6.300 Signal Processing

Week 3, Lecture B:
Discrete Time Fourier Series

- Fourier series representations for discrete-time signals
- CTFS vs DTFS
- Application of DTFS

Quiz 1:
Tuesday September 30,
2-4pm 50-340

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall25

Brief Review

Continuous Time Fourier Series

Synthesis:

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}$$

Analysis:

$$X[k] = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi kt}{T}} dt$$

Discrete Time Sinusoids

$$x[n] = A\cos(\Omega n + \Phi)$$

- n is always an integer!
- Aliasing and base-band

Today: Apply the FS ideas to DT signals and introduce the **DT Fourier Series**

Check Yourself

What is the fundamental (shortest) period of each of the following DT signals?

1. $f_1[n] = \cos\left(\frac{\pi n}{12}\right)$
2. $f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right)$
3. $f_3[n] = \cos(n)$

- The period N of a periodic DT signal must be an integer.
- While this is not surprising, it leads to an interesting consequence.

Recall: Continuous-time Fourier Series

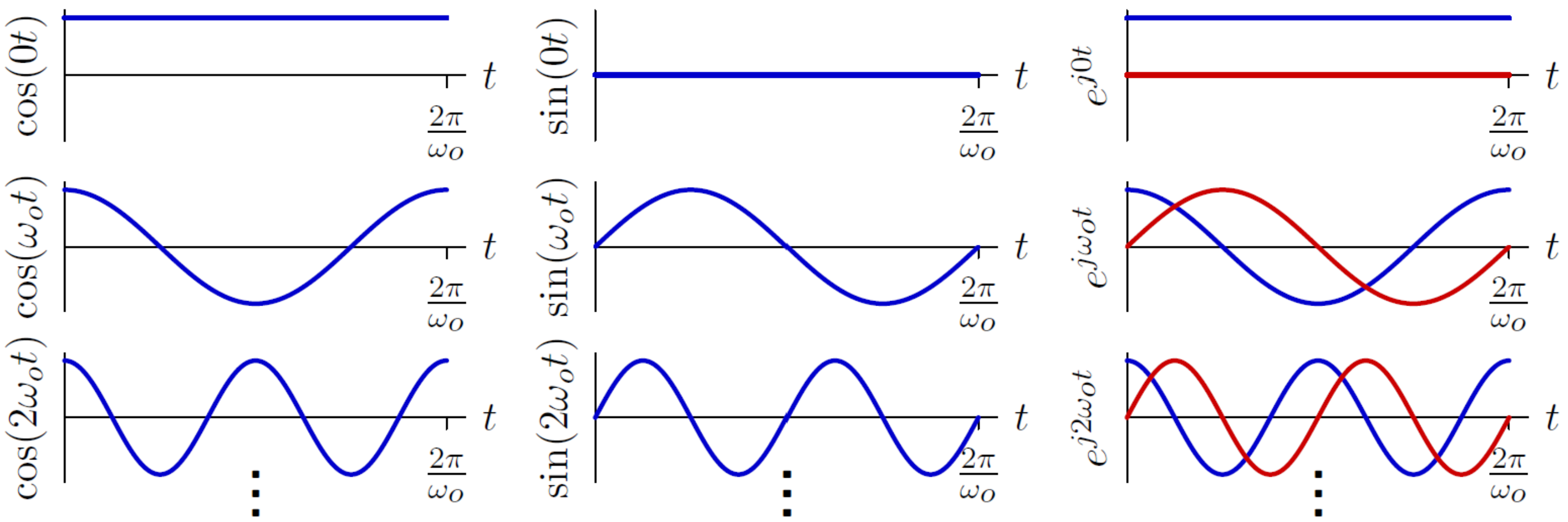
Only **periodic** signals can be represented by Fourier series.

$$f(t) = f(t+T) \;=\; c_0 + \sum_{k=1}^{\infty} c_k \cos k\omega_o t + \sum_{k=1}^{\infty} d_k \sin k\omega_o t \;=\; \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \;=\; \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi k}{T}t}$$

where $\omega_o = \frac{2\pi}{T}$ represents the fundamental frequency.

Real-valued basis functions

Complex basis functions



What is the equivalent constraint for discrete-time signals?

Number of Harmonics

- In the case of CTFS, there can be infinite number of harmonics,

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k t}{T}}$$

- For DT signals with period N, as Ω_o is a submultiple of 2π, there are (only) N distinct complex exponentials $e^{j\Omega_0 kn}$. The rest harmonics alias.

Example of $\mathrm{N} = 8$: $\Omega_0 = \frac{2\pi}{8} = \frac{\pi}{4}$

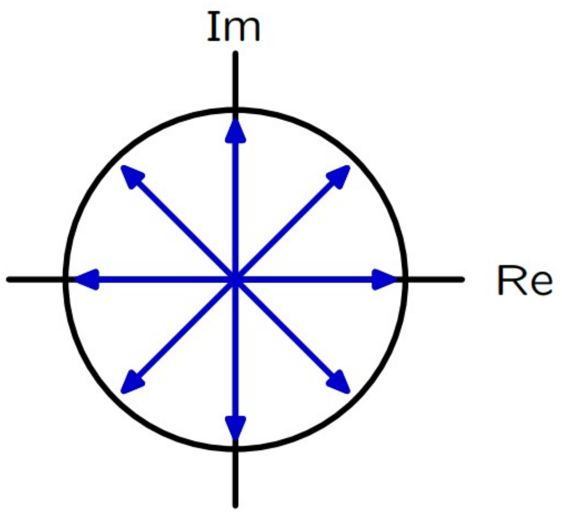
There are only 8 unique harmonics($k\Omega_0$):

$$\frac{0\pi}{4}, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4}$$

or

$$-\frac{3\pi}{4}, -\frac{2\pi}{4}, -\frac{\pi}{4}, \frac{0\pi}{4}, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}$$

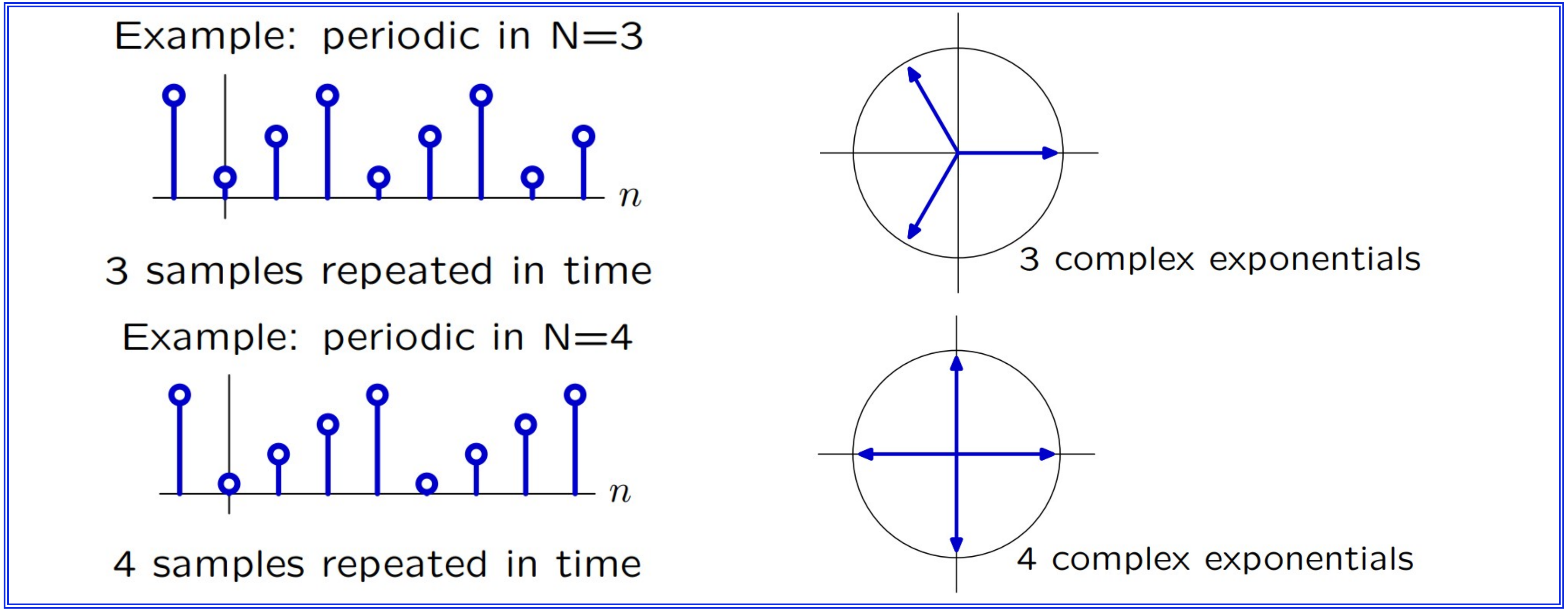
…



Finitely-many Unique Harmonics

There are N distinct complex exponentials with period N.

If a DT signal is periodic with period N, then its Fourier series contains just N terms.



Recall: Continuous-Time Fourier Series

We found the Fourier series coefficients using two key insights.

1. Multiplying complex harmonics of ω_o yields a complex harmonic of ω_o:

$$e^{jk\omega_o t} \times e^{jl\omega_o t} = e^{j(k+l)\omega_o t}$$

2. Integrating a complex harmonic over a period T yields zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_o t} dt \equiv \int_T e^{jk\omega_o t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$= T\delta[k]$$

where $\delta[k]$ is the Kronecker delta function

$$\delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ Fourier components are **orthogonal**.

Discrete-Time Fourier Series

The same two key insights apply to DT analysis.

1. Multiplying complex **DT** harmonics of Ω_o yields a new harmonic of Ω_o:

$$e^{jk\Omega_o n} \times e^{jl\Omega_o n} = e^{j(k+l)\Omega_o n}$$

2. **Summing** a complex harmonic over a period N is zero unless the harmonic is at DC:

$$\sum_{n=n_0}^{n_0+N-1} e^{jk\Omega_o n} \equiv \sum_{n=\langle N\rangle} e^{jk\Omega_o n} = \begin{cases} N & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases} = \begin{cases} N & \text{if } k=0 \\ \dfrac{1-(e^{j\Omega_0 k})^N}{1-e^{j\Omega_0 k}} = 0 & \text{if } k \neq 0 \end{cases}$$

$$= N\delta[k]$$

$\rightarrow$ **DT** Fourier components are **orthogonal**.

Discrete Time Fourier Series

DT Fourier series comprise a weighted sum of just N harmonics.

$$x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\Omega_0 kn}$$

How to find the weights?

DT Fourier components are also orthogonal:

$$\sum_{n=n_0}^{n_0+N} e^{j\Omega_0 kn} \cdot e^{-j\Omega_0 mn} = \sum_{n=<N>} e^{j\Omega_0 (k-m)n} = \sum_{n=0}^{N-1} \left(e^{j\Omega_0 (k-m)}\right)^n$$

$$= \begin{cases} N & if\ k = m \\ \dfrac{1-\left(e^{j\Omega_0 (k-m)}\right)^N}{1-e^{j\Omega_0 (k-m)}} = 0 & if\ k \neq m \end{cases} = N \cdot \delta[k-m]$$

Finding DTFS coefficient

Start with DTFS representation:

$$x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\Omega_0 kn}$$

Then "sift" out one component $X[l]$:

$$\sum_{n=<N>} x[n] e^{-j\Omega_0 ln} = \sum_{n=<N>} \sum_{k=<N>} X[k] e^{j\Omega_0 kn} e^{-j\Omega_0 ln} = \sum_{k=<N>} X[k] \sum_{n=<N>} \left(e^{j\Omega_0 (k-l)}\right)^n$$

$$= \sum_{k=<N>} X[k] \cdot N \cdot \delta[k-l]$$

$$= N \cdot X[l]$$

$$X[k] = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\Omega_0 kn}$$

Periodicity with Fourier Series Coefficient $X[k]$

Consider a signal x[n] that is periodic in N, and consider finding the (k + N)th Fourier Series coefficient:

$$X[k] = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\Omega_0 kn}$$

$$\begin{aligned}
X[k+N] &= \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi(k+N)n}{N}} \\
&= \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi kn}{N}} e^{-j\frac{2\pi Nn}{N}} \\
&= \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi kn}{N}} e^{-j2\pi n} \\
&= \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi kn}{N}} \\
&= X[k]
\end{aligned}$$

Discrete Time Fourier Series

A periodic DT signal with N samples produces a periodic sequence of N Fourier series coefficients.

$$x[n] = x[n+N] = \sum_{k=k_0}^{k_0+N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$X[k] = X[k+N] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\Omega_0 kn}$$

DTFS has just N coefficients, whereas CTFS had infinitely many!

Fourier Series Summary

CT and DT Fourier Series are similar, but DT Fourier Series have just N coefficients while CT Fourier Series have an infinite number.

Continuous-Time Fourier Series

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k t}{T}}$$

Synthesis equation

$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi k t}{T}} dt$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n+N] = \sum_{k=k_0}^{k_0+N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$\Omega_0 = \frac{2\pi}{N}$$

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\Omega_0 kn}$$

Analysis equation

Check yourself

- What are the Fourier Series coefficients of the following signal?

$$x[n] = \sum_{k=<N>} X[k]e^{j\Omega_0 kn}$$

$$x[n] = 1 + \cos(\frac{2\pi}{5}n)$$

$$X[k] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\Omega_0 kn}$$

Check yourself

- What are the Fourier Series coefficients of the following signal?

$$x[n] = 1 + \sin(\frac{\pi}{4}n)$$

Participation question for Lecture

Check yourself

- What are the Fourier Series coefficients of the following signal?

$$x[n] = \begin{cases} 1 & \text{if } n \bmod 10 \equiv 0 \\ 0 & \text{otherwise} \end{cases}$$

Check yourself

- What are the Fourier Series coefficients of the following signal with a period of N=10?

$$x[n] = 0.5$$

Properties of DTFS: Linearity

- Consider $y[n] = Ax_1[n] + Bx_2[n]$, where $x_1[n]$ and $x_2[n]$ are periodic in N. What are the DTFS coefficients $Y[k]$, in terms of $X_1[k]$ and $X_2[k]$?

First, $y[n]$ must also be periodic in N

$$Y[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} (Ax_1[n] + Bx_2[n])e^{-j\frac{2\pi}{N}kn}$$

$$= A\frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x_1[n]e^{-j\frac{2\pi}{N}kn} + B\frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x_2[n]e^{-j\frac{2\pi}{N}kn}$$

$$= AX_1[k] + BX_2[k]$$

If $y[n] = Ax_1[n] + Bx_2[n], then\ Y[k] = AX_1[k] + BX_2[k]$

Properties of DTFS: Time flip

- Consider $y[n] = x[-n]$, where $x[n]$ is periodic in N. What are the DTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y[n]$ must also be periodic in N

$$Y[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[-n]e^{-j\frac{2\pi k}{N}n}$$

Let $m = -n$

$$Y[k] = \frac{1}{N}\sum_{m=-n_0}^{-(n_0+N-1)} x[m]e^{-j\frac{2\pi k}{N}(-m)}$$

$$= \frac{1}{N}\sum_{m=-n_0}^{-n_0-N+1} x[m]e^{-j\frac{2\pi(-k)}{N}m} = X[-k]$$

If $y[n] = x[-n]$, $then\ Y[k] = X[-k]$

Flipping in time flips in frequency.

Properties of DTFS: Time Shift

- Consider $y[n] = x[n-m]$, where $x[n]$ is periodic in N, m is an integer. What are the DTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y[n]$ must also be periodic in N

$$Y[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n-m]e^{-j\frac{2\pi k}{N}n}$$

Let $l = n - m, then\ \ n = l + m$

$$Y[k] = \frac{1}{N}\sum_{l=n_0-m}^{n_0-m+N-1} x[l]e^{-j\frac{2\pi k}{N}(l+m)} = e^{-j\frac{2\pi k}{N}m}\cdot\frac{1}{N}\cdot\sum_{l=n_0-m}^{n_0-m+N-1} x[l]e^{-j\frac{2\pi k}{N}l}$$

$$= e^{-j\frac{2\pi k}{N}m}\cdot X[k]$$

If $y[n] = x[n-m],\ \ then\ Y[k] = e^{-j\frac{2\pi km}{N}}X[k]$

Shifting in time changes phase of Fourier Series Coefficient.

Properties of DTFS: Complex-conjugate Coefficients

If x[n] is real-valued periodic signal, $X[k] = X^*[-k]$.

$$X[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

$$X[-k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi(-k)}{N}n}$$

$$X[-k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{j\frac{2\pi k}{N}n}$$

$$X^*[-k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi k}{N}n} = X[k]$$

Properties of DTFS: Symmetric and Antisymmetric Parts

- A real-valued signal $x[n]$ written in terms of the symmetric and antisymmetric parts: $x[n] = x_S[n] + x_A[n]$

$$x_S[n] = \frac{1}{2}(x[n] + x[-n]) \overset{\text{DTFS}}{\Longleftrightarrow} \frac{1}{2}(X[k] + X[-k]) = \frac{1}{2}(X[k] + X^*[k])$$

$$= Re(X[k])$$

$$x_A[n] = \frac{1}{2}(x[n] - x[-n]) \overset{\text{DTFS}}{\Longleftrightarrow} \frac{1}{2}(X[k] - X[-k]) = \frac{1}{2}(X[k] - X^*[k])$$

$$= j \cdot Im(X[k])$$

The real part of X[k] comes from the symmetric part of the signal,
the imaginary part of X[k] comes from the antisymmetric part of the signal

Example of a DTFS: Sirens

Seebeck used a siren to generate sounds (~1841) by passing a jet of compressed air through holes in a spinning disk.

Example of a DTFS: Sirens

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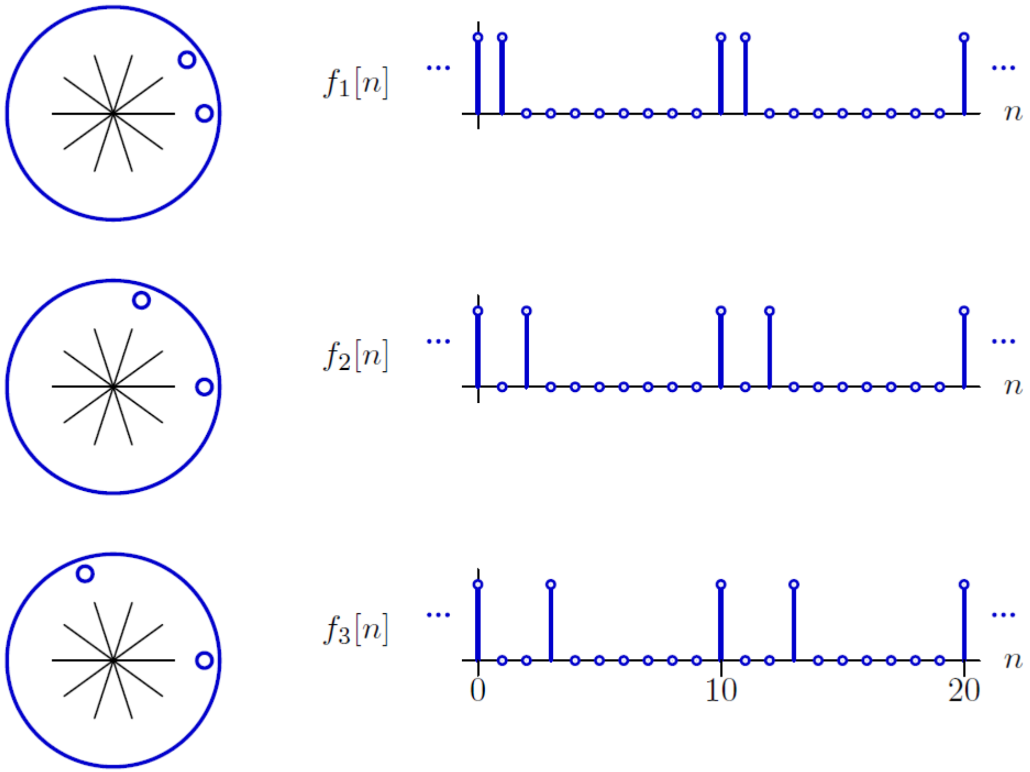
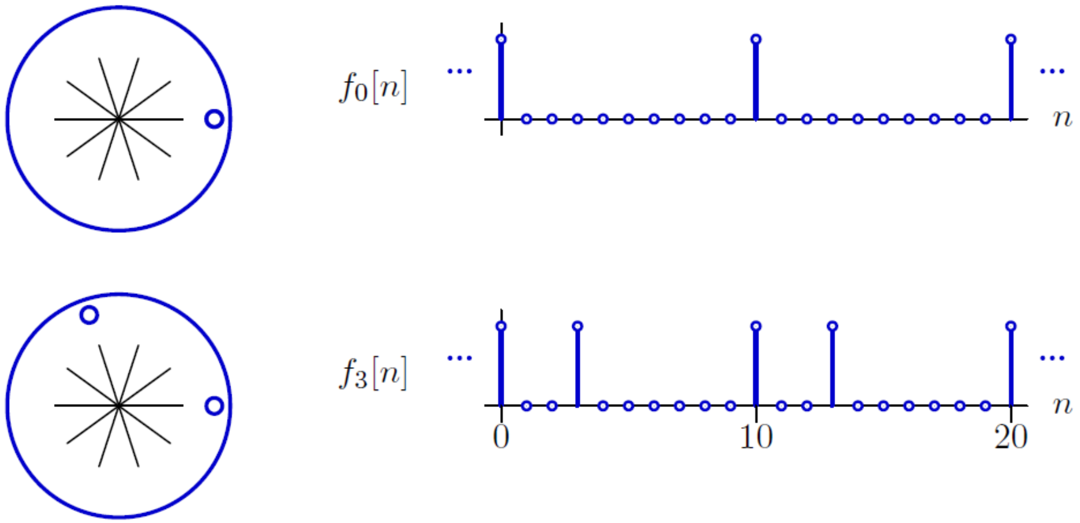


The pattern of holes determined the pattern of pulses in each period. The speed of spinning controlled the number of periods per second.

Example of a DTFS: Sirens

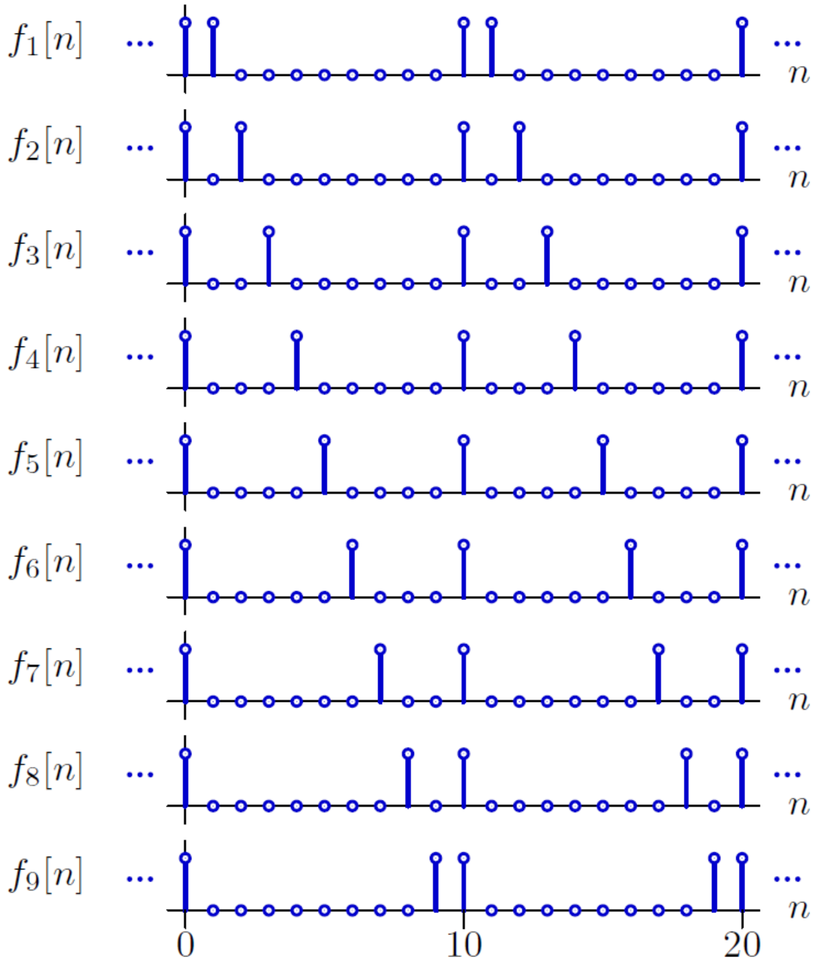
Strangely, adding a second hole per period didn't seem to affect the pitch.



Pitch should be different if it is determined by the intervals between pulses.

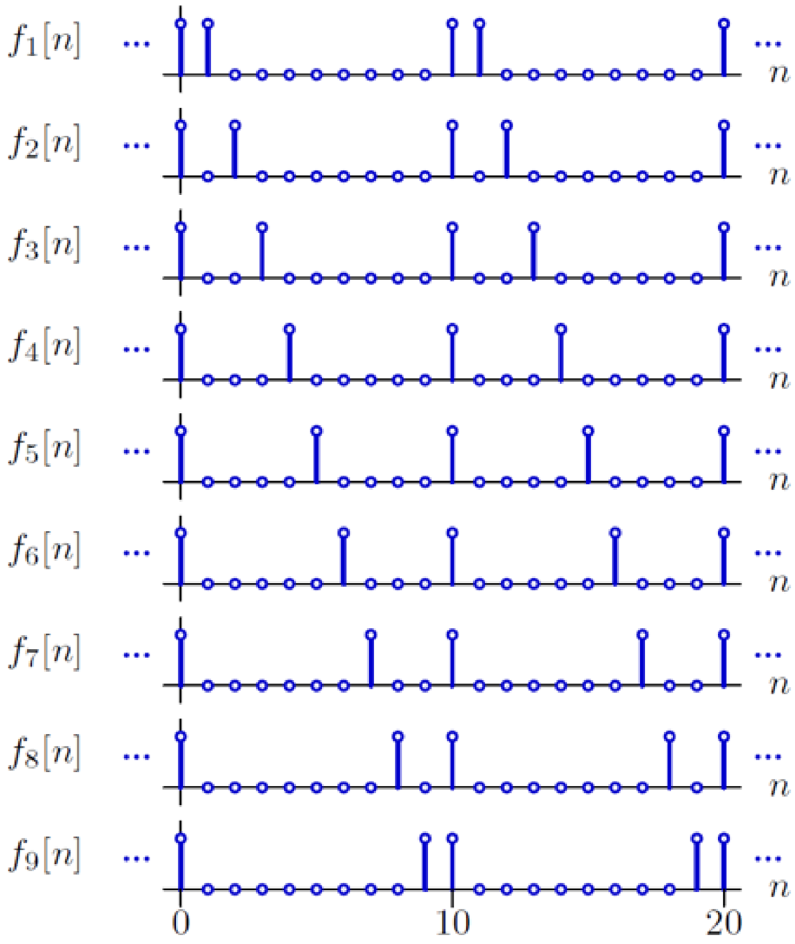
Example of a DTFS: Sirens

There was one very interesting exception.



But hearing this exception required precise alignment of the siren's holes.

Fourier Interpretation



Find the k^{th} coefficient of the i^{th} signal.

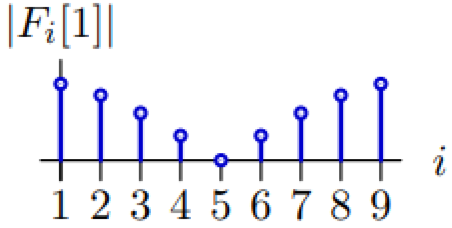
$$F_i[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f_i[n]\, e^{-j\frac{2\pi k}{N}n} = \frac{1}{10} \sum_{n=0}^{9} f_i[n]\, e^{-j\frac{2\pi k}{10}n} = \frac{1}{10}\left(1 + e^{-j\frac{2\pi k}{10}i}\right)$$

DC: the $k=0$ term is $2/10$ for all i

$$F_i[0] = \frac{1}{10}\left(1 + e^{-j\frac{2\pi 0}{10}i}\right) = \frac{2}{10}$$

Fundamental: $k=1$ term

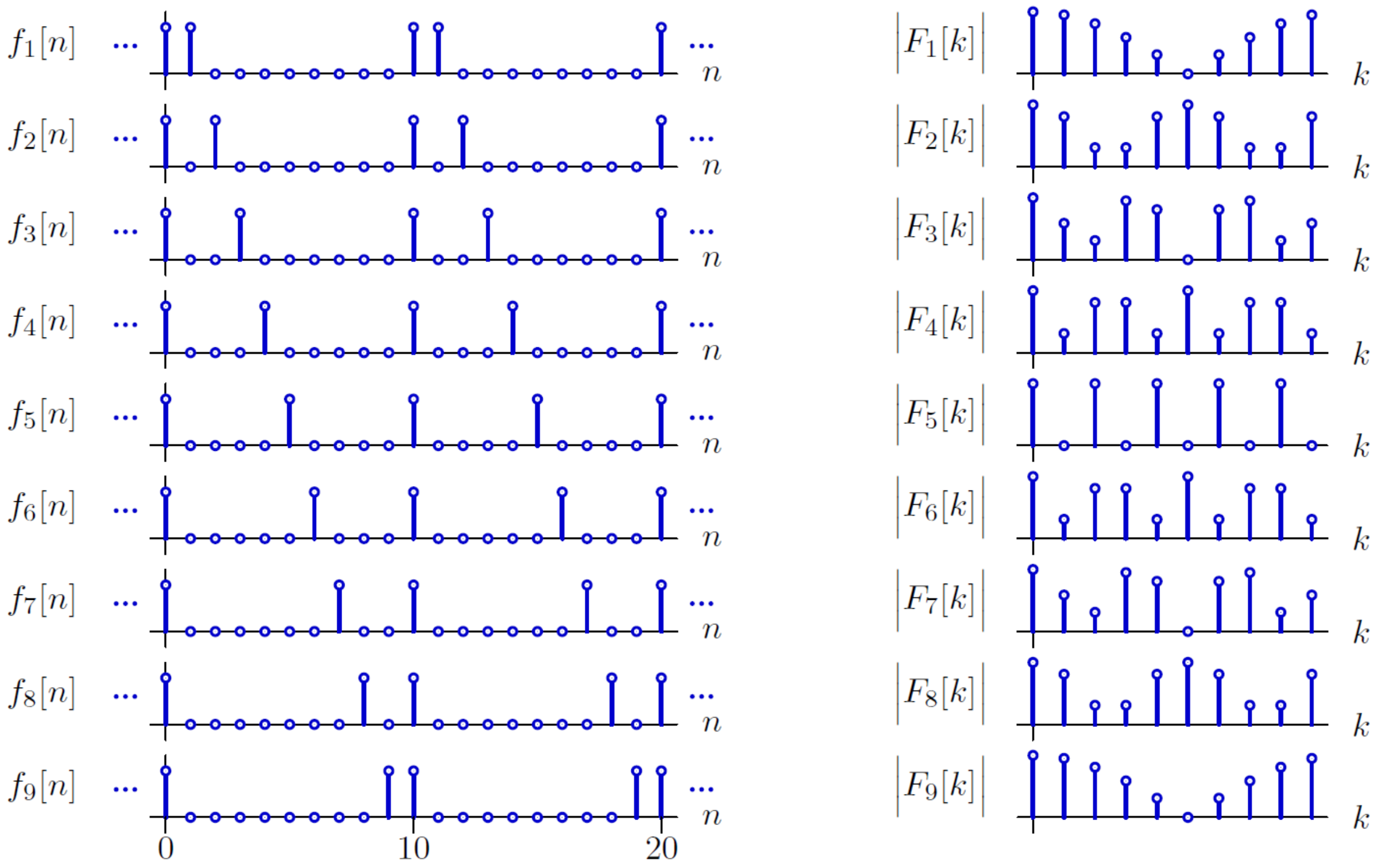
$$F_i[1] = \frac{1}{10}\left(1 + e^{-j\frac{2\pi}{10}i}\right)$$



Notice that $f_5[n]$ has no fundamental component!

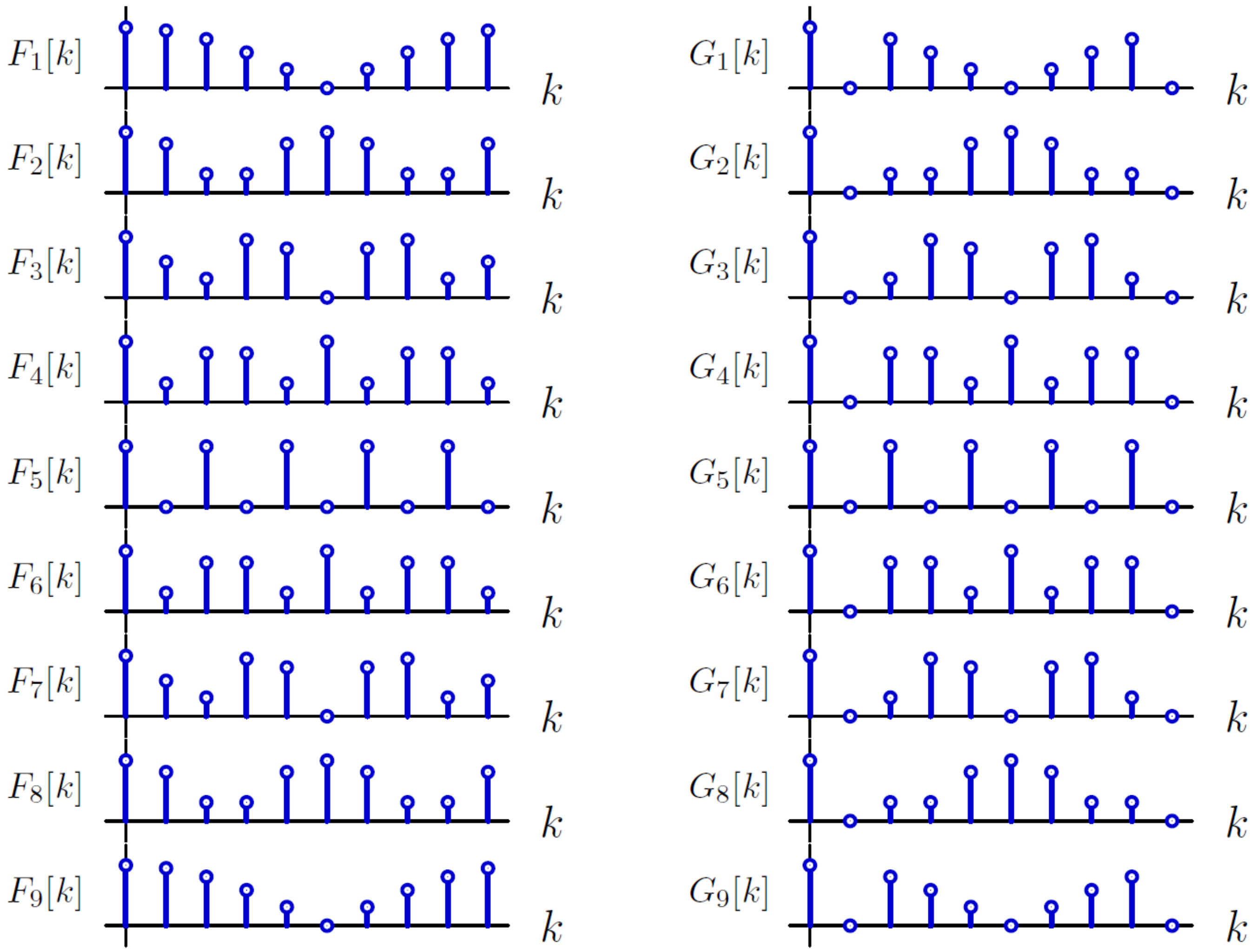
Fourier Series

Notice that $f_5[n]$ has no fundamental component!



Fourier Series With and Without the Fundamental

Resynthesize each waveform without its fundamental component.



Although perception of the fundamental is weakened, it is not gone!

Fourier Series With and Without the Fundamental

Seebeck designed an extremely clever **experiment** to test pitch perception.

Ohm analyzed an important **theory** (from Fourier) and argued that harmonics are present even in the pulsatile sounds generated by a siren.

Neither Seebeck nor Ohm could convincingly account for experimental results that demonstrated the dominance of the fundamental, even when it was weak or missing.

Progress in understanding the "missing fundamental" awaited Helmholtz, who demonstrated the importance of "combination tones" in the ear.

Summary

- We developed Fourier Series for discrete time signals.
- We compared CTFS and DTFS.
- Discrete-Time Fourier Series:

$$x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\frac{2\pi}{N}kn}$$

$$X[k] = X[k+N] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\Omega_0 kn}$$

discrete in time => periodic in frequency