6.3000: Signal Processing

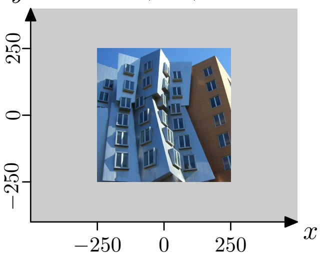
Fourier Series (Trigonometric Form)

Representing Signals as Fourier Series

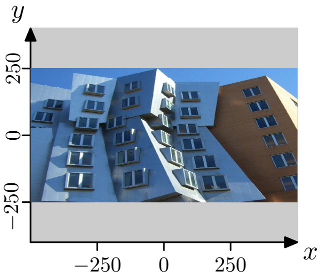
- Synthesis: Making a Signal from Components
- Analysis: Finding the Components

September 5, 2024

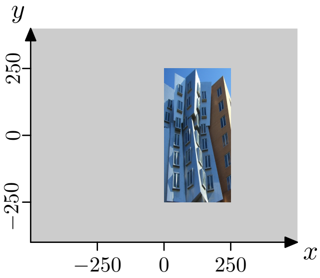
Start with Some Basic Transformations



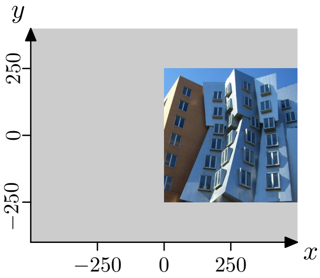
How many images match the expressions beneath them?



$f_1(x,y) = f(2x, y)$?

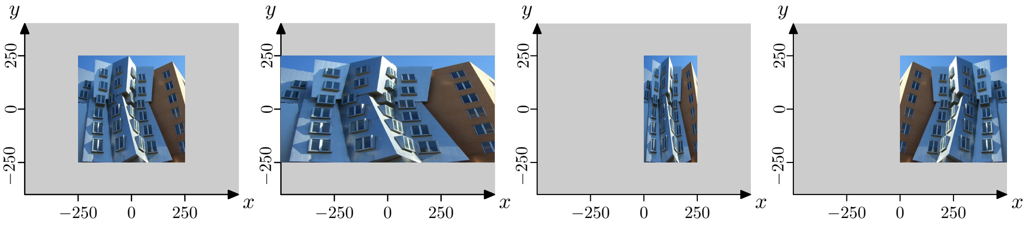


$f_2(x,y) = f(2x - 250, y)$?



$f_3(x,y) = f(-x - 250, y)$?

Start with Some Basic Transformations



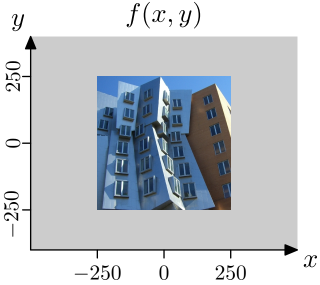
$f(x,y)$ | $f_1(x,y) = f(2x,y)$? | $f_2(x,y) = f(2x-250,y)$? | $f_3(x,y) = f(-x-250,y)$?

$$x = 0 \quad \rightarrow f_1(0,y) = f(0,y) \qquad \checkmark$$
$$x = 250 \quad \rightarrow f_1(250,y) = f(500,y) \qquad \times$$

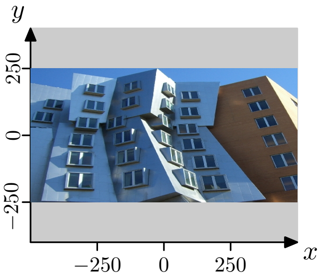
$$x = 0 \quad \rightarrow f_2(0,y) = f(-250,y) \qquad \checkmark$$
$$x = 250 \quad \rightarrow f_2(250,y) = f(250,y) \qquad \checkmark$$

$$x = 0 \quad \rightarrow f_3(0,y) = f(-250,y) \qquad \times$$
$$x = 250 \quad \rightarrow f_3(250,y) = f(-500,y) \qquad \times$$

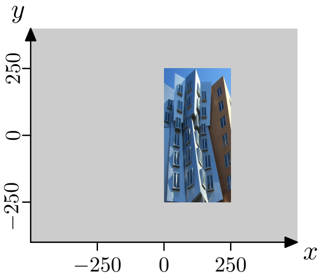
Start with Some Basic Transformations



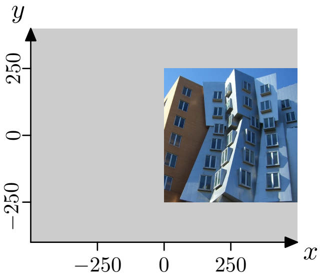
How many images match the expressions beneath them? 1



~~$f_1(x, y) = f(2x, y)$?~~



$f_2(x, y) = f(2x - 250, y)$?



~~$f_3(x, y) = f(-x - 250, y)$?~~

Fourier Series

Fourier representations are a major theme of this subject.

The basic ideas were described in lecture:

Synthesis Equation (making a signal from components):

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

Analysis Equations (finding the components):

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos\left(\frac{2\pi kt}{T}\right)dt\ ; \quad k \geq 1$$

$$d_k = \frac{2}{T}\int_T f(t)\sin\left(\frac{2\pi kt}{T}\right)dt\ ; \quad k \geq 1$$

Warm Up

Find the Fourier series coefficients (c_k and d_k) for

$$f(t) = \cos(t)$$

Warm Up

Find the Fourier series coefficients (c_k and d_k) for

$$f(t) = \cos(t)$$

We can find c_k and d_k directly from the synthesis equation:

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

The function $f(t)$ is periodic in time with period

$$T = 2\pi\,.$$

The coefficients can be found by matching the expression on the left with that on the right:

$$c_k = \begin{cases} 1 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$d_k = 0$$

There is a single non-zero Fourier coefficient: $c_1 = 1$.

Warm Up

Alternatively, we can calculate c_k and d_k from the analysis equations:

$$f(t) = \cos(t)$$

$$c_0 = \frac{1}{2\pi}\int_T f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(t)\,dt = \frac{1}{2\pi}\sin(t)\Big|_{-\pi}^{\pi} = 0$$

For $k > 0$:

$$c_k = \frac{2}{T}\int_T f(t)\cos\left(\frac{2\pi kt}{T}\right)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(kt)\,dt$$

$$c_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\frac{1}{2}\cos(2t)\right)dt = \frac{1}{\pi}\left(\frac{t}{2}+\frac{1}{4}\sin(2t)\right)\Bigg|_{-\pi}^{\pi} = 1$$

For $k > 1$:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\cos\left((k+1)t\right)+\cos\left((k-1)t\right)\Big)dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin\left((k+1)t\right)}{k+1}+\frac{\sin\left(k-1)t\right)}{k-1}\right]_{-\pi}^{\pi} = 0$$

$$d_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(kt)\,dt = 0 \quad \text{(integrand is anti-symmetric)}$$

Fourier Series Coefficients

How many of the following functions have **exactly one** non-zero Fourier series coefficient?

- $f_1(t) = \cos^2 t$
- $f_2(t) = \sin t \cos t$
- $f_3(t) = 4\cos^3 t - 3\cos t$
- $f_4(t) = \cos(12t)\cos(4t)\cos(2t)$

Fourier Series Coefficients

How many of the following functions have **exactly one** non-zero Fourier series coefficient? 2: $f_2(t)$ and $f_3(t)$

$$f_1(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t)$$

→ 2 non-zero components: c_0 and c_1. (could this also be c_0 and c_2?)

$$f_2(t) = \sin(t)\cos(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}\sin(0) = \frac{1}{2}\sin(2t)$$

→ 1 non-zero component: d_1.

$$f_3(t) = 4\cos^3(t) - 3\cos(t) = \cos(t)\Big(4\cos^2(t) - 3\Big)$$
$$= \cos(t)\Big(2\cos(2t) - 1\Big) = \cos(t) + \cos(3t) - \cos(t) = \cos(3t)$$

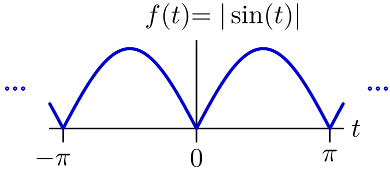
→ 1 non-zero component: c_1.

$$f_4(t) = \cos(12t)\cos(4t)\cos(2t) = \cos(12t)\left(\frac{1}{2}\cos(6t) + \frac{1}{2}\cos(2t)\right)$$
$$= \frac{1}{4}\cos(18t) + \frac{1}{4}\cos(6t) + \frac{1}{4}\cos(14t) + \frac{1}{4}\cos(10t)$$

→ 4 non-zero components: c_3, c_5, c_7, and c_9.

Rectified Sine Wave

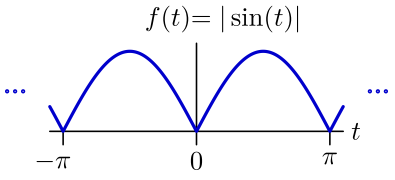
Consider a Fourier series representation of the following function.



- What is the approximate value of c_0?
- Which non-DC Fourier coefficient has the largest absolute value? What's the sign of that coefficient?
- Determine an expression for the Fourier coefficients of $f(t)$.
- Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

Rectified Sine Wave

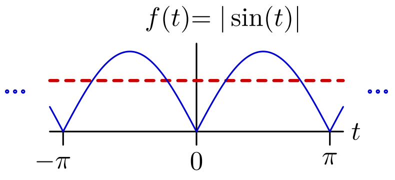
Consider a Fourier series representation of the following function.



Q: What is the approximate value of c_0?

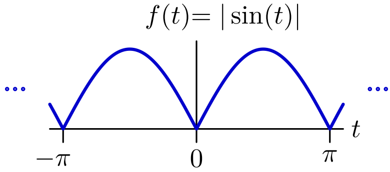
A: c_0 is the average value, which is clearly greater than $\frac{1}{2}$ but less than 1.

More exactly, $c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{\pi}\int_0^\pi \sin(t)dt = -\frac{\cos(t)}{\pi}\Big|_0^\pi = \frac{2}{\pi} \approx 0.64$



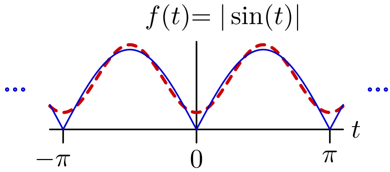
Rectified Sine Wave

Consider a Fourier series representation of the following function.



Q: Which non-DC Fourier coefficient has the largest absolute value? What's the sign of that coefficient?

A: biggest deviations from mean at $t = 0$ and $t = \frac{\pi}{2} \rightarrow -\cos(t)$.

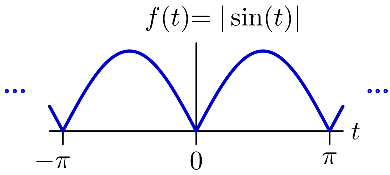


$$c_1 = \frac{2}{T}\int_T f(t)\cos\left(\frac{2\pi t}{T}\right)dt = \frac{2}{\pi}\int_0^\pi \sin(t)\cos(2t)\,dt = \frac{1}{\pi}\int_0^\pi \left(\sin(3t) - \sin(t)\right)$$

$$= \frac{1}{\pi}\left[-\frac{\cos(3t)}{3} + \cos(t)\right]_0^\pi = -\frac{4}{3\pi} \approx -0.42$$

Rectified Sine Wave

Consider a Fourier series representation of the following function.



Determine an expression for the Fourier coefficients of $f(t)$.

The function is symmetric about $t = 0$, so $d_k = 0$ for all k.

$$c_0 = \frac{1}{\pi}\int_0^{\pi} \sin(t)dt = -\frac{\cos(t)}{\pi}\bigg|_0^{\pi} = \frac{2}{\pi}$$

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(2kt)\,dt \qquad ; k \geq 1 \\ &= \frac{1}{\pi}\int_0^{\pi} \Big(\sin\left((2k+1)t\right) - \sin\left((2k-1)t\right)\Big)\,dt \\ &= \frac{1}{\pi}\left[-\frac{\cos\left((2k+1)t\right)}{2k+1} + \frac{\cos\left((2k-1)t\right)}{2k-1}\right]_0^{\pi} = \frac{-4/\pi}{4k^2 - 1} \end{aligned}$$

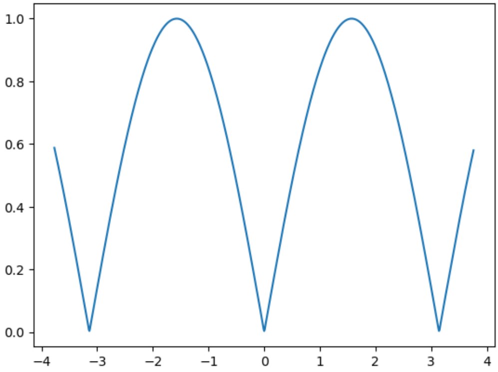
Verify Fourier Series of Rectified Sine Wave Numerically

Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

Verify Fourier Series of Rectified Sine Wave Numerically

Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

```
from math import cos, pi
from matplotlib.pyplot import plot, show
ff = []
tt = []
t = -1.2*pi
while t<1.2*pi:
    ff.append(2/pi+sum([-4/pi/(4*k*k-1)*cos(2*k*t) for k in range(1,100)]))
    tt.append(t)
    t += 0.01
plot(tt,ff)
show()
```



Trig Table

```
sin(a+b) = sin(a) cos(b) + cos(a) sin(b)
sin(a-b) = sin(a) cos(b) - cos(a) sin(b)
cos(a+b) = cos(a) cos(b) - sin(a) sin(b)
cos(a-b) = cos(a) cos(b) + sin(a) sin(b)
tan(a+b) = (tan(a)+tan(b))/(1-tan(a) tan(b))
tan(a-b) = (tan(a)-tan(b))/(1+tan(a) tan(b))

sin(A) + sin(B) =  2 sin((A+B)/2) cos((A-B)/2)
sin(A) - sin(B) =  2 cos((A+B)/2) sin((A-B)/2)
cos(A) + cos(B) =  2 cos((A+B)/2) cos((A-B)/2)
cos(A) - cos(B) = -2 sin((A+B)/2) sin((A-B)/2)

sin(a+b) + sin(a-b) =  2 sin(a) cos(b)
sin(a+b) - sin(a-b) =  2 cos(a) sin(b)
cos(a+b) + cos(a-b) =  2 cos(a) cos(b)
cos(a+b) - cos(a-b) = -2 sin(a) sin(b)

2 cos(A) cos(B) = cos(A-B) + cos(A+B)
2 sin(A) sin(B) = cos(A-B) - cos(A+B)
2 sin(A) cos(B) = sin(A+B) + sin(A-B)
2 cos(A) sin(B) = sin(A+B) - sin(A-B)
```