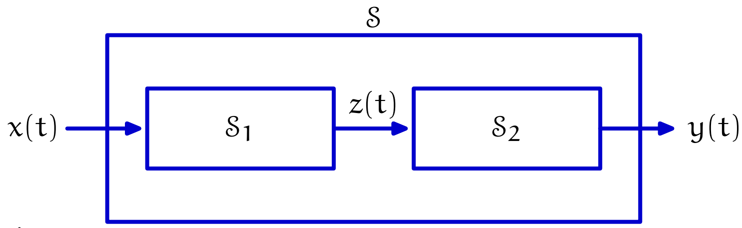
1 System Identification (20 points)

Part a. Two linear, time-invariant, continuous-time systems, $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected so that the output of $\mathcal{S}_1$ is the input to $\mathcal{S}_2$ as shown in the following figure.



The impulse response of $\mathcal{S}_1$ is

$$h_1(t) = e^{-t}u(t)$$

and the impulse response of $\mathcal{S}_2$ is

$$h_2(t) = e^{-2t}u(t)$$

where $u(t)$ represents the unit-step function:

$$u(t) = \begin{cases} 1 & \text{if } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a differential equation of the form

$$a_0y(t) + a_1y'(t) + a_2y''(t) + \cdots = b_0x(t) + b_1x'(t) + b_2x''(t) + \cdots$$

to relate $x(t)$ and $y(t)$, where $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ represent constants, primes represent derivatives, double primes represent double derivatives, and the ellipses represent higher order terms.

Note that your answer should not contain any $z(t)$ terms.

Enter your differential equation in the box below.

$$\boxed{2y(t) + 3y'(t) + y''(t) = x(t)}$$

Find the frequency responses of $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$H_1(\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(1+j\omega)t}dt = \frac{1}{1+j\omega}$$

$$H_2(\omega) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(2+j\omega)t}dt = \frac{1}{2+j\omega}$$

The frequency response of $\mathcal{S}$ is the product of H_1 and H_2, which is equal to the ratio of the Fourier transform of $y(t)$ over that of $x(t)$:

$$H(\omega) = H_1(\omega)H_2(\omega) = \frac{1}{(1+j\omega)(2+j\omega)} = \frac{Y(\omega)}{X(\omega)}$$

Therefore

$$X(\omega) = (2 + 3j\omega + (j\omega)^2)Y(\omega)$$

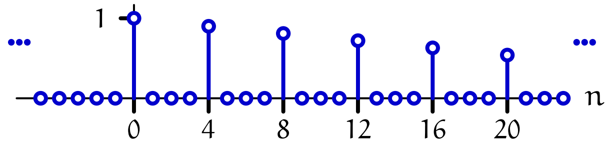
Taking the inverse Fourier transform yields the desired differential equation:

$$2y(t) + 3y'(t) + y''(t) = x(t)$$

Multiplying each term by the same constant results in an equally acceptable solution.

Part b. Consider a linear, time-invariant, discrete-time system with the unit-sample response $h[n]$ given below:

$$h[n] = \sum_{i=0}^{\infty} \alpha^i \delta[n-4i]$$



where $0 < \alpha < 1$.

Determine a difference equation to relate the input $x[n]$ and output $y[n]$ of this system.

Enter your difference equation in the box below.

$$\boxed{y[n] - \alpha y[n-4] = x[n]}$$

Note: Your solution does not have to be closed-form to receive full credit.

Method 1. The convolution sum can be used to find a relation between input and output samples, as follows.

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = \sum_{m=-\infty}^{\infty}\sum_{i=0}^{\infty} \alpha^i \delta[m-4i]x[n-m] = \sum_{i=0}^{\infty}\sum_{m=-\infty}^{\infty} \alpha^i \delta[m-4i]x[n-m] = \sum_{i=0}^{\infty} \alpha^i x[n-4i]$$

$$y[n] = x[n] + \alpha x[n-4] + \alpha^2 x[n-8] + \alpha^3 x[n-12] + \cdots$$

This expression has an infinite number of terms, but it can be simplified by using the linear time-invariant properties of the system.

$$y[n] = x[n] + \alpha x[n-4] + \alpha^2 x[n-8] + \alpha^3 x[n-12] + \cdots$$

$$\alpha y[n-4] = \alpha x[n-4] + \alpha^2 x[n-8] + \alpha^3 x[n-12] + \alpha^4 x[n-16] + \cdots$$

$$y[n] - \alpha y[n-4] = x[n]$$

Method 2. An alternative approach is to work in the frequency domain. The frequency response of the system is the Fourier transform of the unit-sample response.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{m=0}^{\infty} \alpha^i \delta[n-4i]e^{-j\Omega n} = \sum_{m=0}^{\infty} \alpha^i e^{-j\Omega 4i} = \frac{1}{1-\alpha e^{-j4\Omega}}$$

The frequency response is the ratio of the Fourier transforms of y and x:

$$H(\Omega) = \frac{1}{1-\alpha e^{-j4\Omega}} = \frac{Y(\Omega)}{X(\Omega)}$$

$$X(\Omega) = (1-\alpha e^{j4\Omega})Y(\Omega) = Y(\Omega) - \alpha e^{j4\Omega}Y(\Omega)$$

We can find the difference equation by taking the inverse Fourier transforms of each term:

$$y[n] - \alpha y[n-4] = x[n]$$

4 Systems (20 points)

Part a. Let $\mathcal{S}$ represent a linear, time-invariant system with unit-sample response

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } n \text{ is both even and greater than or equal to } 0 \\ 0 & \text{otherwise} \end{cases}$$

If the input to $\mathcal{S}$ is

$$x[n] = \cos(\pi n/4)$$

then the output can be written in the following form:

$$y[n] = A\cos(\pi n/4) + B\sin(\pi n/4)$$

Determine A and B and enter these numbers in the following boxes.

$A = \boxed{\frac{16}{17}}$

$B = \boxed{\frac{4}{17}}$

The frequency response of $\mathcal{S}$ is

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n} = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} e^{-j\Omega 2m} = \frac{1}{1-\frac{1}{4}e^{-j2\Omega}}$$

The input signal can be written as

$$x[n] = \cos(\pi n/4) = \text{Re}\left(e^{j\pi n/4}\right)$$

and the corresponding output is then

$$y[n] = \text{Re}\left(H\left(\frac{\pi}{4}\right)e^{j\pi n/4}\right) = \text{Re}\left(\frac{1}{1-\frac{1}{4}e^{-j\pi/2}}e^{j\pi n/4}\right)$$

$$= \text{Re}\left(\frac{1}{1+\frac{1}{4}j}e^{j\pi n/4}\right) = \text{Re}\left(\frac{4}{4+j}e^{j\pi n/4}\right)$$

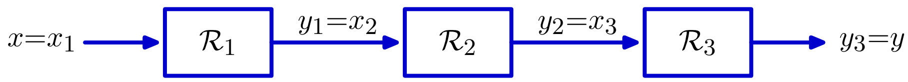
$$= \text{Re}\left(\left(\frac{16-4j}{17}\right)\left(\cos\left(\frac{\pi n}{4}\right) + j\sin\left(\frac{\pi n}{4}\right)\right)\right)$$

$$= \frac{16}{17}\cos\left(\frac{\pi n}{4}\right) + \frac{4}{17}\sin\left(\frac{\pi n}{4}\right)$$

Part b. A system is constructed from three identical subsystems: $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$. The input signal x_i and output signal y_i of each subsystem are related by the following differential equation:

$$y_i(t) + \frac{d}{dt}y_i(t) = x_i(t)$$

where i is 1, 2, and 3 for $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$, respectively. The three subsystems are connected so that the output of $\mathcal{R}_1$ becomes the input to $\mathcal{R}_2$, and the output of $\mathcal{R}_2$ becomes the input to $\mathcal{R}_3$ as shown in the following diagram.



If the input to the combined system is

$$x(t) = x_1(t) = \cos(t)$$

then the output can be written in the following form:

$$y(t) = y_3(t) = C\cos(t - \phi)$$

Determine C and ϕ, and enter these numbers in the following boxes.

$C = \boxed{\frac{1}{2\sqrt{2}}}$

$\phi = \boxed{\frac{3\pi}{4}}$

Determine the frequency response of each subsystem by taking the Fourier transform of its differential equation:

$$Y_i(\omega) + j\omega Y_i(\omega) = X_i(\omega)$$

Then the frequency response of the i^{th} subsystem is

$$H_i(\omega) = \frac{Y_i(\Omega)}{X_i(\Omega)} = \frac{1}{1 + j\omega}$$

The frequency response of the composite system is the product of those for each subsystem:

$$H(\omega) = \frac{Y(\Omega)}{X(\Omega)} = \left(\frac{1}{1 + j\omega}\right)^3$$

The input signal can be written as a sum of complex exponentials:

$$x(t) = \cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}$$

and the resulting output signal is then

$$y(t) = \frac{1}{2}H(1)e^{jt} + \frac{1}{2}H(-1)e^{-jt} = \frac{1}{2}\left(\frac{1}{1+j}\right)^3 e^{jt} + \frac{1}{2}\left(\frac{1}{1-j}\right)^3 e^{-jt}$$

$$= \text{Re}\left(\left(\frac{1}{1+j}\right)^3 e^{jt}\right) = \text{Re}\left(\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^3 e^{jt}\right) = \text{Re}\left(\frac{1}{2\sqrt{2}}e^{j(t-3\pi/4)}\right) = \frac{1}{2\sqrt{2}}\cos\left(t - \frac{3\pi}{4}\right)$$

2 Convolutions (27 points)

Part a. Let f_1 represent the following signal:

$$f_1[n] = \begin{cases} (-1)^n & \text{if } 0 \leqslant n < 6 \\ 0 & \text{otherwise} \end{cases}$$

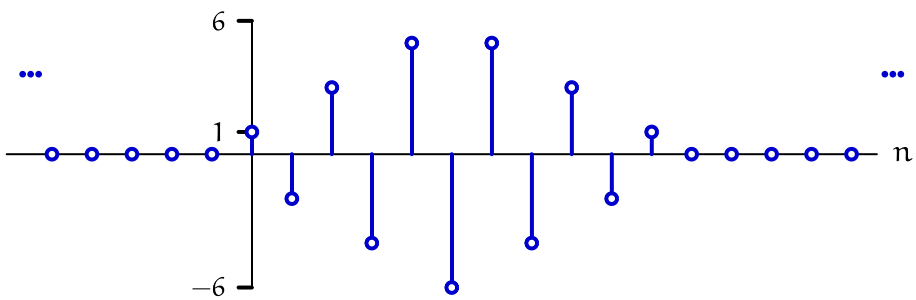
and let f_2 represent the signal that results when f_1 is convolved with itself:

$$f_2[n] = (f_1 * f_1)[n]$$

Determine the first 15 samples of f_2 and enter their values in the boxes below.

$f_2[0]$:	1	$f_2[5]$:	−6	$f_2[10]$:	1
$f_2[1]$:	−2	$f_2[6]$:	5	$f_2[11]$:	0
$f_2[2]$:	3	$f_2[7]$:	−4	$f_2[12]$:	0
$f_2[3]$:	−4	$f_2[8]$:	3	$f_2[13]$:	0
$f_2[4]$:	5	$f_2[9]$:	−2	$f_2[14]$:	0

$$g[n] = (f * f)[n] = \sum_{n=-\infty}^{\infty} f[m]f[n-m]$$



Part b. Let g_1 represent a signal whose even-numbered samples are 1 and whose odd-numbered samples are zero. Let g_2 represent the signal that would result from the following procedure:

- use an analysis width $N = 6$ (i.e., $0 \leqslant n < 6$) to calculate the DFT G_1 of g_1,
- let $G_2[k] = G_1^2[k]$, and
- take the inverse DFT of G_2 and multiply by $N = 6$ to find g_2.

Determine $g_2[0]$ through $g_2[5]$ and enter their values in the boxes below.

$g_2[0]$: $\boxed{3}$

$g_2[1]$: $\boxed{0}$

$g_2[2]$: $\boxed{3}$

$g_2[3]$: $\boxed{0}$

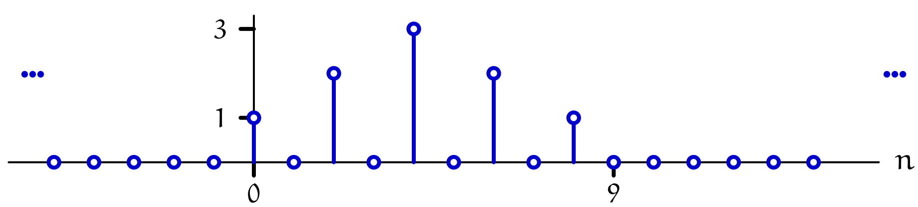
$g_2[4]$: $\boxed{3}$

$g_2[5]$: $\boxed{0}$

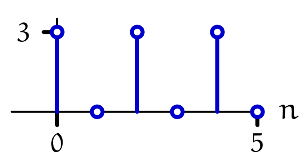
The procedure described for computing g_2 is equivalent to circularly convolving the first N samples of g_1 with itself using an analysis window $N = 10$. Circular convolution is equivalent to conventional convolution followed by aliasing as follows:

$$(g_1 \circledast g_1)[n] = \sum_{m=-\infty}^{\infty} (g_1 * g_1)[n + iN]$$

We can compute the conventional convolution of g_1 with itself using superposition or flip-and-shift. Either way, we get the following result.



After aliasing, the following signal results:

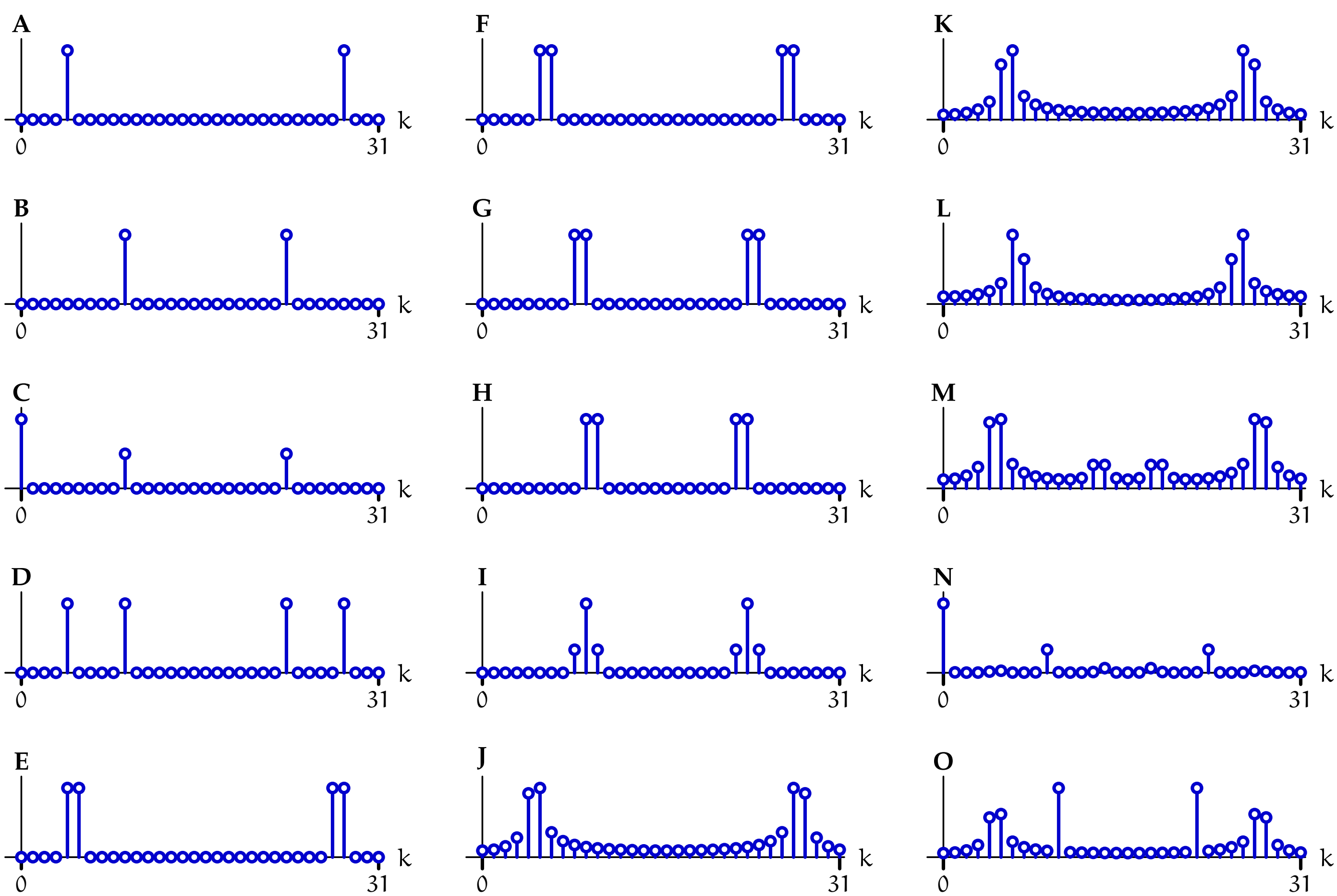


3 DFTs of Sinusoids (32 points)

Each of the following eight signals ($x_0[n]$ through $x_7[n]$) are defined by functions of discrete time n.

$x_0[n] = \cos(9\pi n/32)$:	J	$x_4[n] = \cos(2\pi n/5)$:	L
$x_1[n] = \cos(9\pi n/16)$:	B	$x_5[n] = \left\|\cos(9\pi n/32)\right\|$:	N
$x_2[n] = \cos(9\pi n/26)$:	K	$x_6[n] = \cos^2(9\pi n/32)$:	C
$x_3[n] = \cos(9\pi n/36)$:	A	$x_7[n] = \cos^3(9\pi n/32)$:	M

Discrete Fourier Transforms (DFTs) of length $N = 32$ are computed for each of these signals. Identify which of the plots below shows the magnitude of each of these DFTs as a function of frequency index k. Then write the letter of that plot in the corresponding box above.



The vertical scales for these plots differ. Each has been normalized so that its peak magnitude is 1.

2 Sampling

Consider taking a signal $x(\cdot)$ that is periodic in $T = 1$ second and sampling at a sampling rate of 6 samples per second to obtain DT signal $x[\cdot]$ that is periodic in $N = 6$. Analyzing the resulting DT signal, you find that:

- $x[\cdot]$ is a symmetric function of n.
- $x[n]$ is positive for all values of n.
- $x[0] + x[1] + x[2] + x[3] + x[4] + x[5] = 3$.
- $x[0] - x[1] + x[2] - x[3] + x[4] - x[5] = 1$.
- most of the Fourier series coefficients $X[\cdot]$ are 0; only two out of every 6 coefficients are nonzero.

What are two distinct CT functions $x(\cdot)$ that could have produced the results shown above?

We can start by thinking about the DTFS coefficients (with $N = 6$) of the sampled signal. The fact that $\sum_{n=\langle N \rangle} x[n] = 3$ tells us that the DC component is:

$$X[0] = \frac{1}{6}\left(\sum_{n=\langle 6 \rangle} x[n]\right) = \frac{1}{6}(3) = \frac{1}{2}$$

The other important fact has to do with the alternating sum. We are given that $\sum_{n=\langle 6 \rangle} x[n](-1)^n = 1$ This tells us about the component at $k = 3$:

$$X[3] = \frac{1}{6}\sum_{n=\langle 6 \rangle} x[n]e^{-j\pi n} = \frac{1}{6}\left(\sum_{n=\langle 6 \rangle} x[n](-1)^n\right) = \frac{1}{6}(1) = \frac{1}{6}$$

Since those are the only two non-zero DTFS coefficients, we can determine an expression for the discretized signal:

$$x[n] = \frac{1}{2} + \frac{1}{6}\cos(\pi n)$$

Now we need to determine a CT signal that, when sampled at a rate of 6 samples per second, gives us the expression above.

The DC component is unaffected by sampling, so we just need to find a CT cosine that, when sampled appropriately, gives us $\cos(\pi n)$.

We want a DT cosine where $\Omega = \pi$ (rad/sample), and we are given that $f_s = 6$ (samples/second). So we need ω (rad/sec) = $\Omega f_s = 6\pi$.

Thus, one CT signal that has these properties is given by $x(t) = \frac{1}{2} + \frac{1}{6}\cos(6\pi t)$.

We can find another CT signal that aliases down to the proper value by adding 2π to Ω. In order to make $\Omega = 3\pi$, we need $\omega = 18\pi$. Thus, another CT signal that has the given properties is $x(t) = \frac{1}{2} + \frac{1}{6}\cos(18\pi t)$

$x(t) =$ $\boxed{\frac{1}{2} + \frac{1}{6}\cos(6\pi t)}$

or

$x(t) =$ $\boxed{\frac{1}{2} + \frac{1}{6}\cos(18\pi t)}$