6.300: Signal Processing (Fall 2025)

Handout: Quiz #1 Story Sheet
Author: Titus K. Roesler (tkr@mit.edu)

The Signal-Processing Story So Far

- **09/04:** Signal Processing — identifying, analyzing, manipulating signals
- **09/09:** Fourier Series (Sinusoids) — series representation for periodic CT signals
- **09/11:** Fourier Series (Exponentials) — series representation for periodic CT signals
- **09/16:** Sampling and Aliasing — discretization: from continuous time to discrete time
- **09/18:** Discrete-Time Fourier Series — series representation for periodic DT signals
- **09/23:** Continuous-Time Fourier Transform — frequency representation for aperiodic CT signals
- **09/25:** Discrete-Time Fourier Transform — frequency representation for aperiodic DT signals

Mathematics Review

Dimensional Analysis

$$T_0 \text{ (seconds)} \times f_s \text{ (samples / second)} = N_0 \text{ (samples)}$$

$$\omega_0 \text{ (radians / second)} \div f_s \text{ (samples / second)} = \Omega_0 \text{ (radians / sample)}$$

CT cyclical frequency $f_0 = 1/T_0$ — cycles per second or hertz (Hz)
CT angular frequency $\omega_0 = 2\pi/T_0 = 2\pi f_0$ — radians per second
DT angular frequency $\Omega_0 = \omega_0/f_s = 2\pi f_0/f_s = 2\pi f_0 T_s = 2\pi/N_0$ — radians per sample

Geometric Series

$$\sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z} \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z|<1$$

Binomial Theorem

$$(\alpha+\beta)^n = \binom{n}{0}\alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots + \binom{n}{n-1}\alpha\beta^{n-1} + \binom{n}{n}\beta^n$$

$$\binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!} \text{ where } n! \equiv (n)(n-1)(n-2)\cdots(3)(2)(1)$$

Write down any other formulas you think are especially important to remember.

Continuous-Time Fourier Series

Continuous-Time Fourier Series

$f(t) = f(t+T)$ is a T-periodic function, and $\omega_0 = 2\pi/T$ denotes the fundamental angular frequency.

Continuous-Time Fourier Series in Trigonometric Form

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)$$

$$\text{where } c_0 = \frac{1}{T}\int_T f(t)\,dt \text{ and } c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_0 t)\,dt \text{ and } d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_0 t)\,dt$$

c_0, the "direct current" (DC) term, represents the average value of $f(t)$ over a single period.

Continuous-Time Fourier Series in Complex Exponential Form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \text{ where } a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt \qquad \left(\text{e.g., } a_0 = \frac{1}{T}\int_T f(t)dt\right)$$

Frequency

Time and frequency are inversely proportional.

cyclical: $f_0 = \frac{1}{T}$ (cycles per second, or hertz) **angular:** $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$ (radians per second)

Complex Variables

Think of complex variables geometrically — as points in the complex plane.

$$z = \underbrace{\mathrm{Re}\{z\} + j\,\mathrm{Im}\{z\}}_{\text{rectangular}} = \underbrace{re^{j\phi}}_{\text{polar}} \text{ where } \underbrace{r = \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2}}_{\text{magnitude of } z} \text{ and } \underbrace{\tan(\phi) = \frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}}_{\text{angle or phase of } z}$$

Euler's Formula

Euler's formula relates the rectangular-coordinate and polar-coordinate descriptions of complex variables.

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \cos(\theta) = \mathrm{Re}\{e^{j\theta}\} = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \mathrm{Im}\{e^{j\theta}\} = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Sampling and Aliasing

From Continuous to Discrete

We refer to the process of discretizing time or space as *sampling.*

$$x[n] = x(n\Delta) \text{ where } n \in \mathbb{Z} \text{ and } \Delta \text{ denotes the sampling period, sampling interval, or time step}$$

We refer to the process of discretizing amplitude as *quantization.* (e.g., rounding)

$$\hat{x}[n] = Q\{x[n]\} \text{ where } Q\{\cdot\} \text{ denotes a quantization operator}$$

$$\text{e.g., } Q_\Delta\{x[n]\} = \Delta \left\lfloor \frac{x[n]}{\Delta} + \frac{1}{2} \right\rfloor \text{ for some constant } \Delta > 0, \text{ where } \lfloor\cdot\rfloor \text{ denotes the floor function}$$

Digital signals are discrete in both time and amplitude. Digital systems such as laptops process digital signals. *Discrete-time* signals are discrete in time — but not necessarily in amplitude. In 6.300, we won't study quantization in great depth. We'll focus on *discrete-time signal processing.*

Sampling and Aliasing

We *sample* a continuous-time signal $x(t)$ every Δ seconds to obtain a discrete-time signal $x[n]$.

$$x[n] = x(n\Delta) \text{ where } n \in \mathbb{Z} \text{ and } \Delta \text{ denotes the sampling period, sampling interval, or time step}$$

Note that $x[n]$ is a function of the integer n, which is enclosed in square brackets. In contrast, $x(t)$ is a function of the real variable t, which is enclosed in parentheses. With this notation, $x[n] \neq x(n)$ in general — when you write $x(n)$, you're implicitly saying that $\Delta = 1$.

Aliasing

Sampling involves throwing away information. If we don't sample frequently enough, the information within our signal will be distorted: Frequencies will "fold in" on each other.

Nyquist-Shannon sampling theorem: Let $f_{\max}$ denote the highest frequency in $x(t)$. The minimum sampling rate that prevents aliasing is $2f_{\max}$ — twice the highest frequency in $x(t)$.

Frequencies

Always keep the dimensions of quantities in mind.

$$\textbf{CT cyclical: } f = \frac{1}{T} \qquad \textbf{CT angular: } \omega = 2\pi f = \frac{2\pi}{T} \qquad \textbf{DT angular: } \Omega = \frac{2\pi f}{f_s} = \frac{\omega}{f_s} = \frac{2\pi}{N}$$

The argument to a trigonometric or exponential function must be expressed in radians.

- units$\{2\pi f t\}$ = (radians/cycle) × (cycles/second) × (seconds) = radians
- units$\{\omega t\}$ = (radians/second) × (seconds) = radians
- units$\{\Omega n\}$ = (radians/sample) × (samples) = radians

Discrete-Time Fourier Series

Fourier Series Formulæ

Continuous-Time Fourier Series (CTFS)

$f(t)$ is a T-periodic function with fundamental angular frequency $\omega_0 = 2\pi/T$.

Synthesis: $f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ **Analysis:** $a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}$

Discrete-Time Fourier Series (DTFS)

$f[n]$ is an N-periodic sequence with fundamental angular frequency $\Omega_0 = 2\pi/N$.

Synthesis: $f[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$ **Analysis:** $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_0 n}$

A Few Properties of Fourier Series

Here are a few properties of Fourier series that we'll use often in this class. We'll learn more over time.

Linearity

$f_1(t) \iff F_1[k]$ $f_2(t) \iff F_2[k]$ $f(t) = \alpha f_1(t) + \beta f_2(t) \iff F[k] = \alpha F_1[k] + \beta F_2[k]$

$f_1[n] \iff F_1[k]$ $f_2[n] \iff F_2[k]$ $f[n] = \alpha f_1[n] + \beta f_2[n] \iff F[k] = \alpha F_1[k] + \beta F_2[k]$

Time Shift

$f(t) \iff F[k]$ $f(t-t_0) \iff F[k]e^{-jk\omega_0 t_0} = |F[k]|e^{j(\angle F[k] - k\omega_0 t_0)}$

$f[n] \iff F[k]$ $f[n-n_0] \iff F[k]e^{-jk\Omega_0 n_0} = |F[k]|e^{j(\angle F[k] - k\Omega_0 n_0)}$

Time Flip

$f(t) \iff F[k]$ $f(-t) \iff F[-k]$

$f[n] \iff F[k]$ $f[-n] \iff F[-k]$

Conjugate Symmetry (Hermitian Symmetry)

Real-valued signals have conjugate-symmetric Fourier series coefficients.

real-valued $f(t) \iff F[k]$ such that $F^*[k] = F[-k]$ where * denotes complex conjugation

real-valued $f[n] \iff F[k]$ such that $F^*[k] = F[-k]$ where * denotes complex conjugation

Continuous-Time Fourier Transform

Continuous-Time Fourier Transform (CTFT)

The continuous-time Fourier transform may be conceptualized as the continuum limit of a continuous-time Fourier series. Infinitely-many discrete harmonics $k\omega_0$ cluster infinitely-close together to form a continuous frequency spectrum: $k\omega_0$ (function of integer k) $\mapsto \omega$ (function of real-valued ω).

Synthesis: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ **Analysis:** $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Fourier Transform Pairs

$\delta(t) \iff 1$ (for all ω) $\quad$ 1 (for all t) $\iff 2\pi\delta(\omega)$

$\delta(t - t_0) \iff e^{-j\omega t_0}$ $\quad$ $e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0)$

Fourier Transform Properties

You fill this in! Many properties that we've seen in the context of Fourier series still hold true.

Fourier Series vs. Fourier Transform for Periodic Signals

Series: $f(t) \iff F[k]$ **Transform:** $f(t) \iff \sum_k \underbrace{2\pi F[k]\delta(\omega - k\omega_0)}_{\text{impulses at harmonics}}$

Discrete-Time Fourier Transform

Discrete-Time Fourier Transform (DTFT)

The discrete-time Fourier transform is the discrete-time analogue of the continuous-time Fourier transform — a Fourier transform for discrete-time signals. Infinitely-many discrete harmonics $k\Omega_0$ cluster infinitely-close together to form a continuous frequency spectrum: $k\Omega_0 \mapsto \Omega$.

Synthesis: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$ **Analysis:** $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$

Discrete-Time Fourier Transform Pairs

$\delta[n] \iff 1$ (for all Ω) $\quad$ 1 (for all n) $\iff 2\pi\delta(\Omega \bmod 2\pi)$

$\delta[n-n_0] \iff e^{-j\Omega n_0}$ $\quad$ $e^{j\Omega_0 n} \iff 2\pi\delta(\Omega - \Omega_0 \bmod 2\pi)$

Discrete-Time Fourier Transform Properties

You fill this in! Many properties that we've seen in the context of Fourier series still hold true.

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