
6.302	Fundamentals of Music Processing	fall
6.700	Discrete-Time Signal Processing	fall
6.701	Digital Image Processing	spring
6.S955	Machine Learning for Signal Processing	spring

Related Subjects

6.200	Circuits and Electronics	fall, spring
6.310	Dynamical Systems and Control	fall, spring
6.480	Biomedical Imaging with MRI	fall
6.C27	Computational Imaging	fall
18.065	Matrix Methods	spring
21M.569	Acoustics, Synthesis, and Audio Effects	spring