6. 300 Signal Processing

Week 4, Lecture A:
Continuous Time Fourier Transform

- Definition
- Example
- Impulse function δ(t)

Quiz 1: Tuesday September 30, 2-4pm 50-340

- Closed book except for one page of written notes (8.5'' x 11'' both sides)
- No electronic devices (No headphones, cell phones, calculators, …)
- Coverage up to Week #3 (DTFS); no HW4, a practice quiz will be put on our website.
- Quiz review session: 9/28 1-3pm in 4-370

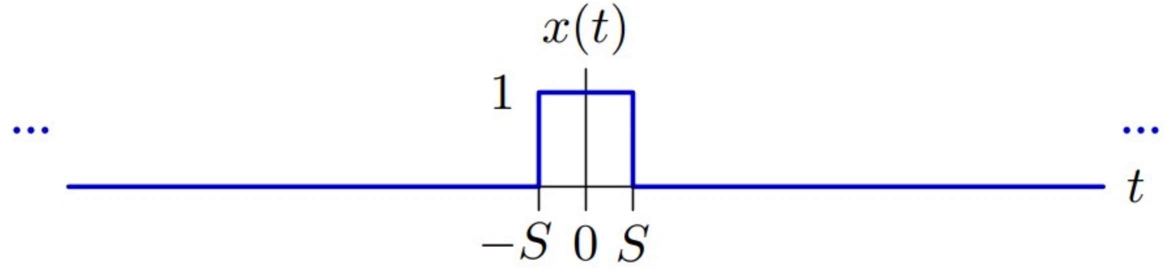
From Periodic to Aperiodic

- Previously, we have focused on Fourier representations of periodic signals: e.g., sounds, waves, music, ...
- However, most real-world signals are not periodic.

Today: generalizing Fourier representations to include aperiodic signals -> **Fourier Transform**

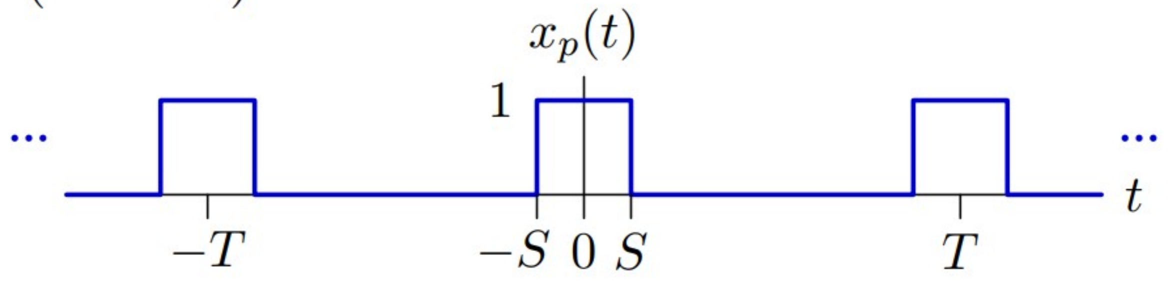
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Strategy: make a periodic version of $x(t)$ by summing shifted copies:

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$$



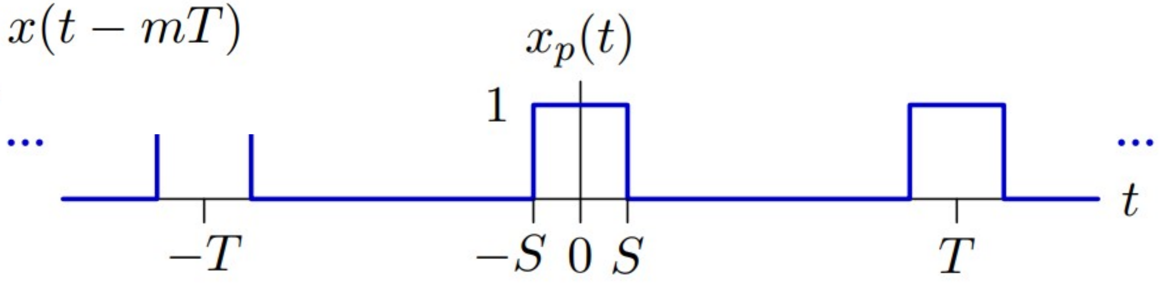
Since $x_p(t)$ is periodic, it has a Fourier series (which depends on T)

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as T → ∞

As T → ∞, $x_p(t) \to x(t)$ and Fourier series will approach Fourier transform.

Fourier Representations of Aperiodic Signals

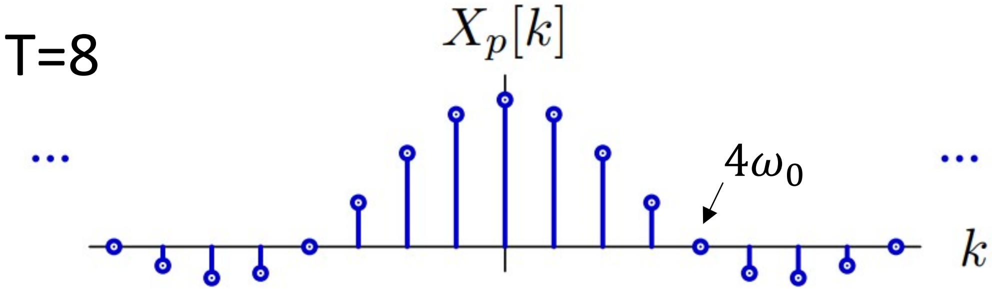
$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$$



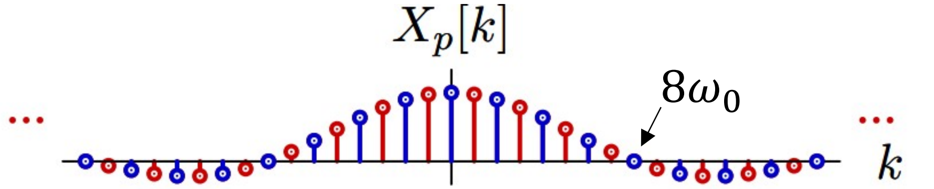
Calculate the Fourier series coefficients $X_p[k]$: $X_p[k] = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) \cdot e^{-j\frac{2\pi}{T}kt}\, dt$

$$X_p[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-j\frac{2\pi}{T}kt}\, dt = \frac{1}{T}\int_{-S}^{S} 1 \cdot e^{-j\frac{2\pi}{T}kt}\, dt = \frac{1}{T} \cdot \left.\frac{e^{-j\frac{2\pi}{T}kt}}{(-j2\pi k/T)}\right|_{-S}^{S} = \frac{2\sin(\frac{2\pi k}{T}S)}{T(\frac{2\pi k}{T})}$$

Plot the resulting Fourier coefficients when S=1 and T=8



What happens if you double the period T?

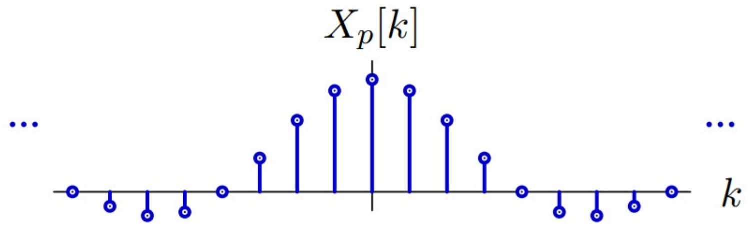


There are twice as many samples per period of the sinc function

The red samples are at new intermediate frequencies

Fourier Representations of Aperiodic Signals

$$X_p[k] = \frac{2\sin(\frac{2\pi k}{T}S)}{T(\frac{2\pi k}{T})}$$

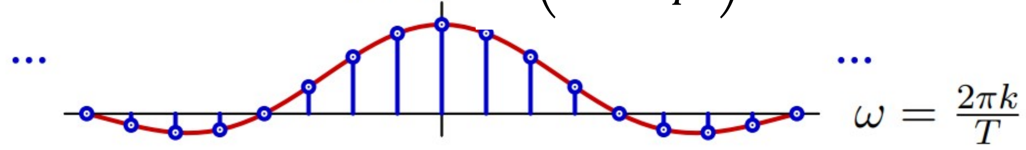


$let\ \omega = \frac{2\pi k}{T}$, Define a new function $X(\omega) = T \cdot X_p[k]$ $= 2\frac{\sin(\omega S)}{\omega}$

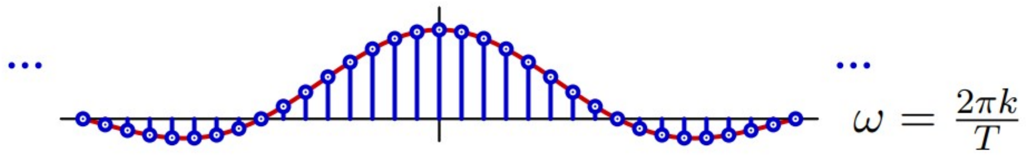
If we consider ω and $X(\omega) = 2\frac{\sin(\omega S)}{\omega}$ to be continuous, $TX_p[k]$ represents a sampled version of the function $X(\omega)$.

$$TX_p[k] = X\left(\omega = \frac{2\pi k}{T}\right)$$

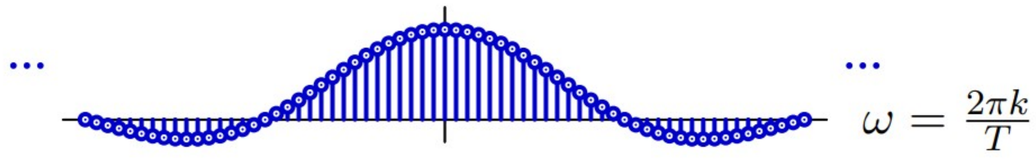
S=1 and T=8:



S=1 and T=16:



S=1 and T=32:



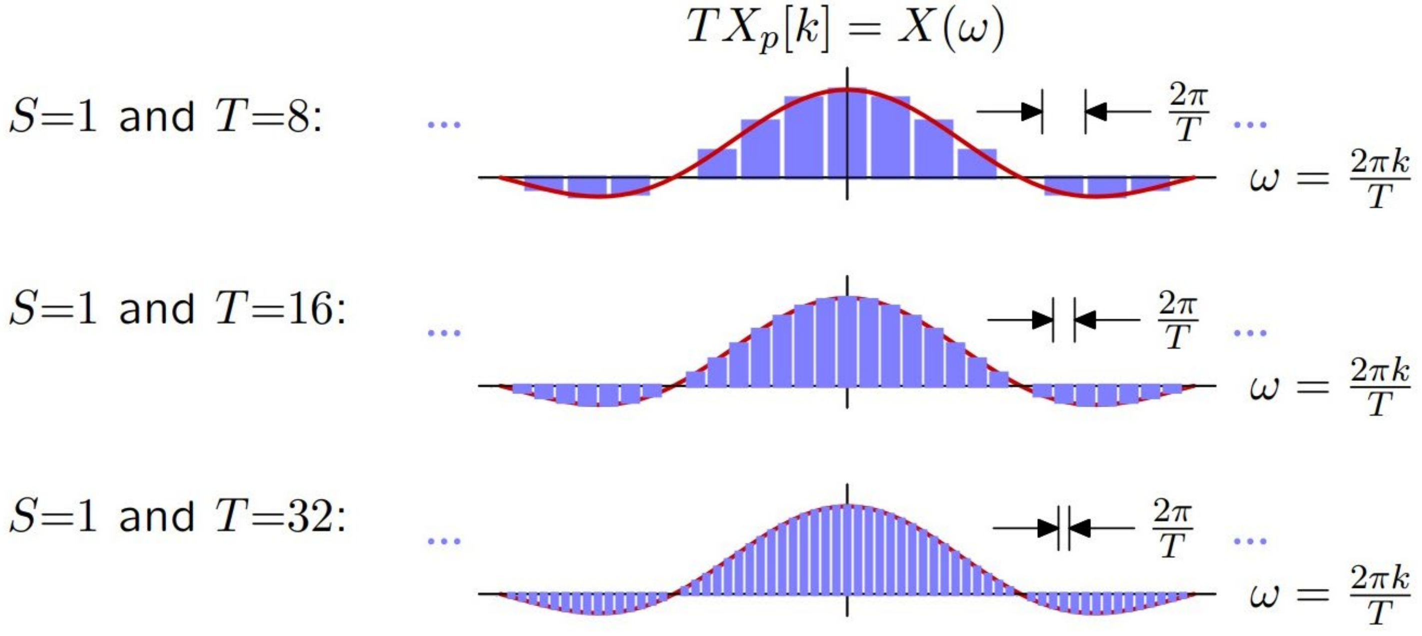
As T increases, the resolution in ω increases.

Fourier Representations of Aperiodic Signals

We can reconstruct $x(t)$ from $\mathrm{X}(\omega)$ using Riemann sums (approximating an integral by a finite sum).

$$x_p(t) = \sum_{k=-\infty}^{\infty} X_p[k] e^{j\frac{2\pi}{T}kt} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} T X_p[k] e^{j\frac{2\pi}{T}kt} \left(\frac{2\pi}{T}\right)$$

$$x(t) = \lim_{T\to\infty} x_p(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_k T X_p[k] e^{j\frac{2\pi}{T}kt} \left(\frac{2\pi}{T}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$



As T → ∞,

- $k\omega_0 = \frac{2\pi k}{T}$ becomes a continuum, $\frac{2\pi k}{T} \to \omega$.
- The sum takes the from of an integral, $\omega_0 = \frac{2\pi}{T} \to d\omega$
- We obtain a spectrum of coefficients: $X(\omega)$.

Fourier Transform

$$x(t) = \lim_{T\to\infty} x_p(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_k T X_p[k] e^{j\frac{2\pi}{T}kt} \left(\frac{2\pi}{T}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Since $\mathrm{X}(\omega) = T \cdot X_p[k]$ $\qquad X_p[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-j\frac{2\pi}{T}kt}\, dt$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

Continuous-Time Fourier Representations

Fourier series and transforms are similar:
both represent signals by their frequency content.

Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$$

Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

Analysis equation

Continuous-Time Fourier Series

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k t}{T}}$$

Synthesis equation

$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T}\int_{T} x(t) e^{-j\frac{2\pi}{T}kt} dt$$

Analysis equation

Continuous-Time Fourier Representations

Periodic signals can be synthesized from a discrete set of harmonics.
Aperiodic signals generally require all possible frequencies.

Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega$$

Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$$

Analysis equation

Continuous-Time Fourier Series

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k t}{T}}$$

Synthesis equation

$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T}\int_{T} x(t) e^{-j\frac{2\pi}{T}kt} dt$$

Analysis equation

Continuous-Time Fourier Representations

All of the information in a periodic signal is contained in one period.
The information in an aperiodic signal is spread across all time.

Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\,d\omega$$

Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\,dt$$

Analysis equation

Continuous-Time Fourier Series

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}$$

Synthesis equation

$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T}\int_{T} x(t)e^{-j\frac{2\pi}{T}kt}dt$$

Analysis equation

Continuous-Time Fourier Representations

Harmonic frequencies $k\omega_0$ are samples of continuous frequency ω

Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$$

Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

Analysis equation

Continuous-Time Fourier Series

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T}t}$$

Synthesis equation

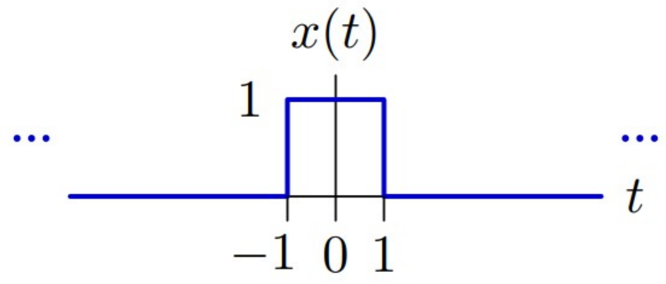
$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T}\int_{T} x(t) e^{-j\frac{2\pi}{T}kt} dt$$

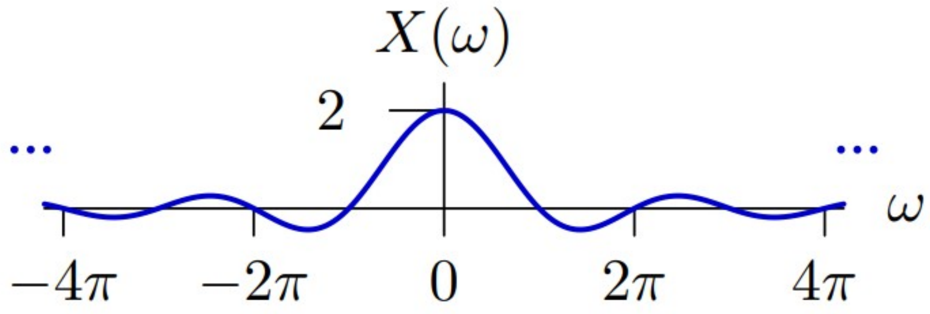
Analysis equation

Fourier Transform of a Rectangular Pulse

$$x(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

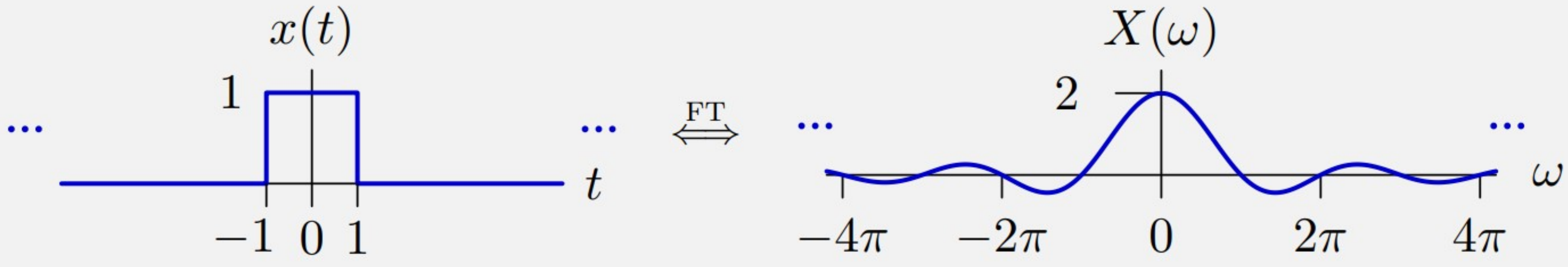


$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt = \int_{-1}^{1} 1 \cdot e^{-j\omega t}\, dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-1}^{1} = 2\frac{\sin(\omega)}{\omega}$$



Fourier Transform of a Rectangular Pulse

The Fourier transform of a rectangular pulse is $2\frac{\sin\omega}{\omega}$.

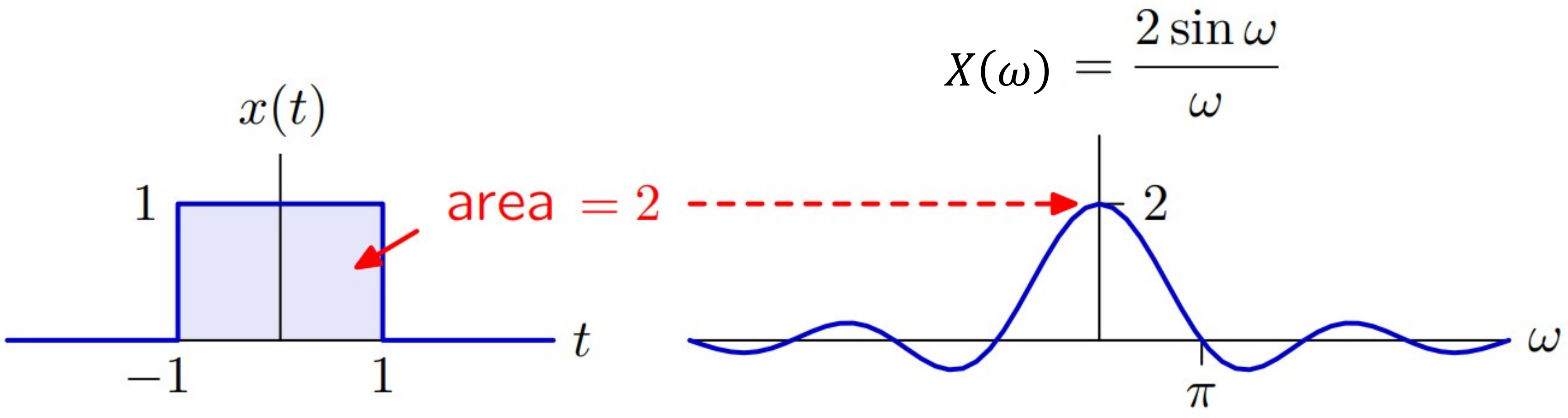


$X(\omega)$ contains all frequencies ω except non-zero multiples of π.

$$X(\omega = m\pi) = \int_{-1}^{1} e^{-j\omega t}dt = \int_{-1}^{1} e^{-jm\pi t}dt = \begin{cases} 2 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Fourier Transform of a Rectangular Pulse

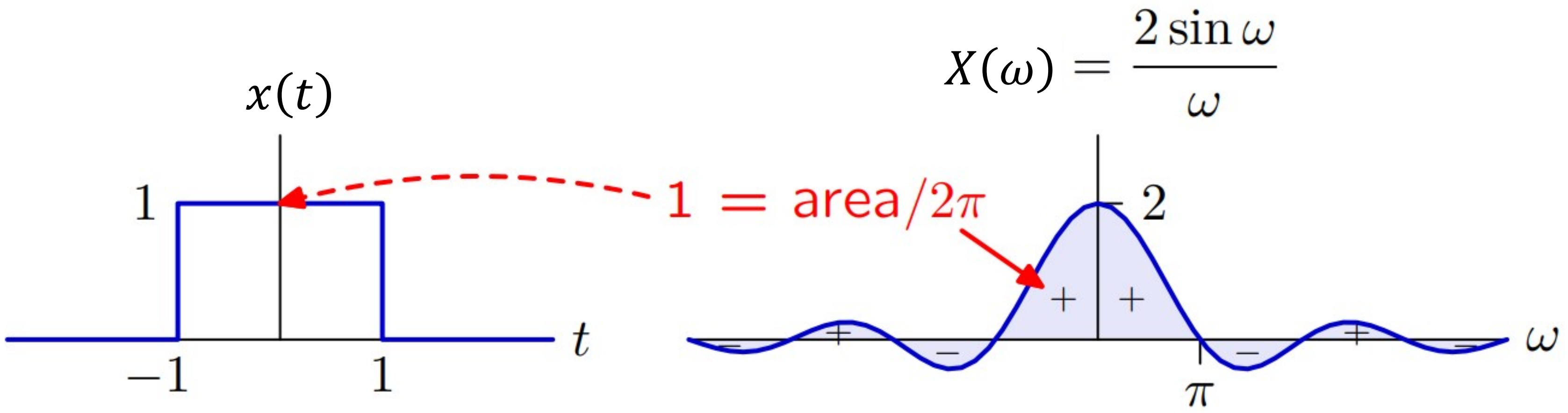
By definition, the value of $X(\omega = 0)$ is the integral of $x(t)$ over all time



$$X(0) = \int_{-\infty}^{\infty} x(t)e^{-j0t}dt = \int_{-\infty}^{\infty} x(t)dt$$

Fourier Transform of a Rectangular Pulse

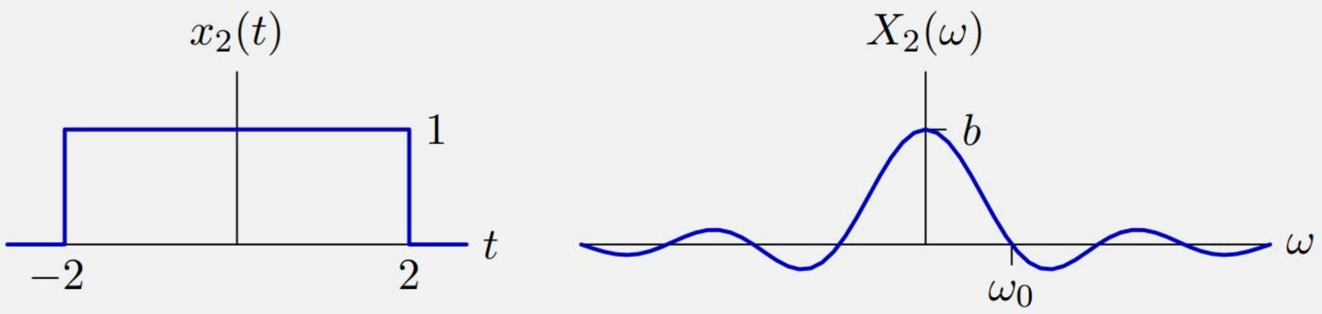
By definition, the value of $x(t=0)$ is the integral of $X(\omega)$ over all frequencies, divided by 2π



$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$$

Check yourself!

Signal $x_2(t)$ and its Fourier transform $X_2(\omega)$ are shown below.

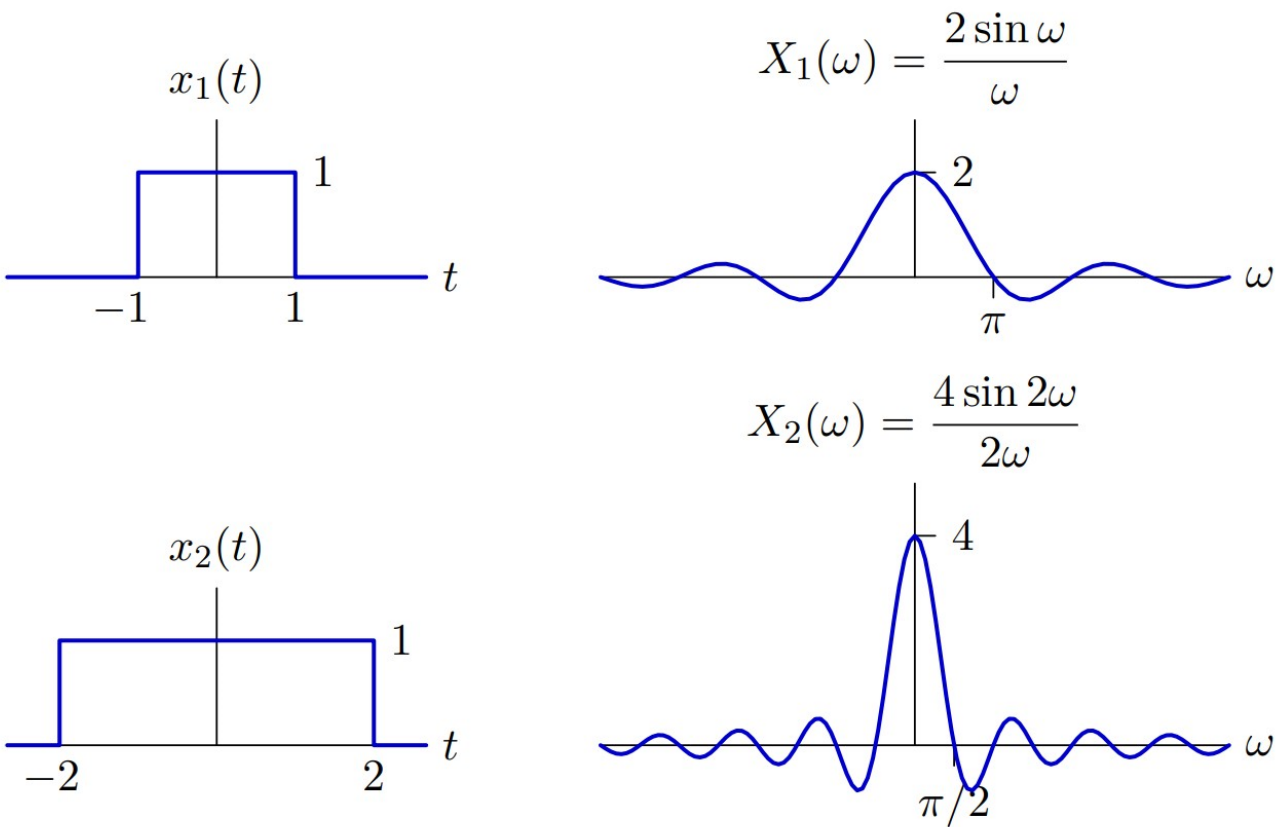


Which of the following is true?

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

Stretching In Time

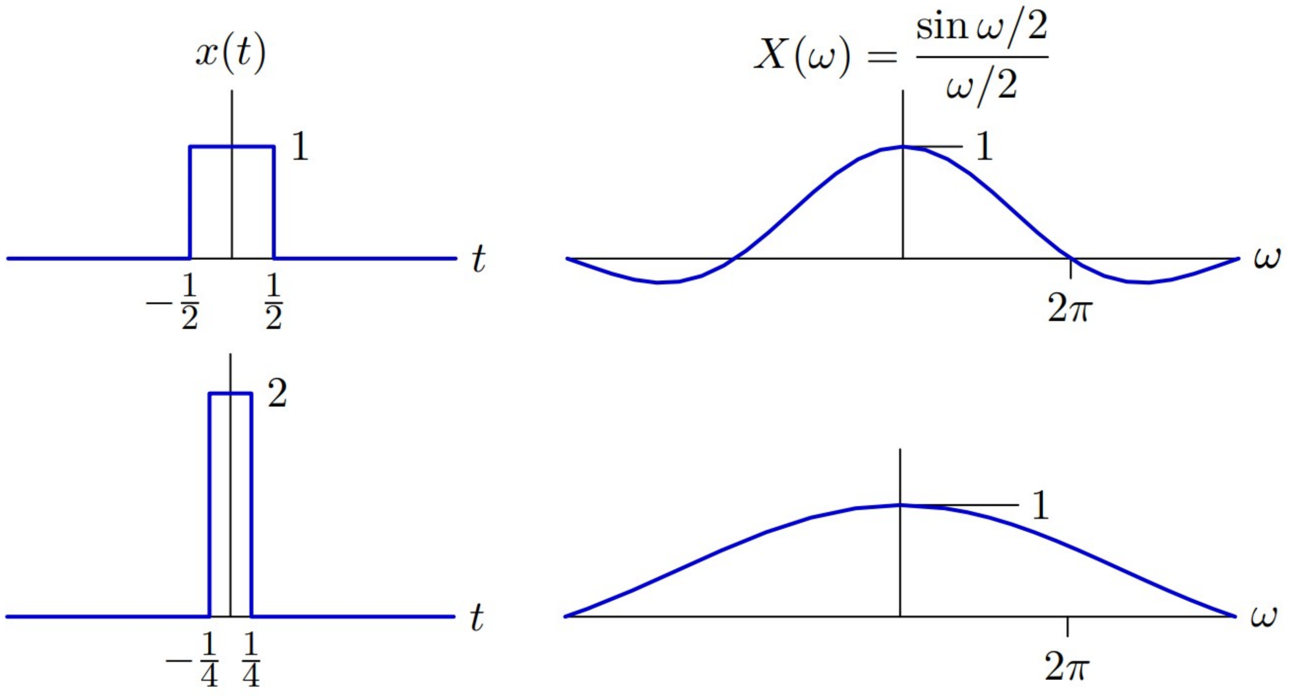
How would $X(\omega)$ scale if time were stretched?



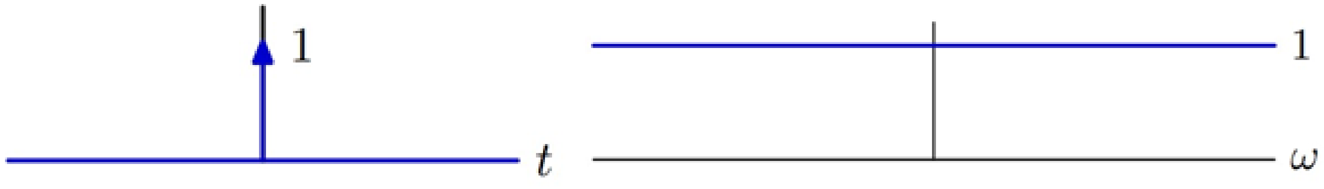
Stretching in time compresses in frequency.

Compressing Time to the Limit

Alternatively, compress time while keeping area = 1:



In the limit, the pulse has zero width but area 1! We represent this limit with the delta function: δ(t).



Math With Impulses

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function δ(t) is useful as a mathematically tractable approximation to a very brief signal.

δ(t) only has a nonzero value at t = 0, but it has finite area: it is most easily described as an integral:

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0_-}^{0_+} \delta(t)dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t-a)\ dt = \int_{a_-}^{a_+} \delta(t)\ dt = 1$$

Importantly, it has the following property (the “sifting property”):

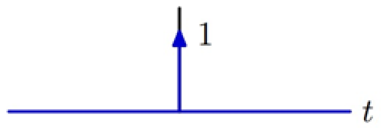
$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$$

$$let\ \tau = t - a,\ \int_{-\infty}^{\infty} \delta(\tau)f(\tau+a)d\tau = \int_{0_-}^{0_+} \delta(\tau)f(a)d\tau = f(a) \cdot \int_{0_-}^{0_+} \delta(\tau)d\tau = f(a)$$

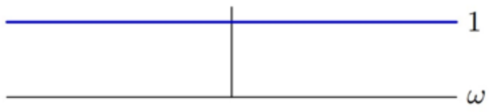
The Fourier Transform of δ(t):

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t}\,dt = \int_{0_-}^{0_+} \delta(t) \cdot e^{-j\omega 0}\,dt \quad = 1$$

$\delta(t)$



$X(\omega)$



Math With Impulses

Find the function whose Fourier transform is a unit impulse.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j\omega t}\, d\omega \;\; = \frac{1}{2\pi}\int_{0_-}^{0_+} \delta(\omega) \cdot e^{j0t}\, d\omega \;\; = \frac{1}{2\pi}$$

$$1 \overset{\text{CTFT}}{\Longleftrightarrow} 2\pi\delta(\omega)$$

Notice the similarity to the previous result:

$$\delta(t) \overset{\text{CTFT}}{\Longleftrightarrow} 1$$

These relations are **duals** of each other:

- A constant in time consists of a single frequency at ω = 0.
- An impulse in time contains components at all frequencies.

Math With Impulses

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Find the function whose Fourier transform is a shifted impulse.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) \cdot e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) \cdot e^{j\omega_0 t}\, d\omega$$

$$= \frac{1}{2\pi} e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, d\omega$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

$$e^{j\omega_o t} \overset{\text{CTFT}}{\Longrightarrow} 2\pi\delta(\omega - \omega_o)$$

We can use this result to relate Fourier series to Fourier Transforms.

Math With Impulses

If a periodic signal $f(t) = f(t+T)$ has a Fourier Series representation, then it can also be represented by an equivalent Fourier Transform.

$$e^{j\omega_o t} \overset{\text{FT}}{\Longrightarrow} 2\pi\delta(\omega - \omega_o)$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad F[k]$$

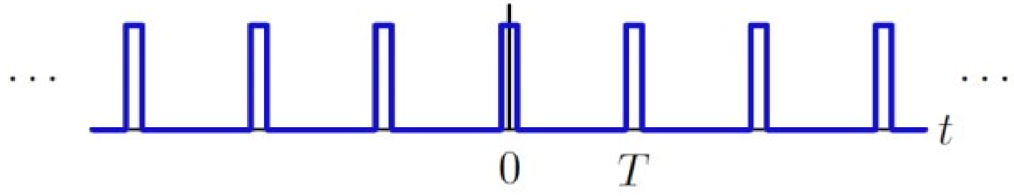
$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k]e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta\left(\omega - \frac{2\pi}{T}k\right) = F(\omega)$$

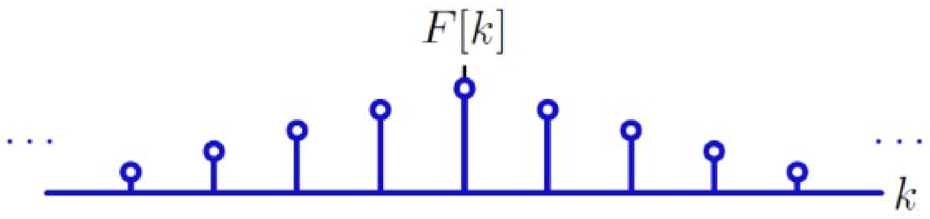
Each term in the Fourier Series is replaced by an impulse in the Fourier transform.

Math With Impulses

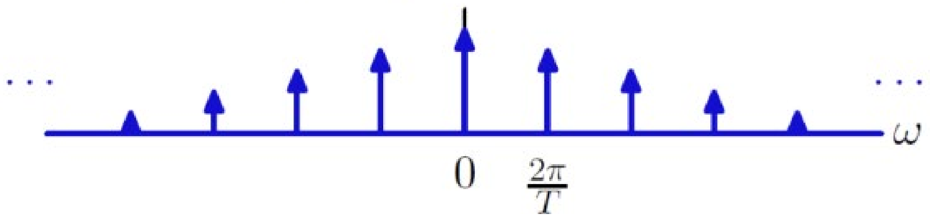
Each Fourier Series term is replaced by an impulse in the Fourier transform.

$$f(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$





$$F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, F[k]\, \delta(\omega - k\frac{2\pi}{T})$$



Summary

- Continuous Time Fourier Transform: Fourier representation to all CT signals!

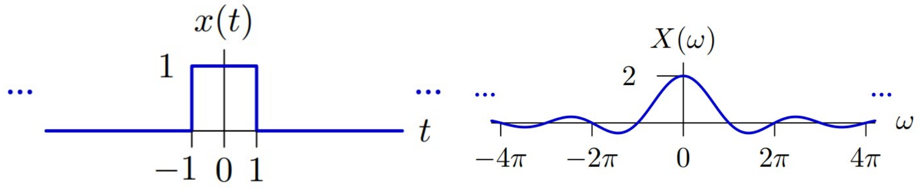
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\,d\omega$$

Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\,dt$$

Analysis equation

- Very useful signals:
  - Rectangular pulse and its Fourier Transform (sinc)
  - Delta function (Unit impulse) and its Fourier Transform



- If a periodic signal $f(t) = f(t+T)$ has a Fourier Series representation, then it can also be represented by an equivalent Fourier Transform.