6.3000: Signal Processing

Superposition and Convolution

$$y(t) = (h * x)(t) = \int h(\tau)x(t-\tau)\,d\tau$$

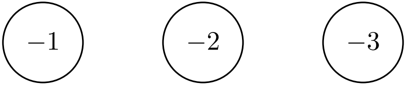
$$y[n] = (h * x)[n] = \sum_m h[m]x[n-m]$$

October 10, 2024

Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

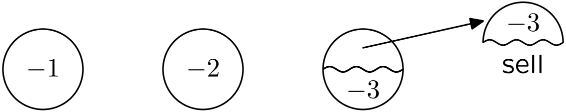
Start with 3 barrels of wine: newest at left, oldest at right.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

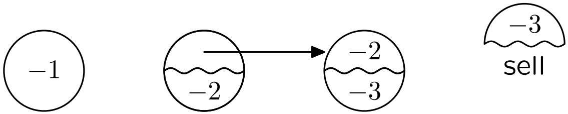
Sell half of the oldest stock.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

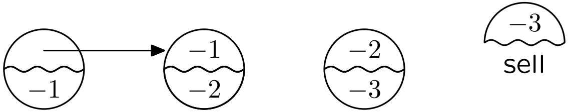
Refill oldest barrel from next-to-oldest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

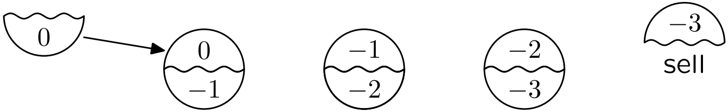
Refill next-to-oldest barrel from youngest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

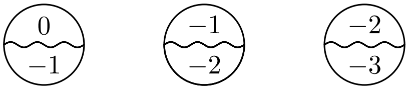
Refill youngest barrel with this year's harvest.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

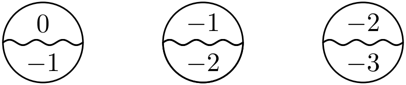
Old and new contents mix; ready for next year.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.



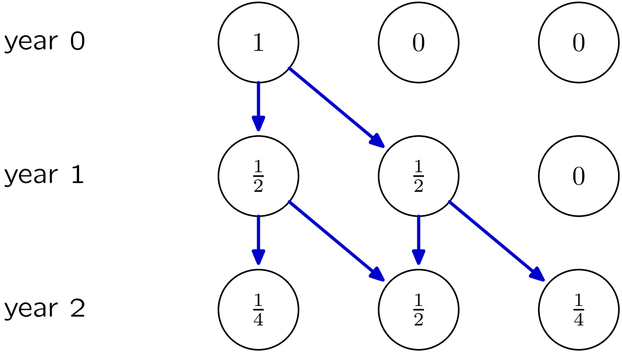
Properties of solera process:

- Mixing produces a more uniform product.
- Mitigates worst-case results of one bad year.
- Blends wines from MANY previous years.

Solera Analysis

We can analyze these effects with a tracer experiment.

Add 1 unit of tracer to new crop; track tracer through the system.

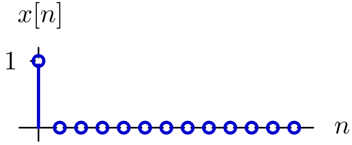


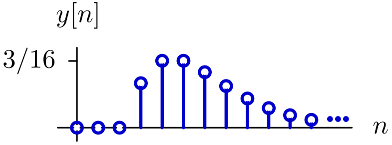
How much tracer will be in each barrel at the end of year 3?

Solera Analysis

Add 1 unit of tracer to new crop; track tracer through the system.

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

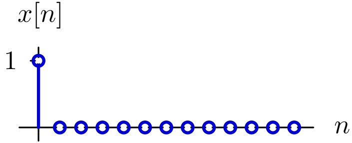


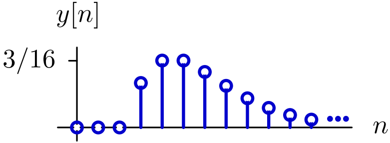


Solera Analysis

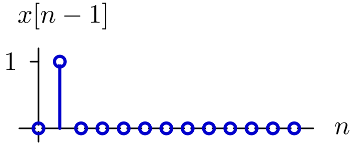
How would results change if tracer were added in year 1 (not 0)?

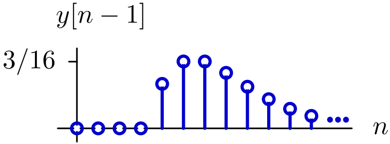
Original response:





Delayed input $\rightarrow$ delayed output:





Delaying the input by a year simply delays the outputs by one year.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

the system is **time invariant** if

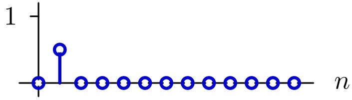
$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

is true for all n_0.

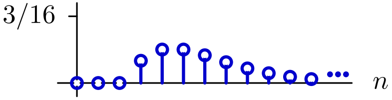
Solera Analysis

Scaling the input amplitudes:

$0.5x[n]$

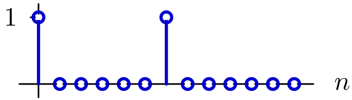


$0.5y[n]$

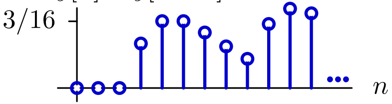


Adding two inputs:

$x[n] + x[n-6]$

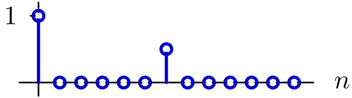


$y[n] + y[n-6]$

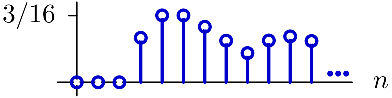


Linearly combining two inputs:

$x[n] + 0.5x[n-6]$



$y[n] + 0.5y[n-6]$



Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the system is linear if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β.

Convolution

If a system is linear and time invariant, the response to an arbitrary input is the convolution of that input with the unit-sample response.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?

1. $\frac{21}{32}$
2. $\frac{1}{2}$
3. $\frac{3}{16}$
4. $\frac{9}{16}$
5. none of above

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

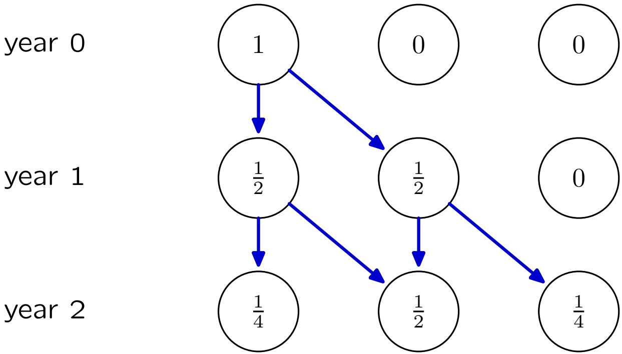
How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2? 2

1. $\frac{21}{32}$
2. $\frac{1}{2}$
3. $\frac{3}{16}$
4. $\frac{9}{16}$
5. none of above

Divide and Conquer

The content of barrel #3 has no direct dependence on barrel #1.

The new content of barrel #3 depends only on itself and barrel #2. All dependence on barrel #1 is through barrel #2.



Since barrel #3 depends only on barrel #2, and barrel #2 depends only on barrel #1, the three barrel system is equivalent to the cascade of three one barrel systems!

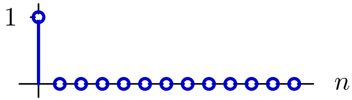
Divide and Conquer

Making a three-barrel system by cascading three one-barrel systems.

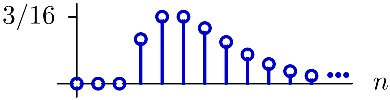
$$x[n] \longrightarrow \boxed{\text{3-barrel solera}} \longrightarrow y[n]$$

Year n	Tracer in $x[n] = \delta[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n] = h_3[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

$x[n] = \delta[n]$

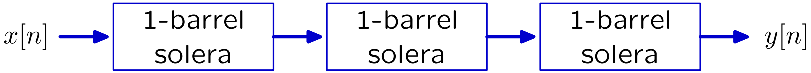


$y[n] = h_3[n]$



Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.



Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.

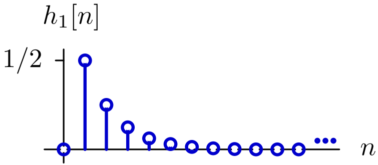
Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.

$$x[n] \rightarrow \boxed{\text{1-barrel solera}} \rightarrow \boxed{\text{1-barrel solera}} \rightarrow \boxed{\text{1-barrel solera}} \rightarrow y[n]$$

Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$



No tracer leaves barrel #1 on the year the tracer is added ($n = 0$).

Half leaves the following year.

Half of the remainder leaves on each subsequent year.

The sum of all that leaves (from $n = 0$ to ∞) is 1 (all of it).

Divide and Conquer

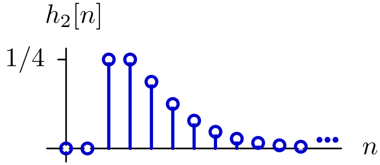
$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1]$$

$$h_2[n] = (h_1 * h_1)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_1[n-m]$$

$$= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] \left(\frac{1}{2}\right)^{n-m} u[n-m-1]$$

The value being summed is zero unless $m-1 \geq 0$ and $n-m-1 \geq 0$. Therefore $1 \leq m \leq n-1$ and $n \geq 2$:

$$h_2[n] = \sum_{m=1}^{n-1} \left(\frac{1}{2}\right)^n = (n-1) \left(\frac{1}{2}\right)^n u[n-2]$$



Divide and Conquer

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] \quad \text{and} \quad h_2[n] = (n-1)\left(\frac{1}{2}\right)^n u[n-2]$$

$$h_3[n] = (h_1 * h_2)[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m]$$

$$= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1](n-m-1)\left(\frac{1}{2}\right)^{n-m} u[n-m-2]$$

The value being summed is zero unless $m-1 \geq 0$ and $n-m-2 \geq 0$. Therefore $1 \leq m \leq n-2$ and $n \geq 3$:

$$h_3[n] = \sum_{m=1}^{n-2}(n-m-1)\left(\frac{1}{2}\right)^n = \frac{(n-1)(n-2)}{2}\left(\frac{1}{2}\right)^n u[n-3]$$