6.300: Signal Processing (Fall 2025)

Handout: Practice Problems for Quiz #1
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Problem: Complex Fourier Series

Determine the coefficients $\{a_k\}_{k=-\infty}^{\infty}$ for a continuous-time Fourier series of the form

$$f(t) = f(t + \tfrac{2\pi}{\omega_0}) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_0 t}$$

for the following periodic functions.

(a) Sinusoids

$$f_1(t) = f_1\left(t + \frac{2\pi}{\omega_0}\right) = \cos(\omega_0 t) + \cos^2(\omega_0 t) + \sin(\omega_0 t) + \sin^2(\omega_0 t)$$

Before launching into calculations, recognize that $\cos^2(\omega_0 t) + \sin^2(\omega_0 t) = 1$.

Then expand $\cos(\omega_0 t) + \sin(\omega_0 t)$ using Euler's formula.

$$1 + \underbrace{\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-jk\omega_0 t}\right)}_{\cos(\omega_0 t)} + \underbrace{\left(\frac{1}{2j}e^{j\omega_0 t} + \frac{-1}{2j}e^{-j\omega_0 t}\right)}_{\sin(\omega_0 t)}$$

From there, read off the Fourier series coefficients.

$$F_1[k] = \begin{cases} 1 & k = 0 \\ \frac{1}{2} \pm \frac{1}{2j} & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Complex (But Really Simple)

$$f_2(t) = f_2(t+4) = \frac{-1}{\left(\sin(\frac{3\pi}{2}t) + j\cos(\frac{3\pi}{2}t)\right)^2}$$

$$\frac{-1}{\left(\sin(\frac{3\pi}{2}t) + j\cos(\frac{3\pi}{2}t)\right)^2} = \frac{-1}{(je^{-j\frac{3\pi}{2}t})^2} = e^{j3\pi t} = e^{j\frac{2\pi}{4}6t} \implies F_2[k] = \begin{cases} 1 & k = 6 \\ 0 & \text{otherwise} \end{cases}$$

(c) Bumps in the Road

$$f_3(t) = f_3(t+2) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

$$F_3[k] = \frac{1}{2}\int_0^1 e^{-jk\pi t}dt = \frac{1}{2}\left[\frac{e^{-jk\pi t}}{-jk\pi}\right]_0^1 = \frac{1}{-jk2\pi}\left[e^{-jk\pi}-1\right] = \frac{e^{-jk\frac{\pi}{2}}}{-jk2\pi}\left[e^{-jk\frac{\pi}{2}}-e^{jk\frac{\pi}{2}}\right] = e^{-jk\frac{\pi}{2}}\left[\frac{\sin(\frac{\pi}{2}k)}{\pi k}\right]$$

(d) Less-Frequent Bumps in the Road

$$f_4(t) = f_4(t+4) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 4 \end{cases}$$

$$F_4[k] = \frac{1}{4}\int_0^1 e^{-jk\frac{\pi}{2}t}dt = \frac{1}{4}\left[\frac{e^{-jk\frac{\pi}{2}t}}{-jk\frac{\pi}{2}}\right]_0^1 = \frac{1}{-jk2\pi}\left[e^{-jk\frac{\pi}{2}}-1\right] = \frac{e^{-jk\frac{\pi}{4}}}{-jk2\pi}\left[e^{-jk\frac{\pi}{4}}-e^{jk\frac{\pi}{4}}\right] = e^{-jk\frac{\pi}{4}}\left[\frac{\sin(\frac{\pi}{4}k)}{\pi k}\right]$$

(e) Stairway to Heaven

$$f_5(t) = f_5(t+4) = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 3 & 2 \le t < 3 \\ 4 & 3 \le t < 4 \end{cases}$$

$f_5(t)$ is a sum of scaled, shifted copies of $f_4(t)$.

$$f_5(t) = f_4(t) + 2f_4(t-1) + 3f_4(t-2) + 4f_4(t-3)$$

$$F_5[k] = F_4[k] + 2e^{-jk\pi/2}F_4[k] + 3e^{-jk\pi}F_4[k] + 4e^{-jk3\pi/2}F_4[k]$$

$$F_5[k] = (1 + 2e^{-jk\pi/2} + 3e^{-jk\pi} + 4e^{-jk3\pi/2})F_4[k] = (1 + 2(-j)^k + 3(-1)^k + 4j^k)F_4[k]$$

Problem: Sampling and Aliasing

We derive a discrete-time function $g[n]$ from a continuous-time function $g(t)$ by sampling at $t = n\Delta$, where $\Delta > 0$ denotes the interval between consecutive samples.

$$g[n] = g(n\Delta)$$

Suppose that we sample $g(t) = \cos(\frac{2\pi}{T}t)$ by setting $\Delta = \frac{1}{4}$ to produce $g[n] = \cos(\frac{\pi}{2}n)$.
If it is possible to do so, list 5 different values for $T > 0$ that could produce the same DT signal $g[n]$. If there are more than 5 possible distinct values, list any 5. If there are fewer 5 different values of $T > 0$ that produce $g[n]$, list all possible values of $T > 0$ and explain why there are fewer than 5 values.

$T \in \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\} \iff \cos(\frac{\pi}{2}n) = \cos(\frac{3\pi}{2}n) = \cos(\frac{5\pi}{2}n) = \cos(\frac{7\pi}{2}n) = \cos(\frac{9\pi}{2}n)$ for integer n

Challenge Problem: More Power to You

(a) A Higher Power

Consider the continuous-time signal

$$x_1(t) = \cos^{100}(\omega_0 t)$$

where $\omega_0 = \frac{2\pi}{T} > 0$ is the fundamental angular frequency. The complex-exponential Fourier series coefficients $X_1[k]$ could be computed from the analysis formula as

$$X_1[k] = \frac{1}{T}\int_T x_1(t)e^{-jk\omega_0 t}dt,$$

though this may be a very tedious calculation if done directly. To potentially simplify any and all calculations ahead, a number of mathematical identities are provided below.

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!}$$

$$\int \cos^m(\theta)d\theta = \frac{1}{m}\Big(\cos(\theta)\Big)^{m-1}\sin(\theta) + \frac{m-1}{m}\int \Big(\cos(\theta)\Big)^{m-2}d\theta$$

$$2\cos^2(\theta) = 1 + \cos(2\theta)$$
$$4\cos^3(\theta) = 3\cos(\theta) + \cos(3\theta)$$
$$8\cos^4(\theta) = 3 + 4\cos(2\theta) + \cos(4\theta)$$
$$16\cos^5(\theta) = 10\cos(\theta) + 5\cos(3\theta) + \cos(5\theta)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$
$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$(\alpha + \beta)^n = \binom{n}{0}\alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots + \binom{n}{n-1}\alpha\beta^{n-1} + \binom{n}{n}\beta^n$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \text{ where } n! \equiv n \cdot (n-1)\cdots 3 \cdot 2 \cdot 1$$

(continued)

By whatever means you wish, determine a simplified closed-form expression for the complex-exponential Fourier series coefficients $X_1[k]$ corresponding to the periodic signal

$$x_1(t) = \cos^{100}(\omega_0 t) \text{ where } \omega_0 = \frac{2\pi}{T} > 0.$$

You may express your answer in terms of factorials and binomial coefficients but not in terms of sums or integrals. **Hint:** Be clever and systematic! Brute force won't work here.

$X_1[k] =$ (See solution on next page.)

Use your expression to determine values of $X_1[k]$ for $k \in \{100, 50, 0, -50, -100\}$.

$X_1[100] =$ (See solution on next page.)

$X_1[50] =$ (See solution on next page.)

$X_1[0] =$ (See solution on next page.)

$X_1[-50] =$ (See solution on next page.)

$X_1[-100] =$ (See solution on next page.)

Use Euler's formula to write $\cos(\omega_0 t)$ as a sum of complex exponentials.

$$\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$x_1(t) = \left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right)^{100}$$

Expand $x_1(t)$ and read off the Fourier series coefficients by pattern-matching the expression to the definition of a Fourier series.

$$x_1(t) = \sum_k X_1[k]e^{jk\omega_0 t}$$

Use the binomial theorem, which allows one to systematically expand a binomial.

$$(\alpha+\beta)^n = \binom{n}{0}\alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \binom{n}{2}\alpha^{n-2}\beta^2 + \cdots + \binom{n}{n-1}\alpha\beta^{n-1} + \binom{n}{n}\beta^n$$

We can apply the binomial theorem by setting

$$\alpha = \frac{1}{2}e^{j\omega_0 t} \text{ and } \beta = \frac{1}{2}e^{-j\omega_0 t} \text{ and } n = 100.$$

We get

$$x_1(t) = \left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right)^{100} = \binom{100}{0}\left(\frac{1}{2}e^{j\omega_0 t}\right)^{100} + \binom{100}{1}\left(\frac{1}{2}e^{j\omega_0 t}\right)^{99}\left(\frac{1}{2}e^{-j\omega_0 t}\right) + \cdots$$

which simplifies to

$$x_1(t) = \frac{1}{2^{100}}\left[\binom{100}{0}e^{j100\omega_0 t} + \binom{100}{1}e^{j98\omega_0 t} + \binom{100}{2}e^{j96\omega_0 t} + \cdots + \binom{100}{100}e^{-j100\omega_0 t}\right]$$

from which we can read off the Fourier series coefficients:

$$X_1[k] = \begin{cases} 2^{-100}\binom{100}{50-k/2} & k \in \{-100, -98, -96, -94, \ldots, 94, 96, 98, 100\} \\ 0 & \text{otherwise} \end{cases}$$

$$X_1[-100] = 2^{-100}\binom{100}{100} \qquad X_1[-50] = 2^{-100}\binom{100}{75} \qquad X_1[0] = 2^{-100}\binom{100}{50}$$

$$X_1[50] = 2^{-100}\binom{100}{25} \qquad X_1[100] = 2^{-100}\binom{100}{0}$$

(b) A Lesser Power

Determine a simplified closed-form expression for $X_2[k]$, the complex-exponential Fourier series coefficients of the periodic signal

$$x_2(t) = -100\,\omega_0 \cos^{99}(\omega_0 t)\sin(\omega_0 t)$$

where, as before, $\omega_0 = 2\pi/T > 0$.
Express your answer in terms of $X_1[k]$ from **(a)**.

$$X_2[k] = jk\omega_0 X_1[k]$$

time-derivative property

(c) Wait a Minute!

Determine a simplified closed-form expression for $X_3[k]$, the complex-exponential Fourier series coefficients of the periodic signal

$$x_3(t) = -100\,\omega_0 \cos^{99}(\omega_0(t - t_0))\sin(\omega_0(t - t_0))$$

where, as before, $\omega_0 = 2\pi/T > 0$. $t_0 > 0$ is a constant.
Express your answer in terms of $X_2[k]$ from **(b)**.

$$X_3[k] = e^{-jk\omega_0 t_0} X_2[k]$$

time-delay property

(d) Mirror, Mirror

Determine a simplified closed-form expression for $X_4[k]$, the complex-exponential Fourier series coefficients of the periodic signal

$$x_4(t) = \text{Symmetric}\{-100\,\omega_0 \cos^{99}(\omega_0(t - t_0))\sin(\omega_0(t - t_0))\}$$

where, as before, $\omega_0 = 2\pi/T > 0$. $t_0 > 0$ is a constant.
Express your answer in terms of $X_3[k]$ from **(c)**.

$$X_4[k] = \tfrac{1}{2}X_3[k] + \tfrac{1}{2}X_3[-k]$$

$\text{Symmetric}\{x_3(t)\} = \frac{1}{2}x(t) + \frac{1}{2}x_3(-t)$