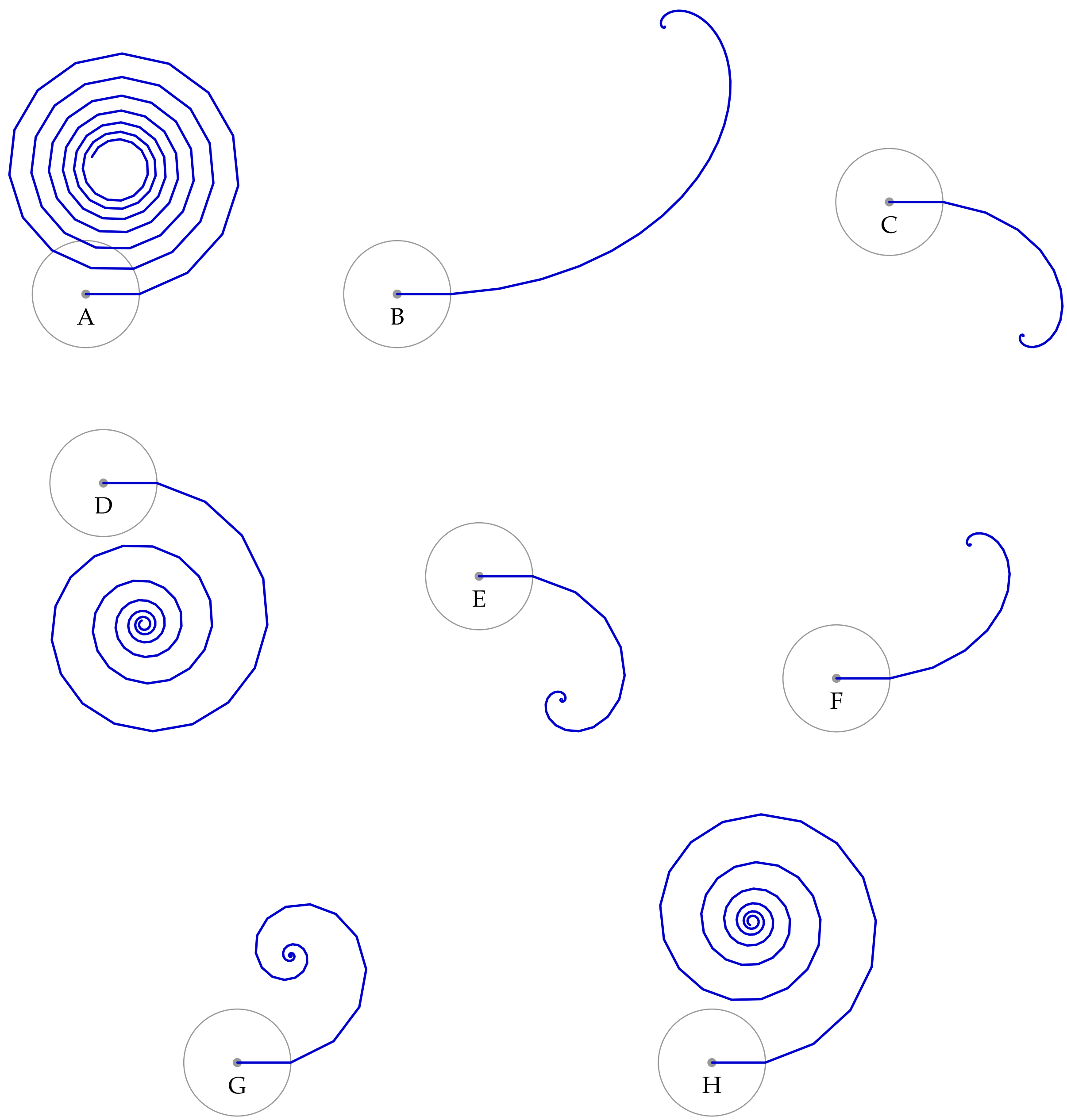
3.4 Geometric Series

Each diagram below shows the unit circle in the complex plane, with the origin labeled with a dot. Each diagram illustrates the following sum for some value of α, with each line segment representing one term of the sum.

$$S = \sum_{n=0}^{100} \alpha^n$$

Determine the diagram for which $\alpha = 0.8 + 0.2j$ and explain your reasoning. There are several things you might consider when thinking about this: what are the magnitude and angle of α? To what value will the sum converge? Etc. Thinking about these (and how they manifest in the diagrams) may help you tell the diagrams apart.



The first two terms in the series representation of S are 1 and $0.8 + 0.2j$.

All of the curves start with a horizontal line to the right that stops at the intersection with the unit circle. Thus all of the curves correctly represent the first term. The second term should have a length that is somewhat shorter than 1 and an angle of $\tan^{-1}(0.25)$. We can immediately discard curves C, D, and E because their second terms have negative imaginary parts, which is wrong.

How quickly should the terms in the series converge? The magnitude of α is $\sqrt{0.8^2 + 0.2^2} \approx 0.8$. The magnitude of the last term in the sum is then approximately $0.8^{100} \approx \frac{1}{2}^{33} \approx \frac{1}{1000}^{3} < 10^{-9}$, which would not be visible in the plots. Thus, the curve should appear to converge to a limit, which eliminates curves A and H.

Furthermore, it says that the sum of 100 terms is very nearly equal to the infinite sum:

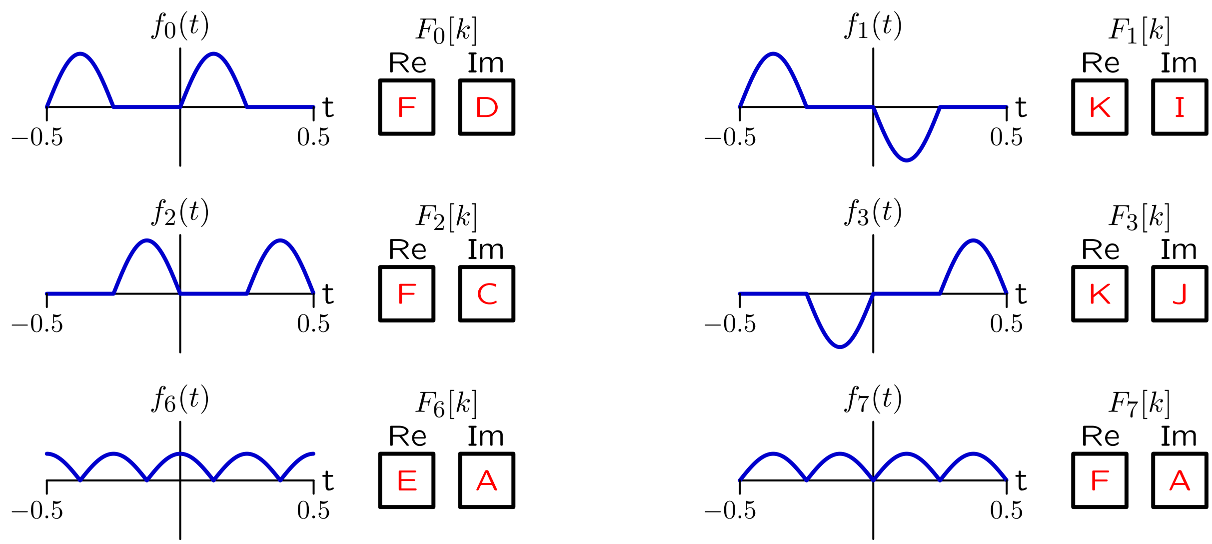
$$
\begin{aligned}
S \approx S_\infty = \sum_{n=0}^{\infty} \alpha^n &= \frac{1}{1-\alpha} \\
&= \frac{1}{1-(0.8+0.2j)} \\
&= \frac{1}{0.2-0.2j} \\
&= \frac{1}{0.2\sqrt{2}e^{-j\pi/4}} \\
&= \frac{5}{\sqrt{2}}e^{j\pi/4}
\end{aligned}
$$

Thus the final values of B and G are wrong (B's ending magnitude is too big and G's ending angle is too big), and the answer is **F**.

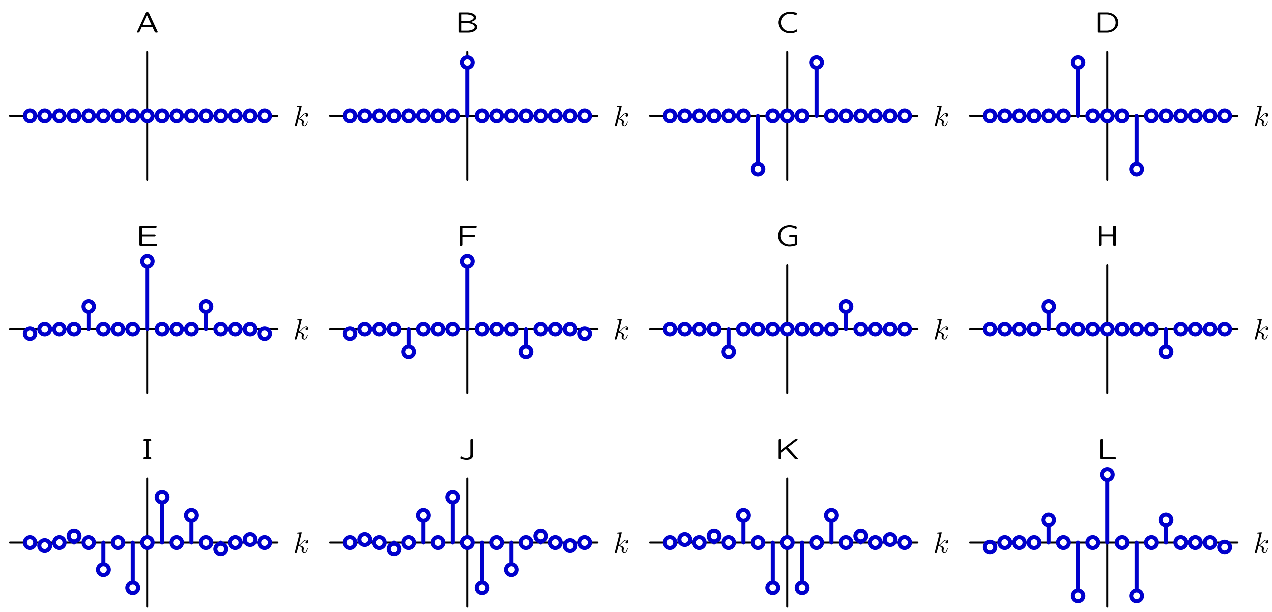
5 Continuous-Time Fourier Series (24 points)

Each of the following six plots shows a one-second interval of a periodic continuous-time signal $f_i(t)$. Each of these signals can be represented by its Fourier series coefficients $F_i[k]$ computed with $T = 1$ as follows:

$$f_i(t) = f_i(t{+}1) = \sum_{k=-\infty}^{\infty} F_i[k]e^{j2\pi kt} \qquad \text{where} \qquad F_i[k] = \int_{-0.5}^{0.5} f_i(t)e^{-j2\pi kt}dt$$



Determine which of the plots below (A–L) represents the real part of $F_i[k]$ for $-8 \leqslant k \leqslant 8$ and enter that letter in the box labeled "Re" (above). Do the same for the imaginary part and enter that letter in the box labeled "Im".



It is helpful to think about $F[k]$ in terms of its trigonomic expansion:

$$F_i[k] = \int_{-0.5}^{0.5} f_i(t)e^{-j2\pi kt} = \int_{-0.5}^{0.5} f_i(t)\Big(\cos(2\pi kt) - j\sin(2\pi kt)\Big)dt$$

The trig expansion shows that if $f_i(t)$ has a large cosine component at $k = k_0$ then $F_i[k]$ will have a positive real component at $k = k_0$. And if $f_i(t)$ has a large sine component at $k = k_0$ then $F_i[k]$ will have a negative imaginary component at $k = k_0$.

Also, since $f_i(t)$ is real-valued, $F[k]$ is conjugate symmetric: the real part of $F[k]$ is symmetric about $k = 0$ and the imaginary part of $F[k]$ is anti-symmetric about $k = 0$.

Part a.

The $\text{Re}(F_0[4]) < 0$ since $\int f_0(t)\cos(2\pi kt)dt < 0$ for $k = 4$. From conjugate symmetry, $\text{Re}(F_0[-4])$ is also < 0. Therefore the answer is F.

The $\text{Im}(F_0[2]) < 0$ since $\int f_0(t)\sin(2\pi kt)dt > 0$ for $k = 2$. From conjugate symmetry, $\text{Im}(F_0[-2])$ is > 0. Therefore the answer is D.

Part b.

The $\text{Re}(F_0[1]) < 0$ since $\int f_0(t)\cos(2\pi kt)dt < 0$ for $k = 1$. From conjugate symmetry, $\text{Re}(F_0[-1])$ is also < 0. Therefore the answer is K.

The $\text{Im}(F_0[1]) > 0$ since $\int f_0(t)\sin(2\pi kt)dt < 0$ for $k = 1$. From conjugate symmetry, $\text{Im}(F_0[-1])$ is < 0. Therefore the answer is I.

Part c.

The $\text{Re}(F_0[4]) < 0$ since $\int f_0(t)\cos(2\pi kt)dt < 0$ for $k = 4$. From conjugate symmetry, $\text{Re}(F_0[-4])$ is also < 0. Therefore the answer is F.

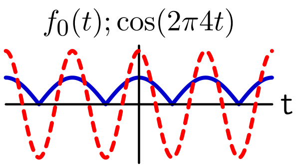
The $\text{Im}(F_0[2]) > 0$ since $\int f_0(t)\sin(2\pi kt)dt < 0$ for $k = 2$. From conjugate symmetry, $\text{Im}(F_0[-2])$ is < 0. Therefore the answer is C.

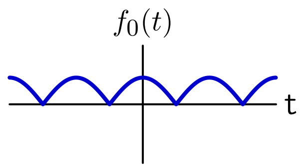
Part d.

The $\text{Re}\,(F_0[1]) < 0$ since $\int f_0(t)\cos(2\pi kt)dt < 0$ for $k = 1$. From conjugate symmetry, $\text{Re}\,(F_0[-1])$ is also < 0. Therefore the answer is K.

The $\text{Im}\,(F_0[1]) > 0$ since $\int f_0(t)\sin(2\pi kt)dt < 0$ for $k = 1$. From conjugate symmetry, $\text{Im}\,(F_0[-1])$ is > 0. Therefore the answer is J.

Part e.

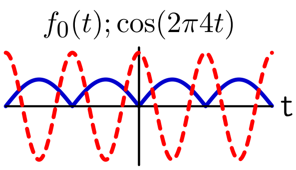


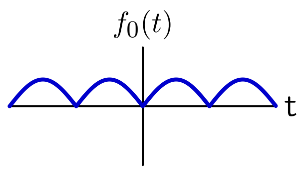


The $\text{Re}\,(F_0[4]) > 0$ since $\int f_0(t)\cos(2\pi kt)dt > 0$ for $k = 4$. From conjugate symmetry, $\text{Re}\,(F_0[-4])$ is also > 0. Therefore the answer is E.

Since $f_6(t)$ is a symmetric function of t, the $\text{Im}\,(F_6[k]) = 0$. Therefore the answer is A.

Part f.





The $\text{Re}\,(F_0[4]) < 0$ since $\int f_0(t)\cos(2\pi kt)dt < 0$ for $k = 4$. From conjugate symmetry, $\text{Re}\,(F_0[-4])$ is also < 0. Therefore the answer is F.

Since $f_6(t)$ is a symmetric function of t, the $\text{Im}\,(F_6[k]) = 0$. Therefore the answer is A.

2 Three Frequencies (24/116 points)

Part a.

Let $f_1(t)$ represent a continuous-time signal that is the sum of sinusoids with frequencies of 200 Hz, 440 Hz, and 500 Hz. Determine if $f_1(t)$ is a periodic function of t. If it is, enter numerical expressions for the fundamental period and fundamental frequency of $f_1(t)$ in the boxes below. If $f_1(t)$ is not periodic, put X's in both boxes and briefly explain.

T(seconds)= $\frac{1}{20}$ seconds f_o(Hz)= 20 Hz

Each of the three frequencies must be integer multiples of f_o:

$$\begin{aligned} 200 &= 2 \times 5 \times 20 &= m_1 f_o \\ 440 &= 2 \times 11 \times 20 &= m_2 f_o \\ 500 &= 5 \times 5 \times 20 &= m_3 f_o \end{aligned}$$

The fundamental frequency is the largest submultiple of the constituent frequencies, which is 20 Hz, since 2×5, 2×11, and 5×5 share no common factors. Therefore the fundamental frequency is 20 Hz and the fundamental period is $T = 1/f_o = 1/20$ seconds.

Part b.

Let $f_2[n]$ represent the discrete-time signal that results when a continuous-time signal that is the sum of sinusoids with frequencies of 200 Hz, 440 Hz, and 500 Hz is sampled at a rate of 1000 samples/second. Determine if $f_2[n]$ is a periodic function of n. If it is, enter numerical expressions for the fundamental period and fundamental frequency of $f_2[n]$ in the boxes below. If $f_2[n]$ is not periodic, put X's in both boxes and briefly explain.

N(samples)= 50 samples Ω_o(radians/sample)= $\frac{2\pi}{50}$ radians/sample

Sampling a 200 Hz sinusoid at 1000 samples/second generates a discrete-time frequency of $2\pi 200/1000$, which is periodic with a period of 5 samples. Sampling a 500 Hz sinusoid at 1000 samples/second generates a discrete-time frequency of $2\pi 500/1000$, which is periodic with a period of 2 samples. Sampling a 440 Hz sinusoid at 1000 samples/second generates a discrete-time frequency of $2\pi 440/1000$, which is periodic with a period of 25 samples. A periodic signal with a period of 5 is also periodic in 25, but a periodic signal with a period of 2 is not periodic in 25. However, all of these signals are periodic with a period of 50 samples. The corresponding Ω_o is $2\pi/50$ radians/sample.