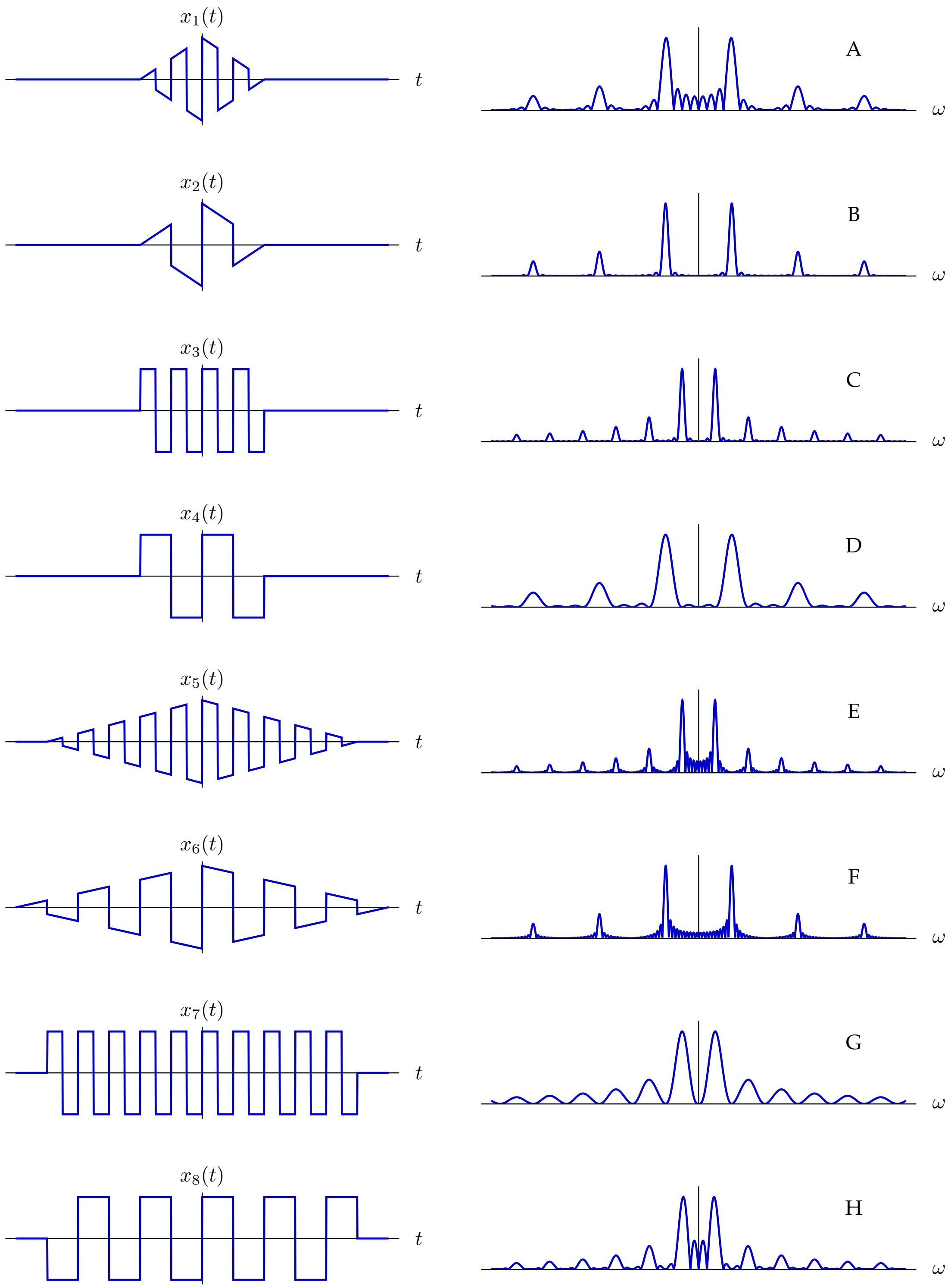
6.300 Problem Set 11

Problem 1: Time and Frequency Patterns

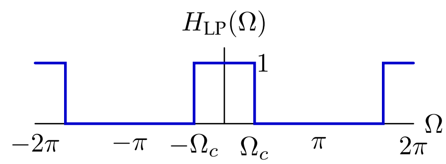
Eight time-domain waveforms are shown in the left panels below. For each signal on the left, determine the corresponding CTFT magnitude plot on the right.



Problem 2: Ideal Filters and "Windowed" Filter Design

Part A: The "Ideal" Low-Pass Filter

A *low-pass* filter passes low frequencies and attenuates high frequencies. An ideal low-pass filter is parameterized by its "cutoff" frequency Ω_c. It allows frequency components below the cutoff frequency to pass through unchanged and eliminates frequencies above the cutoff frequency. The plot below shows the frequency response of an ideal low-pass filter. Notice that $H_{\text{LP}}(\Omega) = 1$ when $|\Omega| < \Omega_c$, and $H_{\text{LP}}(\Omega) = 0$ for frequencies near π.



The ideal low-pass filter shown above is periodic in 2π. Explain why.

The low-pass filter described above is linear and time-invariant. Therefore, its output should be given by the convolution of it input with the unit-sample response of the filter. Determine an expression for this system's unit sample response $h_{\text{LP}}[n]$ in terms of Ω_c and any relevant constants.

Part B: Approximating an Ideal Low-Pass Filter

Notice that the unit-sample response of the ideal low-pass filter described in the previous section has infinite extent. Computing the output using convolution would require an infinite number multiplies and an infinite number of additions! Fortunately, the values of $h_{\text{LP}}[n]$ tend to get smaller as $|n|$ gets large. This observation motivates approximating $h_{\text{LP}}[n]$ as $\hat{h}_{\text{LP}}[n]$:

$$\hat{h}_{\text{LP}}[n] = \begin{cases} h_{\text{LP}}[n] & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

which can be thought of as resulting from multiplication of the original $h_{\text{LP}}[n]$ by a rectangular window $w[n]$.

Specifically, $\hat{h}_{\text{LP}}[n] = w[n]h_{\text{LP}}[n]$ where $w[n]$ is given by $w[n] = \begin{cases} 1 & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$

Determine an expression for $W(\Omega)$, which is the DTFT of our rectangular window.

Hint: The sum of a finite number of geometric terms is $\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$, provided that $a \neq 1$.

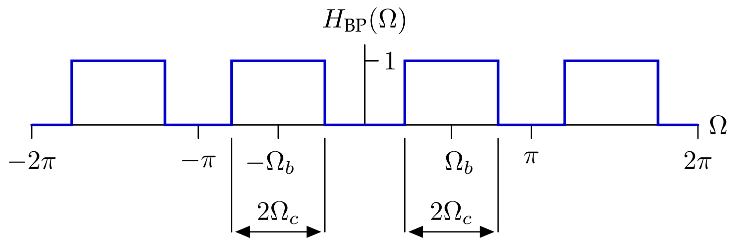
You do not need to find a closed-form expression for $\hat{H}_{\text{LP}}(\Omega)$, but how does it relate to $H_{\text{LP}}(\Omega)$ and $W(\omega)$?

Use Python to make plots of the frequency response of the windowed ideal low-pass filter, $\hat{H}_{\text{LP}}(\Omega)$, with $\Omega_c = \pi/4$ and $\Omega_c = \pi/10$, each with $M = 50$ and $M = 500$. How do Ω_c and M affect the frequency response of the system? Can you explain those results in terms of the relationship between $\hat{H}_{\text{LP}}(\Omega)$, $H_{\text{LP}}(\Omega)$, and $W(\omega)$?

Part C: The "Ideal" Band-Pass Filter

Next we'll consider a different type of filter called a *band-pass filter*. This type of filter passes frequencies in a particular range (not necessarily centered around 0) and eliminates frequencies outside that range.

A band-pass filter is parameterized by the center frequency Ω_b and half-width Ω_c of the band of frequencies that is passed, as illustrated below.



Determine a closed-form expression for the unit-sample response of this system, $h_{\text{BP}}[n]$ in terms of Ω_b, Ω_c, and any necessary constants. How does this relate to $h_{\text{LP}}[n]$ from Part A? Explain these similarities.

To get a sense of the shape of this system's unit sample response, use Python to generate plots of the unit sample response for $\Omega_b = \pi/2$, with $\Omega_c = \pi/10$ and $\Omega_c = \pi/30$, each plotted from from $n = -100$ to $n = 100$.

Part D: Approximating an Ideal Band-Pass Filter

The unit sample response of an ideal band-pass filter is also infinite. Once again, we will consider approximating this filter with a windowed version:

$$\hat{h}_{\text{BP}}[n] = \begin{cases} h_{\text{BP}}[n] & \text{if } -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Use Python to make plots of the frequency response of the windowed ideal band-pass filter with $\Omega_b = \pi/2$ and $\Omega_b = \pi/4$; $\Omega_c = \pi/10$ and $\Omega_c = \pi/30$; and $M = 50$ and $M = 500$.

Explain the effect of Ω_b, Ω_c, and M on the frequency response of the filter.

Problem 3: Designing a Little Filter

Consider a filter whose unit-sample response is given by the following, for some values of a and b:

$$h[n] = \begin{cases} a & \text{if } n = 0 \\ b & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

By adjusting a and b, we can dramatically alter the frequency response of this system. How would you set a and b to make as effective of a low-pass filter as possible? How would you set a and b to make as effective of a high-pass filter as possible? Explain your results and your reasoning.