Quiz #1 Review: Chalk Talk

- **Remarks:** Logistics • Why Fourier? • Calculus vs. Signal Processing
- **Fundamentals:** Euler's Formula • Pattern Matching (Basis Functions)
- **Formulæ:** Fourier Properties • Fourier Transform Pairs • Magnitude and Phase
- **Discrete-Time Signals:** Sampling • Aliasing

Logistics

Here is the structure of the quiz review.

- I'll start with a chalk talk that emphasizes problem-solving strategies.
- Following this, we'll have an open Q&A session.
- Then we'll spend some time working through practice problems.

By the end of the review, I hope that you are fully capable of doing what's listed below.

- Get the "big idea" of Fourier representations.
- Curb the tedium of complex calculations using Euler's formula.
- Determine Fourier series coefficients without computing integrals or sums.
- Eliminate tedious calculations by remembering common Fourier transform pairs and properties.
- Apply geometric reasoning to sketch magnitude and phase plots.
- Understand how to sample continuous-time (CT) signals to produce discrete-time (DT) signals.
- Recognize when aliasing occurs, how aliasing "folds" frequencies, and how to prevent aliasing.

Why Fourier?

Examine the time-domain sequence $\{f[0], f[1], f[2], f[3]\}$ or the Fourier coefficients $\{a_0, a_1, a_2, a_3\}$.

$$\underbrace{\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{bmatrix}}_{\text{signal}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}}_{4\times 4 \text{ Fourier matrix}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\text{coefficients}} \underbrace{= a_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix}}_{\text{linear combination of Fourier basis vectors}}$$

What does the frequency-domain perspective of signals afford us?

Eigenfunctions

By the end of this course, you ought to understand what's written in this box. (Probably not yet.)

The Fourier kernel functions are complex exponentials, and complex exponentials are eigenfunctions of linear, time-invariant (LTI) systems. This is a remarkable property that allows us to easily analyze and design LTI systems that process signals.

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) H(\Omega) e^{j\Omega n} d\Omega$$

We consider three equivalent representations of LTI systems in this class.

- differential equations (CT) and difference equations (DT)
- impulse response (CT) and unit-sample response (DT)
- frequency response

Euler's Formula

$e^{j\theta}$ is to $\cos(\theta) + j\sin(\theta)$ as polar coordinates are to rectangular coordinates.

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad \mathrm{Re}\{e^{j\theta}\} = \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \mathrm{Im}\{e^{j\theta}\} = \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

No need for tables of trigonometric identities — just Euler's formula!

$$\cos(\theta) - j\sin(\theta) = \cos(-\theta) + j\sin(-\theta) = e^{j(-\theta)} = e^{-j\theta} = \left(e^{j\theta}\right)^{-1} = \frac{1}{\cos(\theta) + j\sin(\theta)}$$

$$\left(\cos(\theta) + j\sin(\theta)\right)^n = \left(e^{j\theta}\right)^n = e^{j(n\theta)} = \cos(n\theta) + j\sin(n\theta)$$

Pattern Matching (Basis Functions)

A Fourier series is a sum of complex exponentials. So, if we already have a sum of complex exponentials ...

$$f[n] = f[n+4] = 5 + 7\,e^{j\frac{\pi}{2}n} + 11\,(-1)^n + 13\,e^{j\frac{3\pi}{2}n} \implies a_0 = 5, a_1 = 7, a_2 = 11, a_3 = 13$$

$$g[n] = g[n+4] = 1 + j^n + (-1)^n + (-j)^n \implies a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1$$

$$h[n] = h[n+4] = 5 + 20\cos(\tfrac{\pi}{2}n) = 5 + 10\,e^{j\frac{\pi}{2}n} + 10\,e^{j\frac{3\pi}{2}n} \implies a_0 = 5, a_1 = 10, a_2 = 0, a_3 = 10$$

Fourier Properties

Complex exponentials simplify our calculations. No need to worry about trigonometric identities!

Time Shift Property: If $f(t) \iff F[k]$, then $g(t) = f(t - t_0) \iff G[k] = e^{-jk\frac{2\pi}{T}t_0}F[k]$.

Differentiation Property: If $f(t) \iff F[k]$, then $g(t) = \dfrac{df(t)}{dt} \iff G[k] = jk\omega_0 F[k]$.

$$\text{real } f[n] \iff F[k] \qquad g[n] = 1 - f[n_0 - n] \iff G[k] = \delta[k] - e^{-jk\frac{2\pi}{N}n_0}F[-k]$$

Fourier Transform Pairs

Continuous-Time Fourier Transform

$$\delta(t) \iff 1 \qquad 1 \iff 2\pi\delta(\omega) \qquad e^{j\omega_0 t} \iff 2\pi\delta(\omega - \omega_0) \qquad \delta(t - t_0) \iff e^{-j\omega t_0}$$

$f(t) = \cos(\omega_0 t)$ **CTFS:** $F[k] = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$ **CTFT:** $\sum_k 2\pi F[k]\delta(\omega - k\omega_0)$

Discrete-Time Fourier Transform

All discrete-time Fourier transforms are periodic in 2π.

$$\delta[n] \iff 1 \qquad 1 \iff 2\pi\delta(\Omega \bmod 2\pi) \qquad e^{j\Omega_0 n} \iff 2\pi\delta(\Omega - \Omega_0 \bmod 2\pi) \qquad \delta[n - n_0] \iff e^{-j\Omega n_0}$$

$f[n] = \cos(\Omega_0 n)$ **DTFS:** $F[k] = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$ **DTFT:** $\sum_k 2\pi F[k]\delta((\Omega - k\Omega_0) \bmod 2\pi)$

Calculus vs. Signal Processing

Calculus

You wouldn't want to walk into a calculus quiz without knowing

$$\frac{d\sin(\theta)}{d\theta} = \cos(\theta)$$

by heart, right? Going back to the derivation wastes precious time!

$$\lim_{\phi\to 0} \frac{\sin(\theta+\phi)-\sin(\theta)}{\phi}$$
$$\lim_{\phi\to 0} \frac{\sin(\theta)\cos(\phi)+\sin(\phi)\cos(\theta)-\sin(\theta)}{\phi}$$
$$\lim_{\phi\to 0} \underbrace{\left[\frac{\sin(\phi)}{\phi}\right]}_{1 \text{ as } \phi\to 0} \cos(\theta) + \lim_{\phi\to 0} \underbrace{\left[\frac{\cos(\phi)-1}{\phi}\right]}_{0 \text{ as } \phi\to 0} \sin(\theta)$$

To do well on a calculus quiz, you probably need to know at least a few things by heart.

common functions and their derivatives

$$\frac{d(t^n)}{dt} = nt^{n-1} \qquad \frac{d(e^{\lambda t})}{dt} = \lambda e^{\lambda t} \qquad \frac{d\sin(t)}{dt} = \cos(t)$$

differentiation rules

$$\frac{d[c_1 f(t) + c_2 g(t)]}{dt} = c_1\frac{df}{dt} + c_2\frac{dg}{dt} \qquad \frac{dg(f(t))}{dt} = \frac{dg}{df}\cdot\frac{df}{dt}$$

Signal Processing

Likewise, to do well on a signal processing quiz, you probably need to know at least a few things by heart.

common signals and their Fourier transforms

$$\delta[n-n_0] \iff e^{-j\Omega n_0} \qquad e^{j\Omega_0 n} \iff 2\pi\delta((\Omega-\Omega_0) \bmod 2\pi)$$

Fourier properties

$$c_1x_1[n] + c_2x_2[n] \iff c_1X_1(\Omega) + c_2X_2(\Omega)$$
$$x[n-n_0] \iff e^{-j\Omega n_0}X(\Omega)$$
$$e^{j\Omega_0 n}x[n] \iff X(\Omega-\Omega_0)$$

Magnitude and Phase

$z = a + jb = re^{j\phi}$ where $r = \sqrt{a^2 + b^2}$ and $\tan(\phi) = \dfrac{b}{a}$ (alternatively, $a = r\cos(\phi)$ and $b = r\sin(\phi)$)

Sketch magnitude and phase plots. $1 + j\omega$ $\quad \dfrac{1}{1+j\omega}$ $\quad \dfrac{1}{1+\omega^2}$ $\quad \dfrac{\omega}{1+\omega^2}$ $\quad e^{j\Omega}$ $\quad e^{j\Omega}+1$ $\quad \dfrac{1}{1+re^{j\Omega}}$

Discrete-Time Signals

Sampling

Sample a continuous-time signal $x(t)$ at $t = n\Delta$ to produce a discrete-time signal $x[n]$.

$$x[n] \equiv x(n\Delta) = x(n/f_s)$$

$$x(t) = \cos(\pi t) \rightarrow \text{ sample with } \Delta = 1 \rightarrow x[n] = \cos(\pi n \Delta) = \cos(\pi n) = (-1)^n = \{\ldots, 1, -1, 1, -1, \ldots\}$$

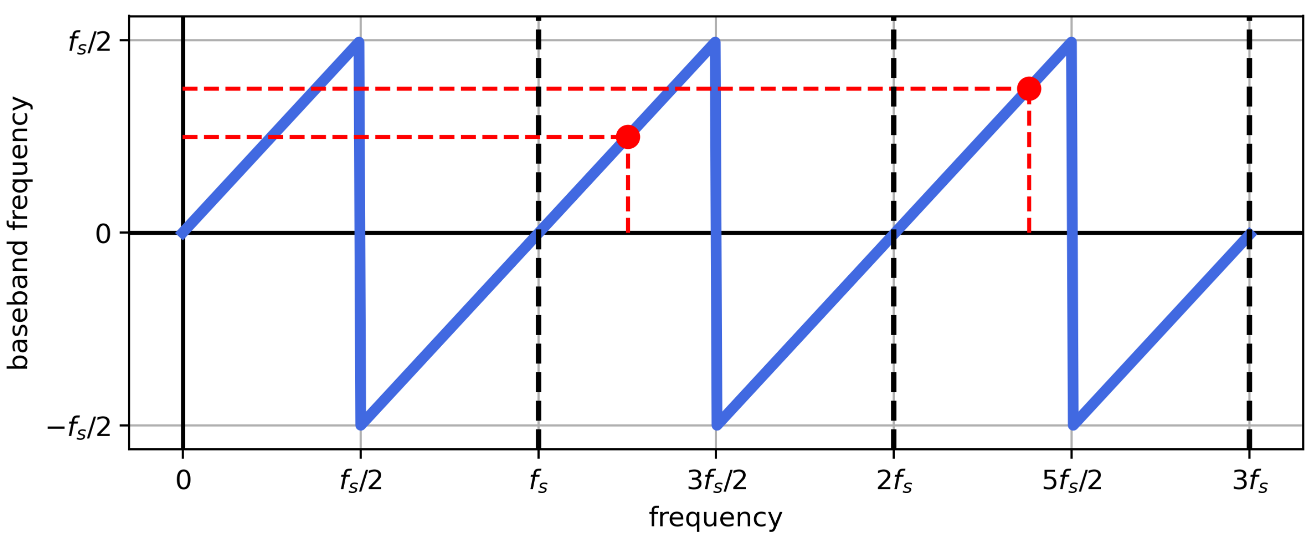
Dimensional Analysis

$$T_0 \text{ (seconds)} \times f_s \text{ (samples / second)} = N_0 \text{ (samples)}$$

$$\omega_0 \text{ (radians / second)} \div f_s \text{ (samples / second)} = \Omega_0 \text{ (radians / sample)}$$

Aliasing: Frequency Folding

Sketch this frequency-folding schematic to think through aliasing. Frequencies that lie outside the baseband range $[-\frac{1}{2}f_s, \frac{1}{2}f_s]$ alias to frequencies within the baseband range.



Nyquist-Shannon sampling theorem: Suppose $x(t)$ is a bandlimited signal — i.e., $X(\omega) = 0$ for $|\omega| = 2\pi|f| > 2\pi f_{\text{max}}$. The minimum sampling rate that prevents aliasing is $f_s = 2f_{\text{max}}$.

- $\frac{1}{2}f_s$ is the Nyquist frequency or "folding frequency."
- $\frac{1}{2}f_s$ (samples per second) $\iff \pi$ (radians per sample)
- To prevent aliasing, sample at twice the highest frequency in the signal. (e.g., If you want to sample frequencies up to $f_{\text{max}} = 22.05$ kHz, you ought to sample at the rate $f_s = 44.1$ kHz.)