6.300 Signal Processing

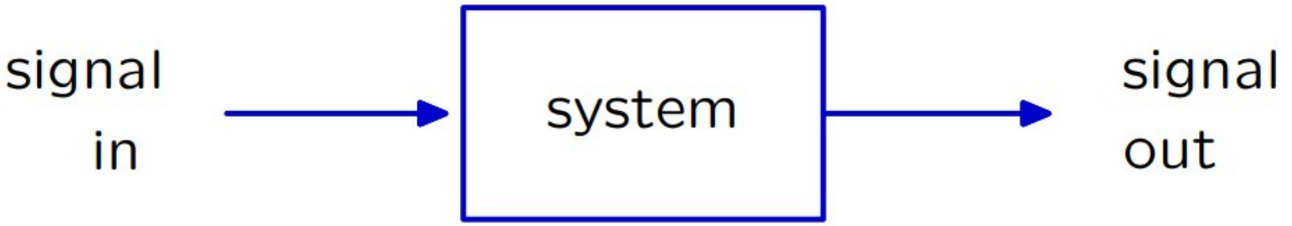
Week 6, Lecture B:
System Abstraction (III): Frequency Response

- Discrete-Time Frequency Response
- Continuous-Time Frequency Response
- Filtering

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall25

The System Abstraction

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.



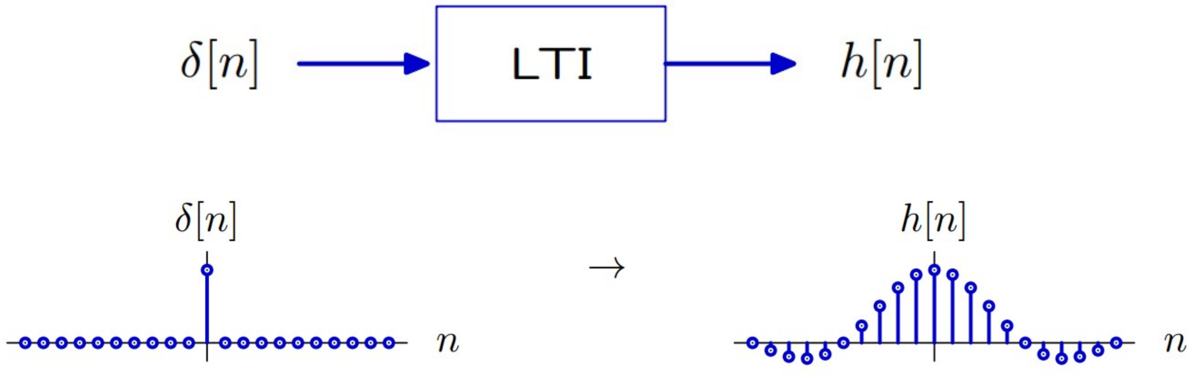
This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.

We previously studied representations based on difference/differential equations and on convolution:

- Difference/Differential Equation: represent system by algebraic constraints on samples
- Convolution: represent a system by its unit-sample/impulse response
- **Filter: represent a system as by its frequency response** ⟵ Today

Represent a System by Its Unit-Sample-Response

Unit-sample response h[n] is a complete description of an LTI system.



The unit-sample signal is the shortest possible non-trivial DT signal.

It is the building block of any arbitrary DT signal $x[n]$: $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

The response to $\delta[n]$ can be used to determine the response to any arbitrary input $x[n]$.

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Frequency Response

The frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to sinusoids.

The idea is to characterize a system by its response to signals of individual frequencies.

Since any arbitrary input can be represented by its frequency components, we can find the response to this arbitrary input from the system's frequency response.

Sinusoids differ from the unit-sample signal in important ways:

- "eternal" (longest possible signals) versus transient (shortest possible)
- comprises a single frequency versus a sum of all possible frequencies

Frequency Response

Using complex exponentials to characterize the frequency response.

$$e^{j\Omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(\Omega)\, e^{j\Omega n}$$

$$y[n] = (x * h)[n] = (h * x)[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] = \sum_{m=-\infty}^{\infty} h[m]\, e^{j\Omega(n-m)}$$

$$= e^{j\Omega n} \cdot \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\Omega m} \quad = H(\Omega)\, e^{j\Omega n}$$

The response to a complex exponential is a complex exponential with the same frequency but possibly different amplitude and phase.

The map for how a system modifies the amplitude and phase of a complex exponential input is the Fourier transform of the unit-sample response.

Frequency Response

The frequency response is a complete characterization of an LTI system.

For any arbitrary input $x[n] \rightarrow$ LTI $\rightarrow y[n]$

$$e^{j\Omega n} \rightarrow \text{LTI} \rightarrow H(\Omega)\, e^{j\Omega n}$$

$$X(\Omega) e^{j\Omega n} \rightarrow \text{LTI} \rightarrow X(\Omega) H(\Omega)\, e^{j\Omega n}$$

Scaling the input by a constant scales the output by the same constant

$$\frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} \mathrm{d}\Omega \rightarrow \text{LTI} \rightarrow \frac{1}{2\pi}\int_{2\pi} X(\Omega) H(\Omega)\, e^{j\Omega n} \mathrm{d}\Omega$$

We know $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} \mathrm{d}\Omega$ and $y[n] = \frac{1}{2\pi}\int_{2\pi} Y(\Omega)\, e^{j\Omega n} \mathrm{d}\Omega$

The Fourier Transform of the output, $Y(\Omega)$, can always be found by multiplying X(Ω) by H(Ω).

Frequency Representation of Convolution

For a LTI system, $y[n] = (x * h)[n]$. Find $Y(\Omega)$.

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} (h * x)[n]\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[m]x[n-m]\, e^{-j\Omega n}$$

$$Y(\Omega) = \sum_{m=-\infty}^{\infty} h[m] \sum_{n=-\infty}^{\infty} x[n-m]\, e^{-j\Omega n}$$

$let\ l = n - m, n = l + m$

$$= \sum_{m=-\infty}^{\infty} h[m] \sum_{l=-\infty}^{\infty} x[l]\, e^{-j\Omega(l+m)}$$

$$= \sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m} \sum_{l=-\infty}^{\infty} x[l]\, e^{-j\Omega l} = H(\Omega)X(\Omega)$$

The frequency response $H(\Omega)$ relates the Fourier transform of the input signal to the Fourier transform of the output signal.

$$(x * h)[n] \overset{\text{DTFT}}{\Longleftrightarrow} H(\Omega)X(\Omega)$$

Time domain convolution,

frequency domain multiplication

Frequency Response

The frequency response can be the most insightful description of a system.

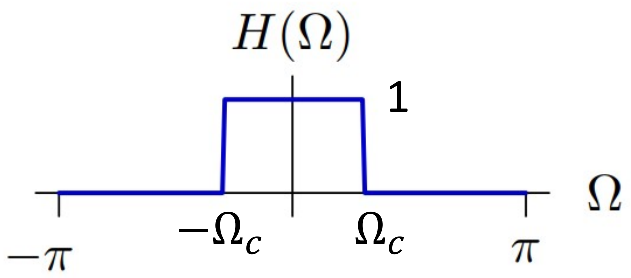
$$Y(\Omega) = H(\Omega)X(\Omega)$$

Each frequency component X(Ω) is scaled by a factor H(Ω), which can be possibly complex.

Multiplication of Fourier transforms can be regarded as **filtering.**

Example:

A low-pass filter passes frequencies near 0 and rejects those near π.

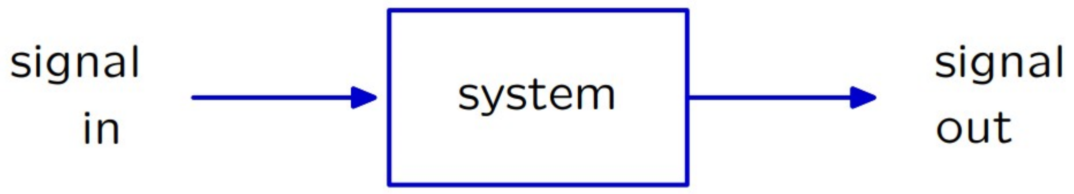


Very natural way to describe audio enhancements:

- bass-boost
- room equalizer
- tone control

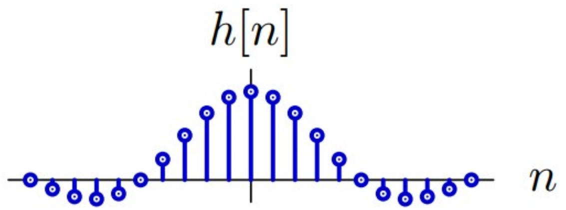
System Abstraction

Three complete representations for linear, time-invariant systems.

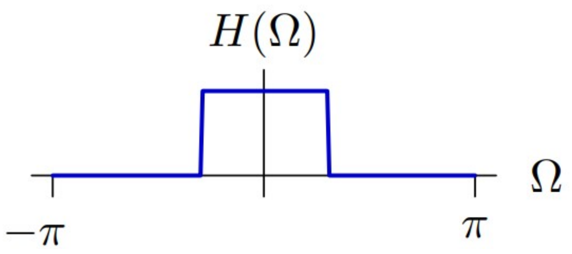


Difference Equations: relating output samples with input samples.

Unit-Sample Response: responses across time for a unit-sample input.



Frequency Response: responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **unit-sample response**!

Example

Find the frequency response of a (causal) system described by the following difference equation: $y[n] - \alpha y[n-1] = x[n], \qquad 0 < \alpha < 1$

Method 1:

Find the unit-sample response and take its Fourier transform.

$$x[n] = \delta[n]$$

Solve the difference equation for y[n].

$$y[n] = x[n] + \alpha y[n-1]$$

For n<0, since $\delta[n < 0] = 0$, for a causal system we have $y[n < 0] = 0$

$$h[0] = \delta[0] + \alpha h[-1] = 1$$

$$h[1] = \delta[1] + \alpha h[0] = \alpha$$

$$h[2] = \delta[2] + \alpha h[1] = \alpha^2$$

$$h[3] = \delta[3] + \alpha h[2] = \alpha^3$$

Can you generalize h[n]? **Participation question for Lecture**

$$\Longrightarrow \quad h[n] = \alpha^n u[n] = \begin{cases} \alpha^n & for\ n \geq 0 \\ 0 & otherwise \end{cases}$$

Example

Find the frequency response of a (causal) system described by the following difference equation:

$$y[n] - \alpha y[n-1] = x[n], \qquad 0 < \alpha < 1$$

Method 1:

Find the unit-sample response and take its Fourier transform.

The frequency response is the Fourier transform of h[n].

$$h[n] = \alpha^n u[n] = \begin{cases} \alpha^n & for\ n \geq 0 \\ 0 & otherwise \end{cases}$$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Example

Find the frequency response of a (causal) system described by the following difference equation:

$$y[n] - \alpha y[n-1] = x[n], \qquad 0 < \alpha < 1$$

Method 2:

Find the response to $e^{j\Omega n}$ directly. $\qquad x[n] = e^{j\Omega n}$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y[n] = H(\Omega)e^{j\Omega n}$$

$$y[n-1] = H(\Omega)e^{j\Omega(n-1)} = H(\Omega)e^{-j\Omega}\, e^{j\Omega n}$$

Substitute into the difference equation.

$$H(\Omega)e^{j\Omega n} - \alpha H(\Omega)e^{-j\Omega}\, e^{j\Omega n} = e^{j\Omega n}$$

$$H(\Omega)(1 - \alpha e^{-j\Omega})\cdot e^{j\Omega n} = e^{j\Omega n}$$

$$\boxed{H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}}$$

Same answer as method 1.

Example

Find the frequency response of a (causal) system described by the following difference equation:

$$y[n] - \alpha y[n-1] = x[n], \qquad 0 < \alpha < 1$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j\Omega n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

$$\sum_{n=-\infty}^{\infty} y[n-1] \cdot e^{-j\Omega n} = e^{-j\Omega} Y(\Omega)$$

Method 3:

Take the Fourier transform of the difference equation.

$$\sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j\Omega n} - \alpha \sum_{n=-\infty}^{\infty} y[n-1] \cdot e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

$$Y(\Omega) - \alpha e^{-j\Omega} Y(\Omega) = X(\Omega)$$

Solve for Y (Ω).

$$Y(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}} \cdot X(\Omega)$$

Since $Y(\Omega) = H(\Omega)X(\Omega)$,

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as method 1 and 2.

Example

A LTI that is described by:

$$y[n] - \alpha y[n-1] = x[n], \qquad 0 < \alpha < 1$$

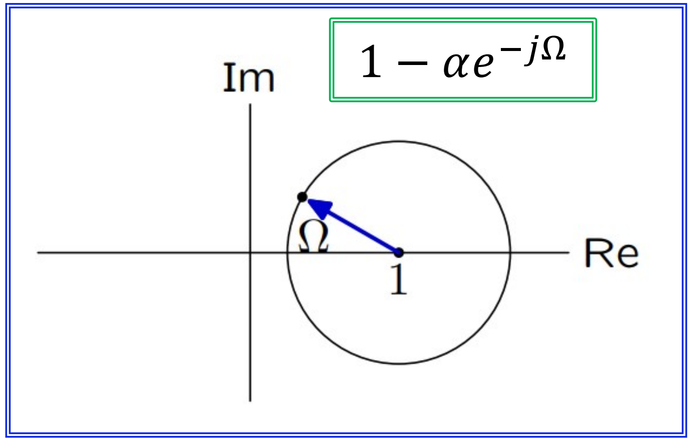
$$h[n] = \alpha^n u[n] = \begin{cases} \alpha^n & for\ n \geq 0 \\ 0 & otherwise \end{cases}$$

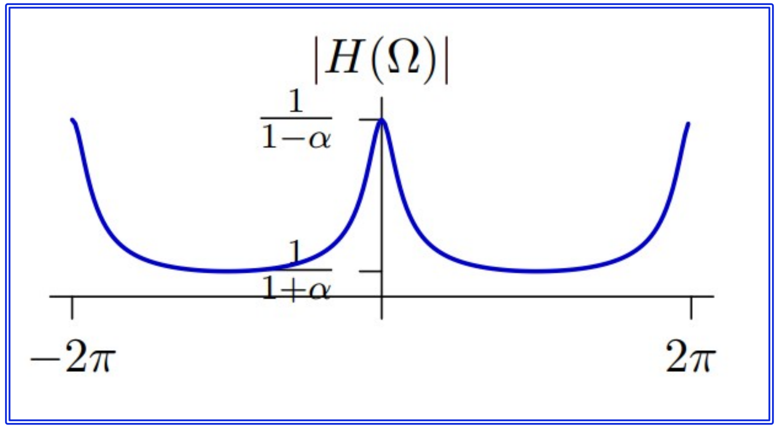
$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

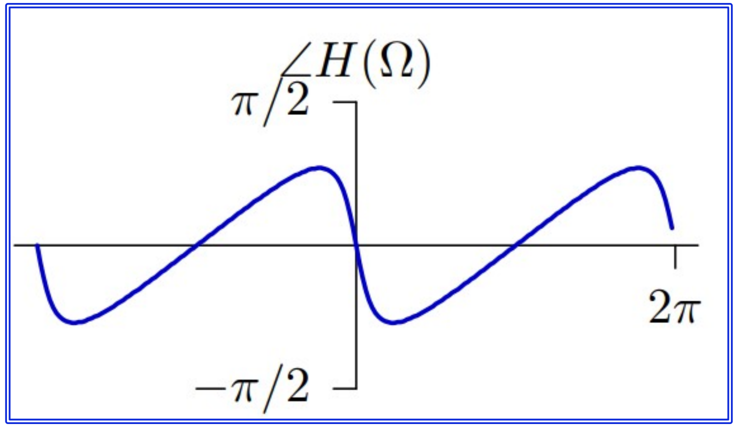
Plot the frequency response.

Note that denominator is sum of 2 complex numbers.

- Amplifies low frequencies
- Attenuates high frequencies
- Adds phase delay



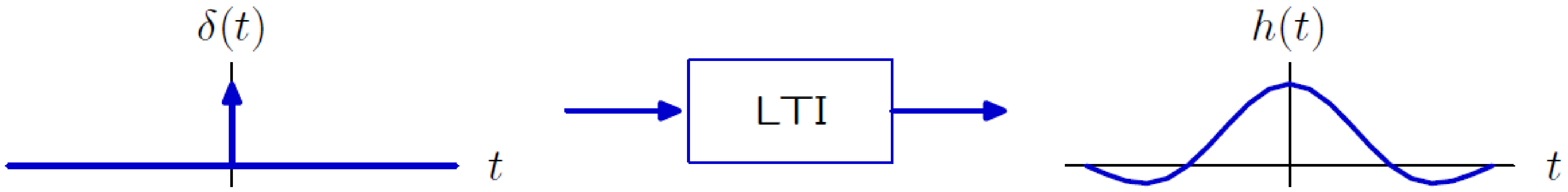




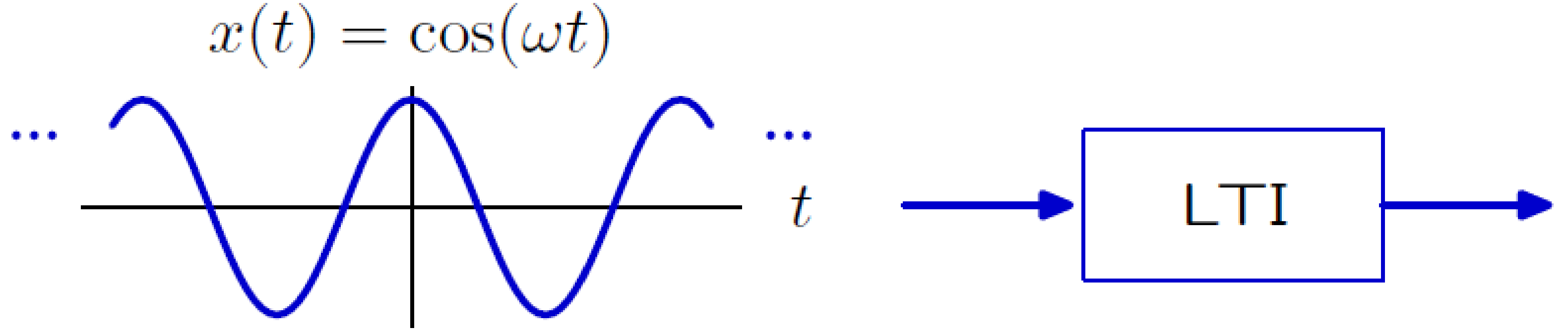
Frequency Response of a Continuous-Time System

Use convolution to characterize the frequency response of a system.

The response of a CT LTI system to the Dirac delta function $\delta(t)$ is the impulse response $h(t)$.



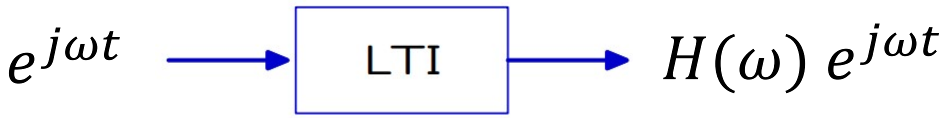
The response $y(t)$ to a sinusoid $x(t) = \cos(\omega t)$ is $y(t) = (\mathrm{x} * \mathrm{h})(\mathrm{t})$.



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Frequency Response

Using complex exponentials to characterize the frequency response.



$$y(t) = (x * h)(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau$$

$$= e^{j\omega t} \cdot \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(\omega)\, e^{j\omega t}$$

The response to a complex exponential is a complex exponential with the same frequency ω but possibly different amplitude and phase given by $H(\omega)$.

The map for how a system modifies the amplitude and phase of a complex exponential input is the Fourier transform of the impulse response.

Frequency Response

The frequency response is a complete characterization of an LTI system.

For any arbitrary input $x(t) \longrightarrow$ LTI $\longrightarrow y(t)$

We know $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$ and $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) \cdot e^{j\omega t}\, d\omega$

$e^{j\omega t} \longrightarrow$ LTI $\longrightarrow H(\omega)\, e^{j\omega t}$

$X(\omega)e^{j\omega t} \longrightarrow$ LTI $\longrightarrow X(\omega)H(\omega)\, e^{j\omega t}$

Scaling the input by a constant scales the output by the same constant

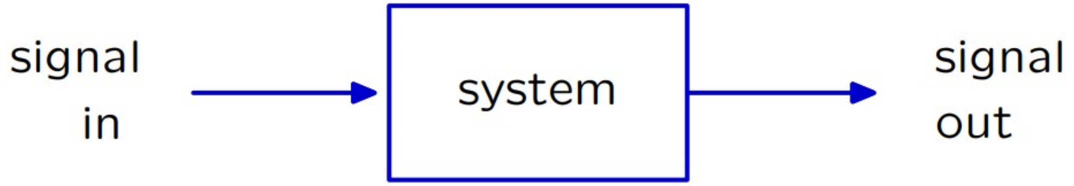
$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega \longrightarrow$ LTI $\longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega) \cdot e^{j\omega t}\, d\omega$

⬆ Linearity implies that the response to a sum is the sum of the responses.

The Fourier Transform of the output $Y(\omega)$ can always be found by multiplying X(ω) by H(ω).

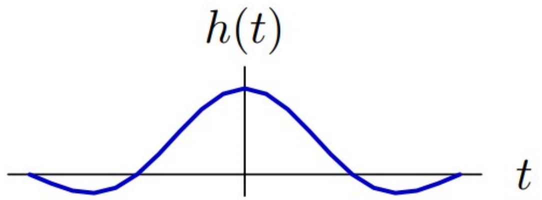
CT System Abstraction

Three complete representations for linear, time-invariant systems.

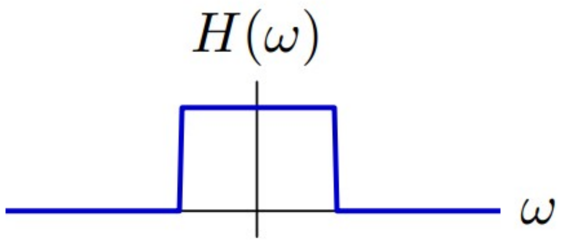


Differential Equations: relating output derivatives with input derivatives.

Impulse Response: responses across time for an impulse input.



Frequency Response: responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **impulse response**!

Example

Find the frequency response of a system described by the following differential equation:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t), \alpha > 0$$

Method 1:

Find the response to $e^{j\omega t}$ directly. $x(t) = e^{j\omega t}$

$$y(t) = H(\omega)e^{j\omega t}$$

$$\frac{dy(t)}{dt} = j\omega H(\omega)e^{j\omega t}$$

Substitute into the differential equation.

$$H(\omega)e^{j\omega t} + \alpha j\omega H(\omega)e^{j\omega t} = 2e^{j\omega t}$$

Since $e^{j\omega t}$ is never 0, we can divide it out.

$$H(\omega)(1 + j\omega\alpha) = 2$$

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

Example

Find the frequency response of a system described by the following differential equation:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t)\,, \alpha > 0$$

Method 2:

Take the Fourier transform of the differential equation.

$$\int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t}\, dt + \alpha \int_{-\infty}^{\infty} \frac{dy(t)}{dt} \cdot e^{-j\omega t}\, dt = 2 \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$Y(\omega) + \alpha j\omega Y(\omega) = 2X(\omega)$$

If $z(t) = \frac{d}{dt} y(t)$, then $Z(\omega) = j\omega \cdot Y(\omega)$

Solve for Y (ω).

$$Y(\omega) = \frac{2}{1 + j\omega\alpha} \cdot X(\omega)$$

Since $Y(\omega) = H(\omega)X(\omega)$,

$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$

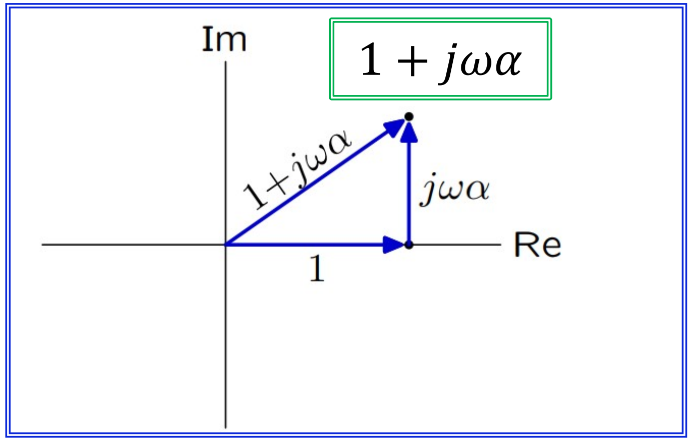
Same answer as method 1.

Example

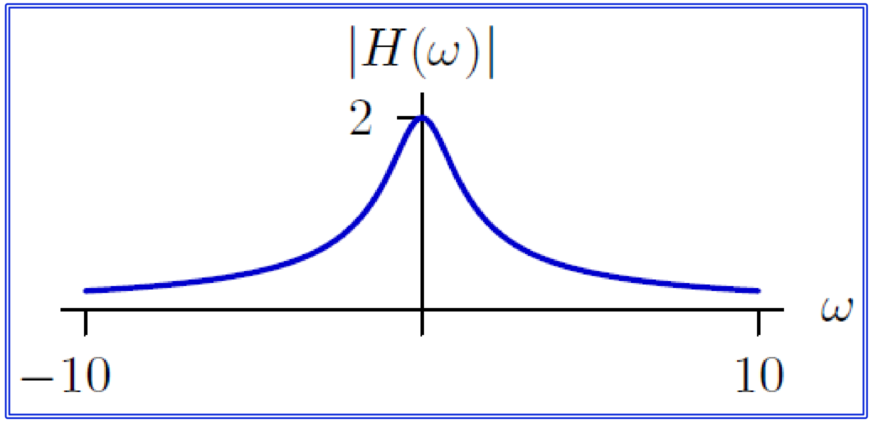
A LTI that is described by:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t), \alpha > 0$$

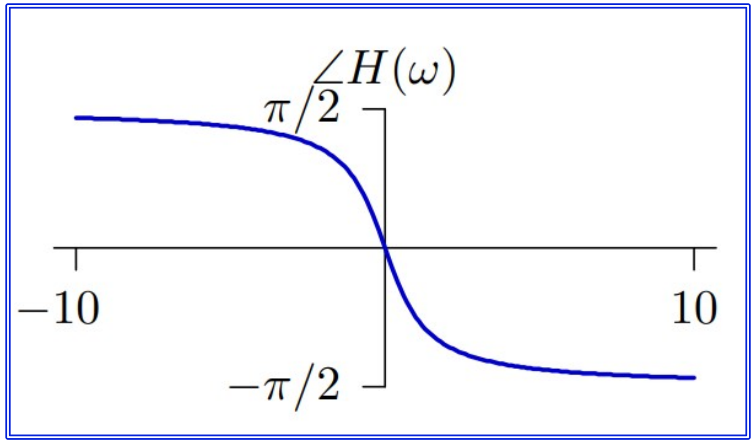
$$H(\omega) = \frac{2}{1 + j\omega\alpha}$$



Plot the frequency response.



Note that denominator is sum of 2 complex numbers.



- Pass low frequencies
- Attenuates high frequencies
- Adds phase delay

Check yourself

Find the frequency response of a rectangular box averager:

$$y(t) = \frac{1}{2}\int_{t-1}^{t+1} x(\tau)d\tau$$

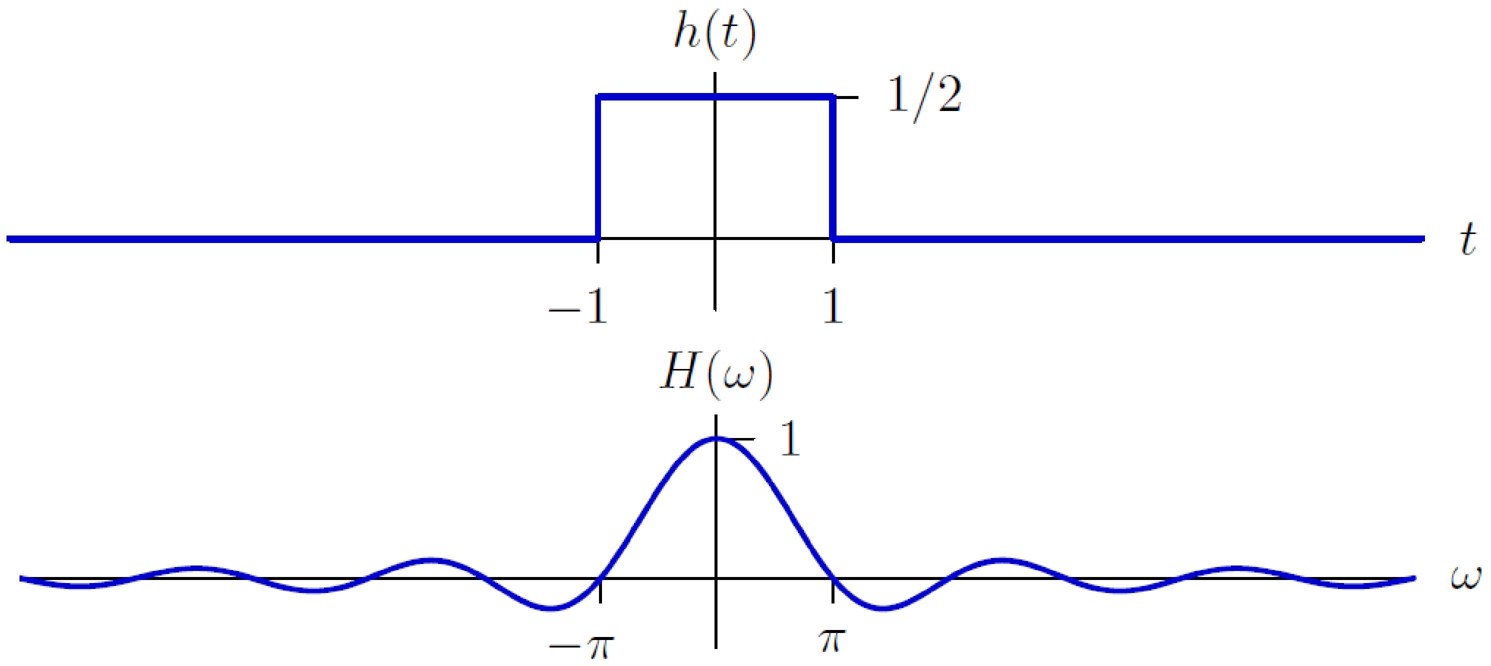
(This CT averager is analogous to the three-point averager in DT.)

Check yourself

Find the frequency response of a rectangular box averager:

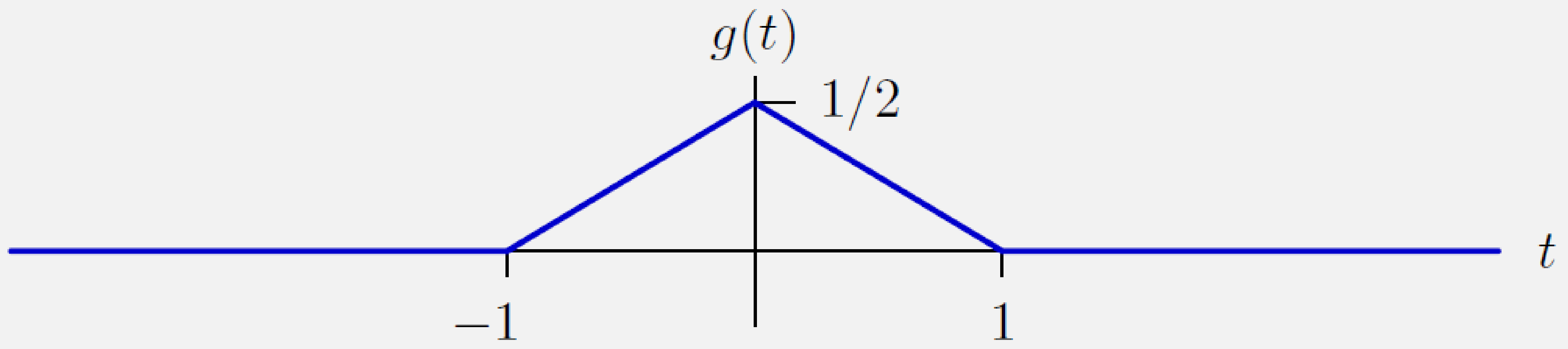
$$y(t) = \frac{1}{2}\int_{t-1}^{t+1} x(\tau)d\tau \qquad h(t) = \frac{1}{2}\int_{t-1}^{t+1} \delta(\tau)d\tau = \begin{cases} \frac{1}{2} & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \frac{1}{2}\int_{-1}^{1} e^{-j\omega t}dt = \frac{\sin(\omega)}{\omega}$$



Check yourself

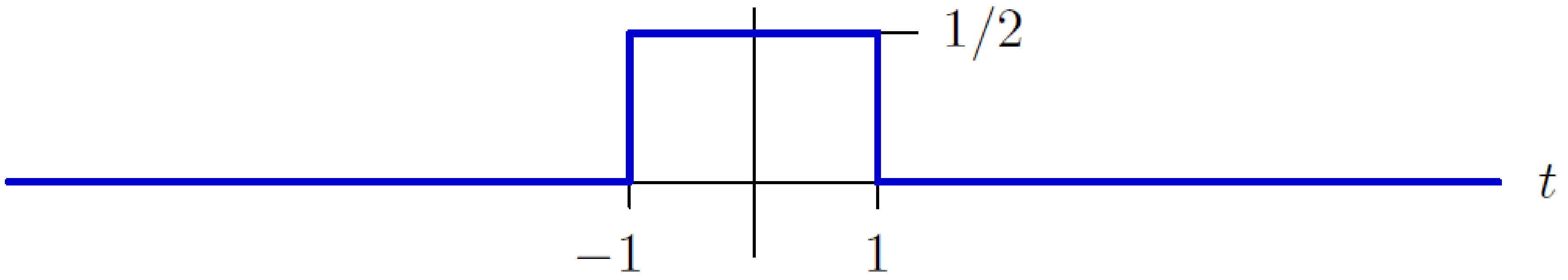
Find the frequency response of a triangular averager:



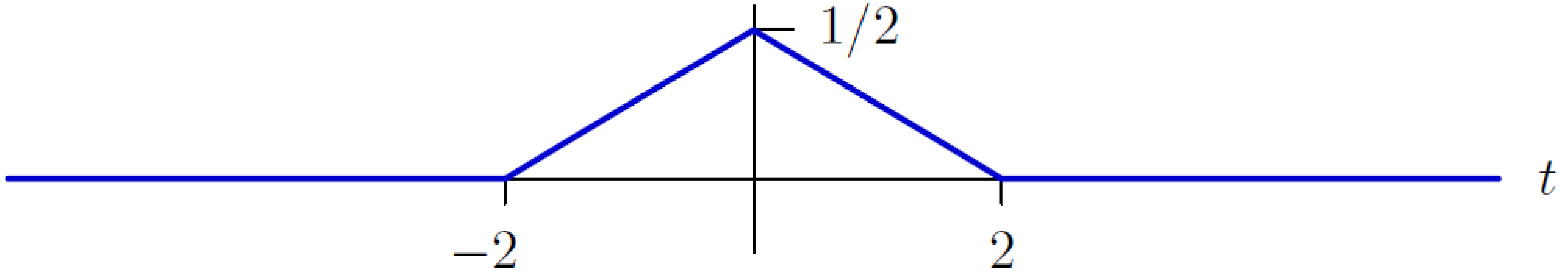
Check yourself

The triangular averager $g(t)$ can be expressed as the cascade of two rectangular averagers $h(t)$.

$h(t)$



$g(t) = (h * h)(t)$



Check yourself

Convolution in time is equivalent to multiplication in frequency.

$$g(t) = (f * f)(t) = \int f(t-\tau) f(\tau) d\tau$$

$$G(\omega) = \int g(t) e^{-j\omega t} dt = \int_t \underbrace{\int_\tau f(t-\tau) f(\tau) d\tau}_{g(t)} e^{-j\omega t} dt$$

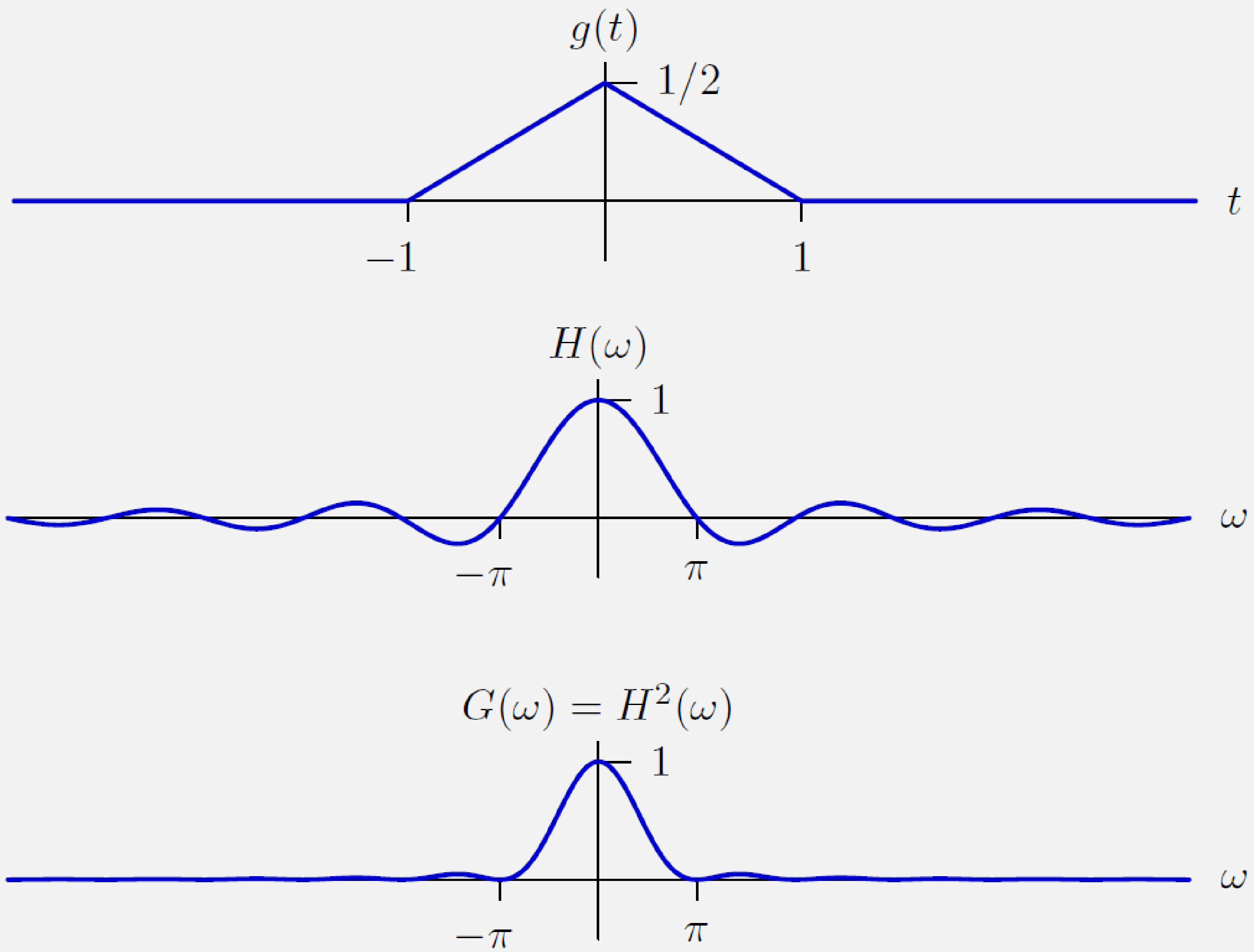
$$= \int_\tau f(\tau) \underbrace{\int_t f(t-\tau) e^{-j\omega t} dt}_{e^{-j\omega\tau} F(\omega)} d\tau$$

$$= F(\omega) \underbrace{\int_\tau f(\tau) e^{-j\omega\tau} d\tau}_{F(\omega)}$$

$$= F^2(\omega)$$

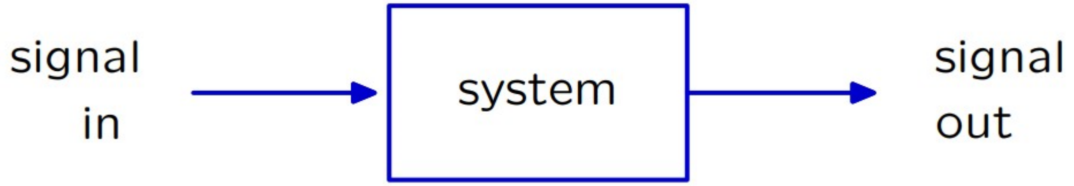
Check yourself

Find the frequency response of a triangular averager:



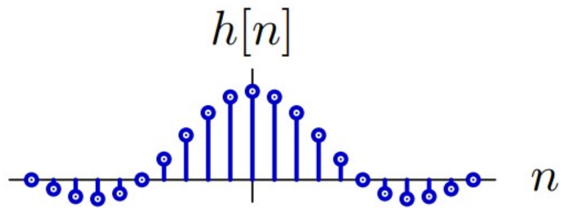
Summary

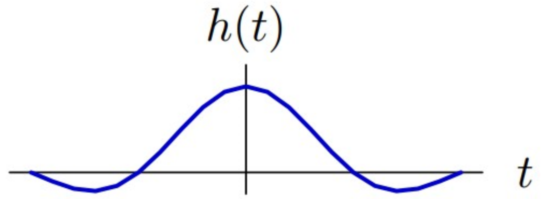
Three complete representations for linear, time-invariant systems.



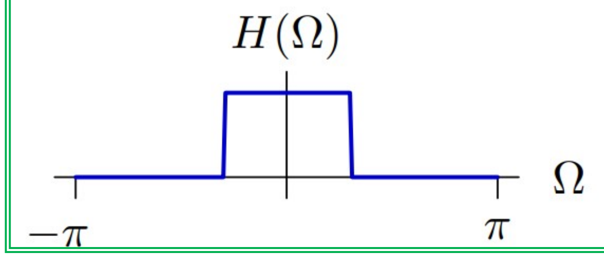
Difference/Differential Equations: relating output with input.

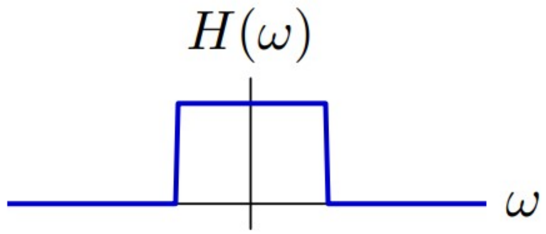
Unit-Sample/Impulse Response: responses across time for an impulse input.





Frequency Response: responses across frequencies for sinusoidal inputs.

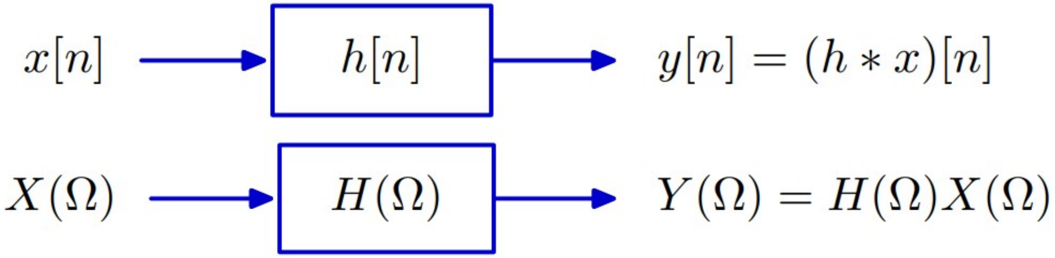




The **frequency response** is Fourier transform of **unit-sample/impulse response**!

Summary

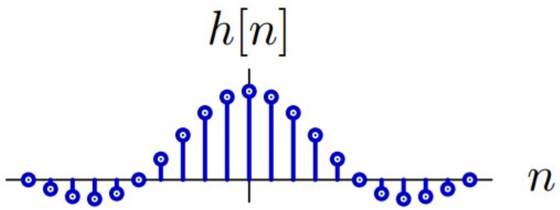
Three complete representations for linear, time-invariant systems.

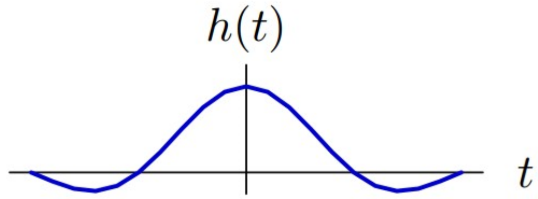


$$(x * h)[n] \overset{\text{DTFT}}{\Longleftrightarrow} H(\Omega)X(\Omega)$$

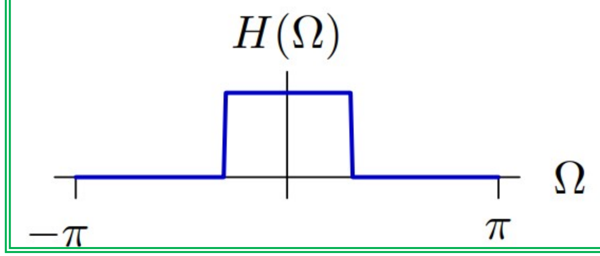
Difference/Differential Equations: relating output with input.

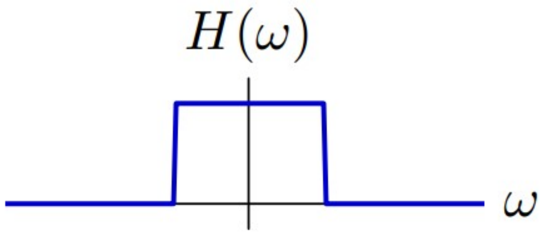
Unit-Sample/Impulse Response: responses across time for an impulse input.





Frequency Response: responses across frequencies for sinusoidal inputs.





The **frequency response** is Fourier transform of **unit-sample/impulse response**!