

6.300 Signal Processing

Week 4, Lecture B: Discrete Time Fourier Transform

- Definition
- Examples
- DT vs CT; FS vs FT
- DT Impulse

Quiz 1: Tuesday September 30, 2-4pm 50-340

- Closed book except for one page of **written** notes (8.5" x 11" both sides)
- No electronic devices (No headphones, cell phones, calculators, ...)
- Coverage up to Week #3 (DTFS); no HW4, a practice quiz will be put on our website.
- Quiz review session: 9/28 1-3pm in 4-370

From Fourier Series to Fourier Transform (DT)

- Last time: use continuous-time Fourier transform to represent arbitrary (aperiodic) CT signals as sums of sinusoidal components

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

Analysis equation

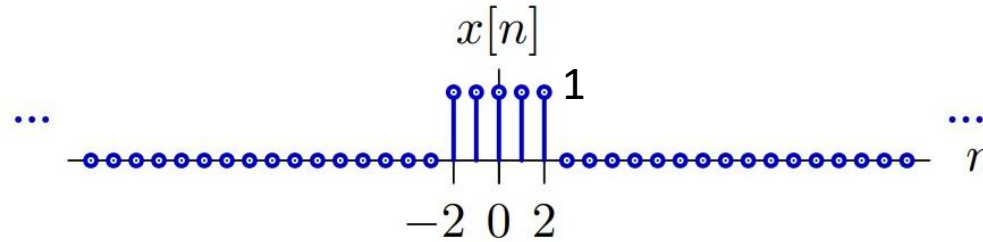
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

Synthesis equation

Today: generalize the **Fourier Transform** idea to **discrete-time** signals.

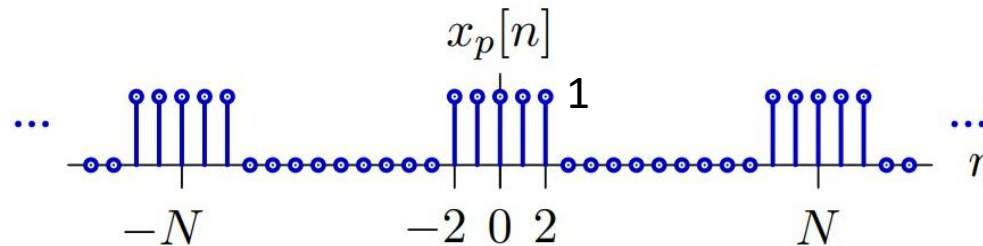
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Strategy: make a periodic version of $x[n]$ by summing shifted copies:

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$



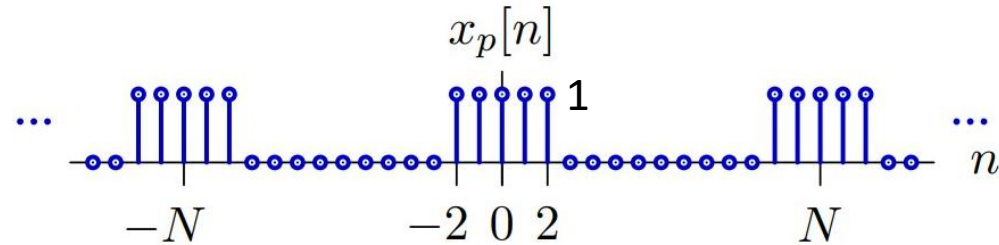
Since $x_p[n]$ is periodic, it has a Fourier series (which depends on N)

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as $N \rightarrow \infty$

As $N \rightarrow \infty$, $x_p[n] \rightarrow x[n]$ and Fourier series will approach Fourier transform.

Fourier Representations of Aperiodic Signals

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

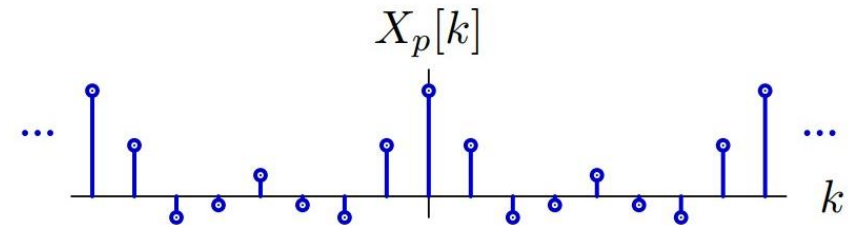


Calculate the Fourier series coefficients $X_p[k]$: $X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] \cdot e^{-j\frac{2\pi}{N}kn}$

$$X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \cdot e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N} \cos\left(\frac{2\pi k}{N}\right) + \frac{2}{N} \cos\left(\frac{4\pi k}{N}\right)$$

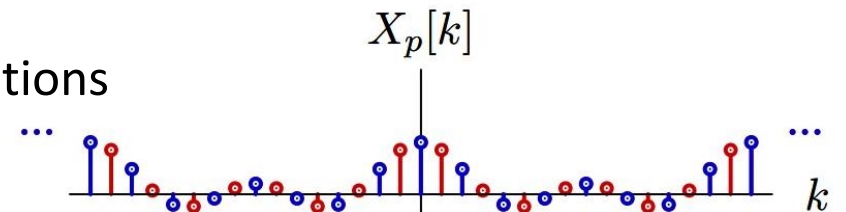
Plot the resulting Fourier Series coefficients for $N=8$.

What happens if you double the period N ?



There will be twice as many samples per period of the cosine functions

The red samples are at new intermediate frequencies

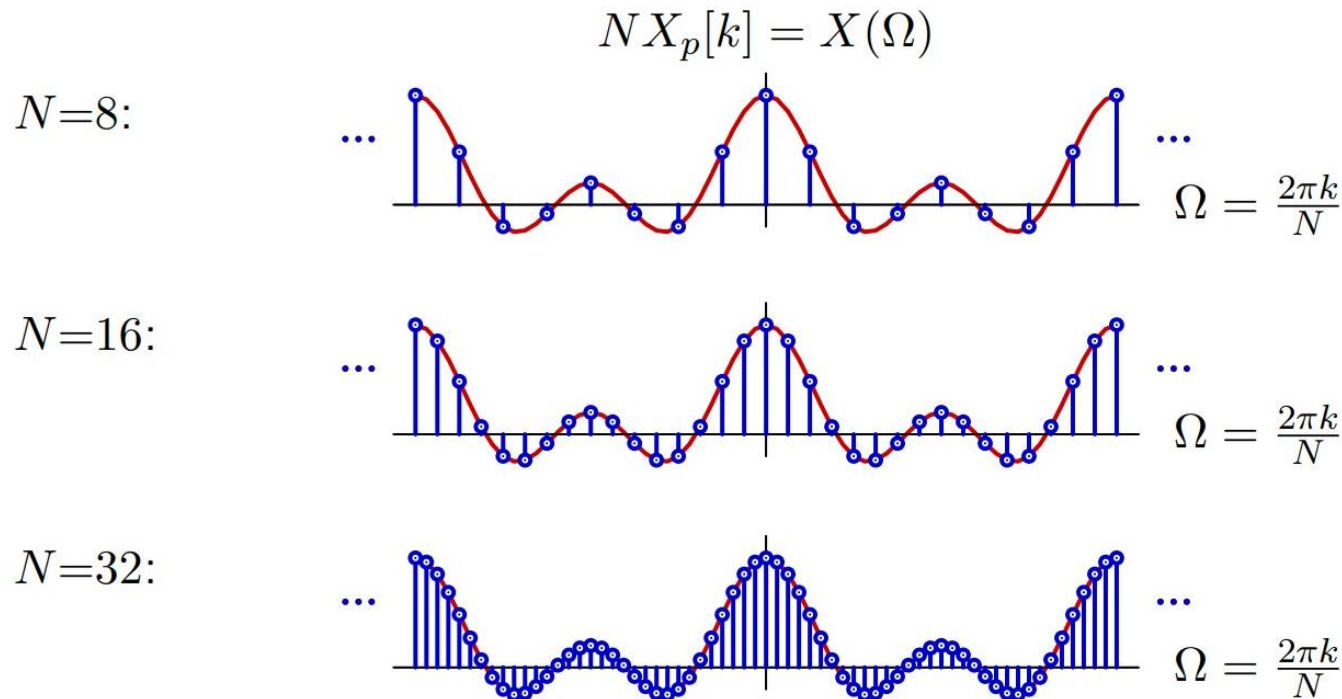


Fourier Representations of Aperiodic Signals

$$X_p[k] = \frac{1}{N} + \frac{2}{N} \cos\left(\frac{2\pi k}{N}\right) + \frac{2}{N} \cos\left(\frac{4\pi k}{N}\right)$$

let $\Omega = \frac{2\pi k}{N}$, Define a new function $X(\Omega) = N \cdot X_p[k] = 1 + 2 \cos(\Omega) + 2 \cos(2\Omega)$

If we consider Ω and $X(\Omega) = 1 + 2 \cos(\Omega) + 2 \cos(2\Omega)$ to be continuous, the discrete function $NX_p[k]$ is a sampled version of $X(\Omega)$.



As N increases, the resolution in Ω increases

Fourier Representations of Aperiodic Signals

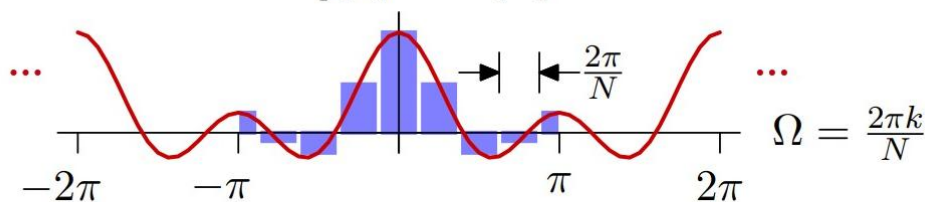
We can reconstruct $x[n]$ from $X(\Omega)$ using Riemann sums (approximating an integral by a finite sum).

$$x_p[n] = \sum_{k=\langle N \rangle} X_p[k] e^{j \frac{2\pi}{N} kn} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} \color{red}{N} X_p[k] e^{j \frac{2\pi}{N} kn} \left(\frac{\color{red}{2\pi}}{\color{red}{N}} \right)$$

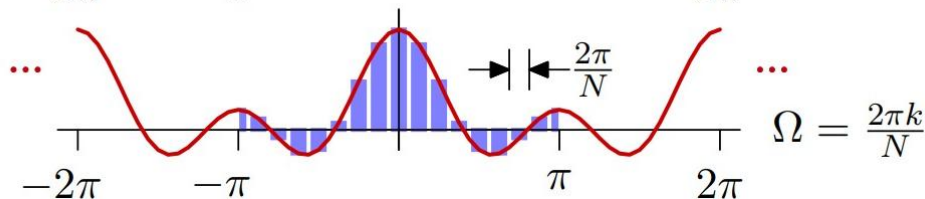
$$x[n] = \lim_{N \rightarrow \infty} x_p[n] = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{k=\langle N \rangle} \color{red}{N} X_p[k] e^{j \frac{2\pi}{N} kn} \left(\frac{\color{red}{2\pi}}{\color{red}{N}} \right) = \frac{1}{2\pi} \int_{\color{red}{2\pi}} X(\Omega) e^{j \Omega n} d\Omega$$

$$N X_p[k] = X(\Omega)$$

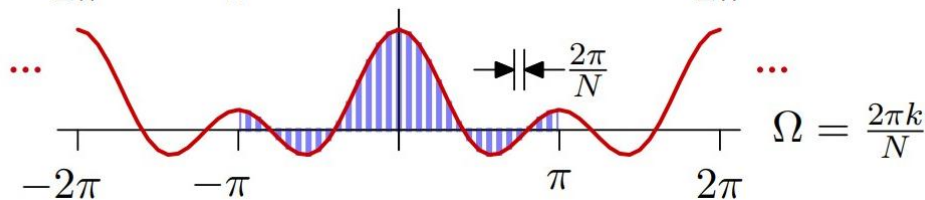
$N=8$:



$N=16$:



$N=32$:



As $N \rightarrow \infty$,

- $k\Omega_0 = \frac{2\pi k}{N}$ becomes a continuum,
 $\frac{2\pi k}{N} \rightarrow \Omega$.
- The sum takes the form of an integral, $\Omega_0 = \frac{2\pi}{N} \rightarrow d\Omega$
- We obtain a spectrum of coefficients: $X(\Omega)$.

Discrete-Time Fourier Transform

$$x[n] = \lim_{N \rightarrow \infty} x_p[n] = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{k=\langle N \rangle} N X_p[k] e^{j \frac{2\pi}{N} kn} \left(\frac{2\pi}{N} \right) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Since $X(\Omega) = N \cdot X_p[k]$

$$X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \cdot e^{-j \frac{2\pi}{N} kn}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Fourier Series and Fourier Transform

Fourier series and transforms are similar:
both represent signals by their frequency content.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n + N] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$X[k] = X[k + N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_0 kn}$$

Analysis equation

$$\Omega_0 = \frac{2\pi}{N}$$

Fourier Series and Fourier Transform

Periodic signals can be synthesized from a discrete set of harmonics.
Aperiodic signals generally require all possible frequencies.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n + N] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$X[k] = X[k + N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_0 kn}$$

Analysis equation

$$\Omega_0 = \frac{2\pi}{N}$$

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period.
The information in an aperiodic signal is spread across all time.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n + N] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$X[k] = X[k + N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_0 kn}$$

Analysis equation

$$\Omega_0 = \frac{2\pi}{N}$$

Fourier Series and Fourier Transform

Harmonic frequencies $k\Omega_0$ are samples of continuous frequency Ω

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n + N] = \sum_{k=\langle N \rangle} X[k] e^{j\Omega_0 k n}$$

Synthesis equation

$$X[k] = X[k + N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_0 k n}$$

Analysis equation

$$\Omega_0 = \frac{2\pi}{N}$$

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{j\Omega n}$.
Because of aliasing, we need only integrate $d\Omega$ over a 2π interval.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Analysis equation

Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

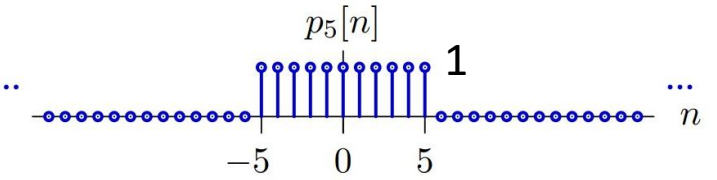
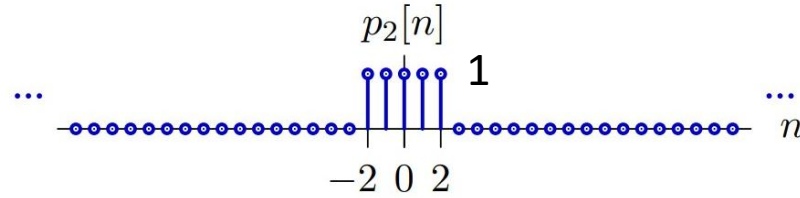
Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

Analysis equation

Fourier Transform of a Rectangular Pulse (width $2S+1$)

$$p_S[n] = \begin{cases} 1 & -S \leq n \leq S \\ 0 & \text{otherwise} \end{cases}$$



$$P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n] \cdot e^{-j\Omega n} = \sum_{n=-S}^S e^{-j\Omega n} \stackrel{\text{let } m = n + S, n = m - S}{=} e^{j\Omega S} \sum_{m=0}^{2S} e^{-j\Omega m}$$

Participation question for Lecture

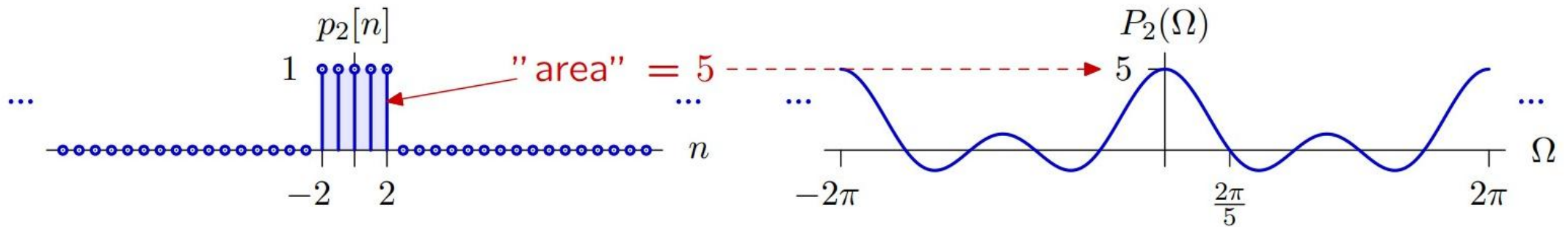
When $\Omega = 0$, (or $2k\pi$), $P_S(\Omega) = ?$

When $\Omega \neq 0$ or $2k\pi$

Fourier Transform of a rectangular pulse

Similar to CT, the value of $X(\Omega)$ at $\Omega = 0$ is the sum of $x[n]$ over all time.

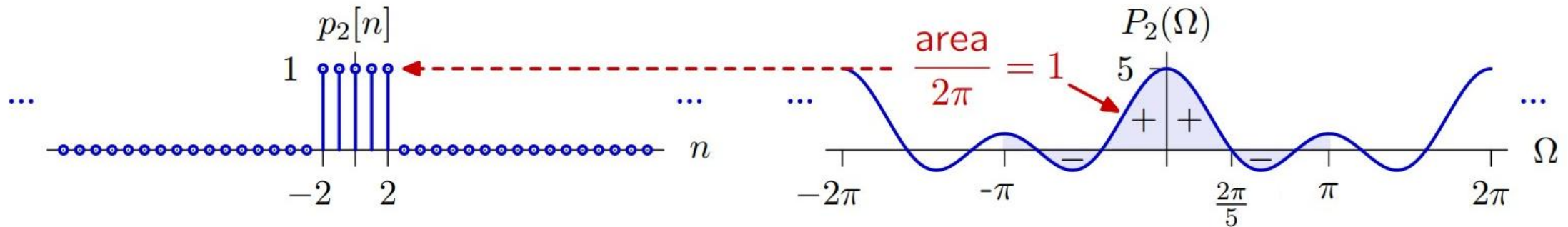
$$X(0) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]$$



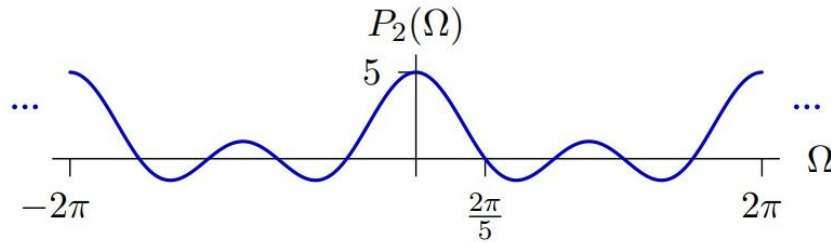
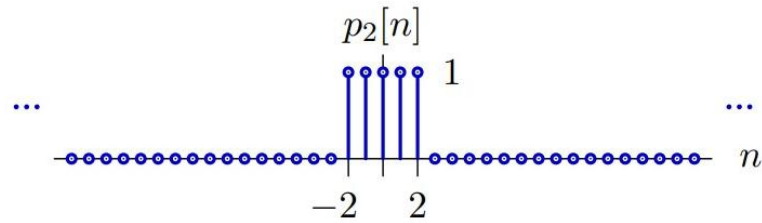
Fourier Transform of a rectangular pulse

The value of $x[0]$ is $1/2\pi$ times the integral of $X(\Omega)$ over $\Omega = [-\pi, \pi]$.

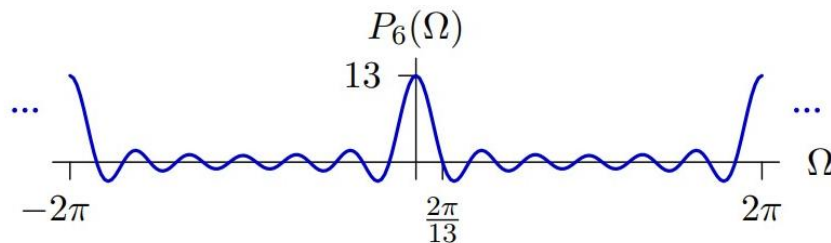
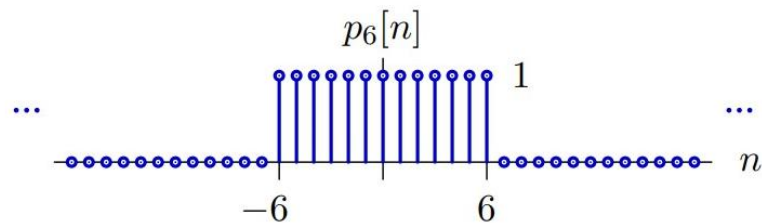
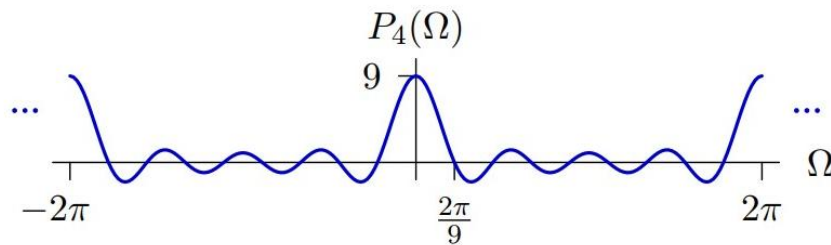
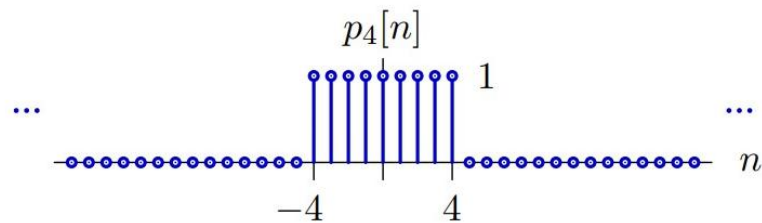
$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) d\Omega$$



Fourier Transforms of Pulses with Different Widths



$$P_S(\Omega) = \frac{\sin(\Omega (S + \frac{1}{2}))}{\sin(\frac{\Omega}{2})}$$



As the function widens in n (time) the Fourier transform narrows in Ω (freq).

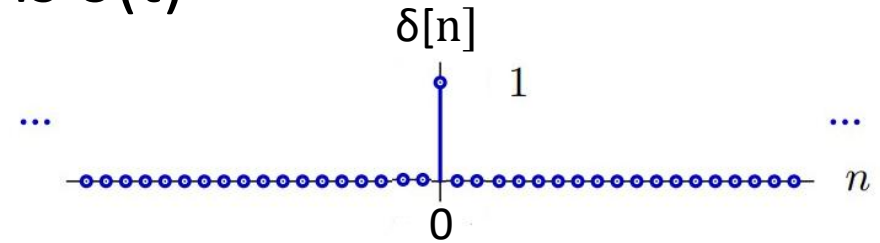
How about going the other way?

In the extreme of $S=0$, the signal becomes a unit impulse $\delta[n]$

DT Impulse

The DT impulse is $\delta[n]$, its CT equivalent is $\delta(t)$

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$



The DTFT of $\delta[n]$:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] \cdot e^{-j\Omega n} = 1$$

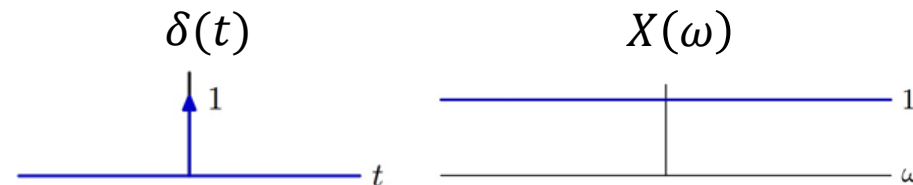
$\delta[n]$ still has the “sifting property:”

$$\sum_{n=-\infty}^{\infty} \delta[n - a] f[n] = f[a]$$

In comparison to its CT counterpart $\delta(t)$:

$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{0-}^{0+} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - a) f(t) dt = f(a)$$



Special Cases

The Fourier transform of the shortest possible CT signal $f(t) = \delta(t)$ is the widest possible CT transform $F(\omega) = 1$.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0} = 1$$

A similar result holds in DT.

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega 0} = 1$$

Special Cases

The Fourier transform of the widest possible CT signal $f(t) = 1$ is the narrowest possible CT transform $F(\omega) = 2\pi\delta(\omega)$.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{-j\omega t} d\omega = \int_{-\infty}^{\infty} \delta(\omega) e^{-j0t} d\omega = 1$$

A similar result holds in DT.

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{-j\Omega n} d\Omega = \frac{1}{2\pi} \int_{2\pi} 2\pi\delta(\Omega) e^{-j\Omega n} d\Omega = \int_{2\pi} \delta(\Omega) e^{-j0n} d\Omega = 1$$

Unit Impulse in Frequency Domain

Because DT Fourier Transforms are periodic in 2π , it becomes an impulse train repeated every 2π .

$$1 \xLeftrightarrow{\text{DTFT}} \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi m)$$

This is in contrast to the CT case:

$$1 \xLeftrightarrow{\text{CTFT}} 2\pi\delta(\omega)$$

Math With Impulses

This is what we learned previously:

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_o) e^{j\omega t} d\omega \\ &= \int_{-\infty}^{\infty} \delta(\omega - \omega_o) e^{j\omega_o t} d\omega \\ &= e^{j\omega_o t} \int_{-\infty}^{\infty} \delta(\omega - \omega_o) d\omega \\ &= e^{j\omega_o t} \end{aligned}$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\omega_o t} \xrightarrow{\text{CTFT}} 2\pi \delta(\omega - \omega_o)$$

The impulse in frequency has infinite value at $\omega = \omega_o$ and is zero at all other frequencies.

Math With Impulses

A similar construction applies in DT.

$$\begin{aligned} f[n] &= \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{2\pi} 2\pi \delta(\Omega - \Omega_o) e^{j\Omega n} d\Omega \\ &= \int_{2\pi} \delta(\Omega - \Omega_o) e^{j\Omega_o n} d\Omega \\ &= e^{j\Omega_o n} \int_{2\pi} \delta(\Omega - \Omega_o) d\Omega \\ &= e^{j\Omega_o n} \end{aligned}$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\Omega_o n} \xrightarrow{\text{DTFT}} 2\pi \delta(\Omega - \Omega_o)$$

The impulse in frequency shows that the transform is infinite at $\Omega = \Omega_o$ and is zero at all other frequencies.

Relations Between Fourier Series and Fourier Transforms

If a periodic signal $f(t) = f(t + T)$ has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

$$e^{j\omega_0 t} \xrightarrow{\text{FT}} 2\pi\delta(\omega - \omega_0)$$

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \begin{array}{c} \text{CTFS} \\ \longleftrightarrow \end{array} \quad F[k]$$

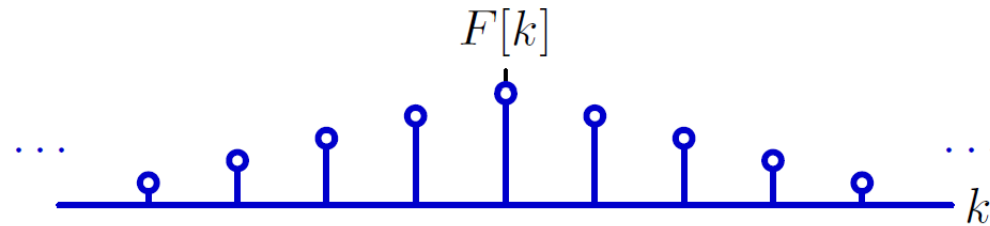
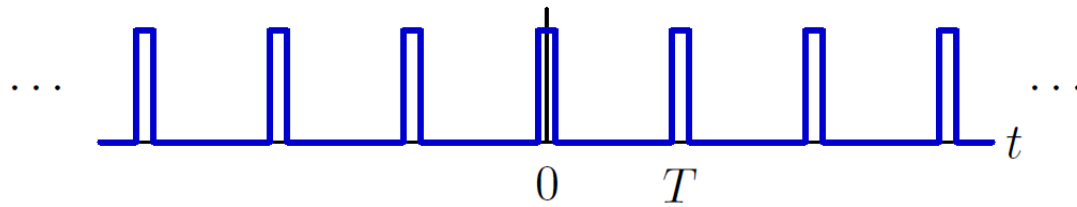
$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \begin{array}{c} \text{CTFT} \\ \longleftrightarrow \end{array} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta\left(\omega - \frac{2\pi}{T}k\right)$$

Each term in the Fourier series is replaced by an impulse in the Fourier transform.

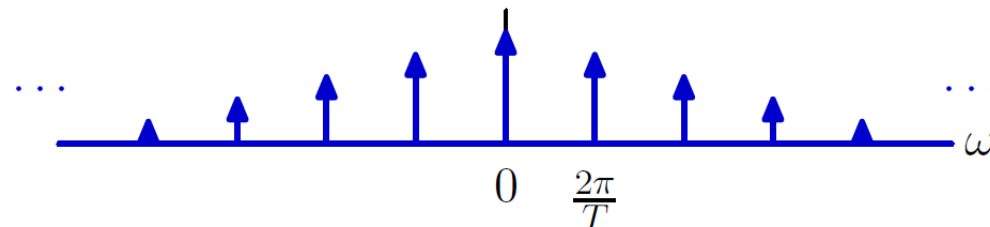
Relations Between Fourier Series and Fourier Transforms

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$$f(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$

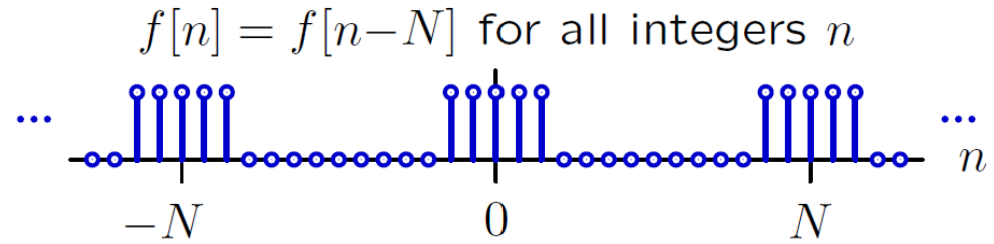


$$F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta(\omega - k \frac{2\pi}{T})$$

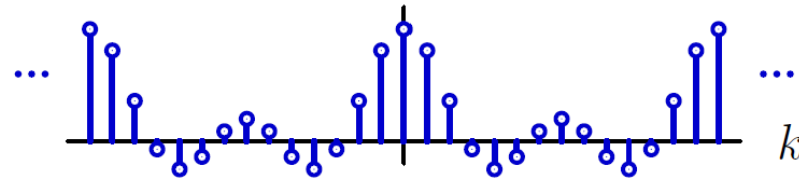


Relations Between Fourier Series and Fourier Transforms

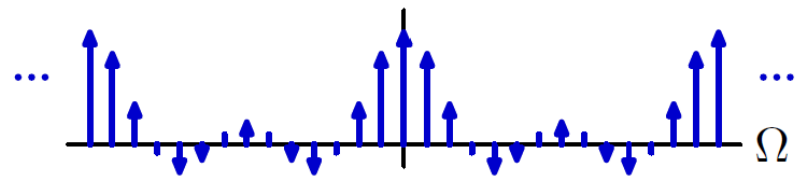
Each Fourier series term is replaced by an impulse in the Fourier transform.



$F[k] = F[k-N]$ for k goes from $-N/2$ to $N/2$



$$F(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k] \delta\left(\Omega - k \frac{2\pi}{N}\right)$$



Periodic DT signals that have Fourier series representations also have Fourier transform representations.

Summary

- Discrete-Time Fourier Transform: Fourier representation to all DT signals!

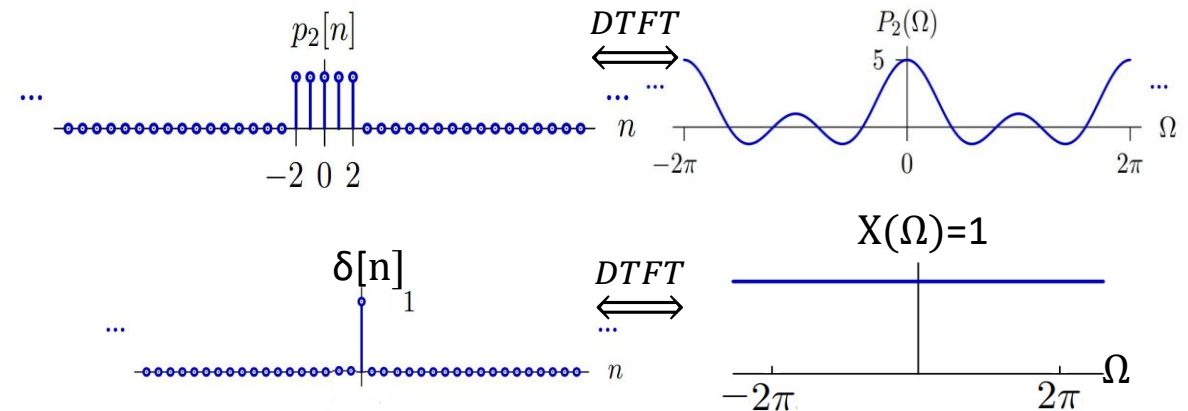
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j\Omega n} d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Analysis equation

- Very useful signals:
 - Rectangular pulse and its FT(sinc)
 - Delta function (Unit impulse) and its FT



- If a periodic signal $f[n] = f[n + N]$ has a Fourier Series representation, then it can also be represented by an equivalent Fourier Transform.