6.300 Quiz #2

Spring 2024

6 questions — 110 minutes — 25% of course grade

Name:

Kerberos:

- Wait until we tell you to begin.
- If you have questions, come to us at the front to ask.
- If you finish the quiz with less than 15 minutes remaining, quietly remain seated and wait to upload your quiz to Gradescope until we call time.

Continuous-Time Fourier Series (CTFS)

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,\mathrm{d}t \qquad x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

Discrete-Time Fourier Series (DTFS)

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n} \qquad x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$$

Continuous-Time Fourier Transform (CTFT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega n}\mathrm{d}\Omega$$

Discrete Fourier Transform (DFT)

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \qquad x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}$$

Problem #1: DTFT and DFT (18 points)

Discrete-Time Fourier Transform (6 points)

(a) Determine the discrete-time Fourier transform (DTFT) $X_1(\Omega)$ of the signal $x_1[n]$, defined as

$$x_1[n] = \begin{cases} 1 & -2 \leq n \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

Discrete Fourier Transform (6 points)

(b) Determine the discrete Fourier transform (DFT) $X_1[k]$ of the signal $x_1[n]$, defined as in **(a)**. Use an analysis window of $N = 32$ samples.

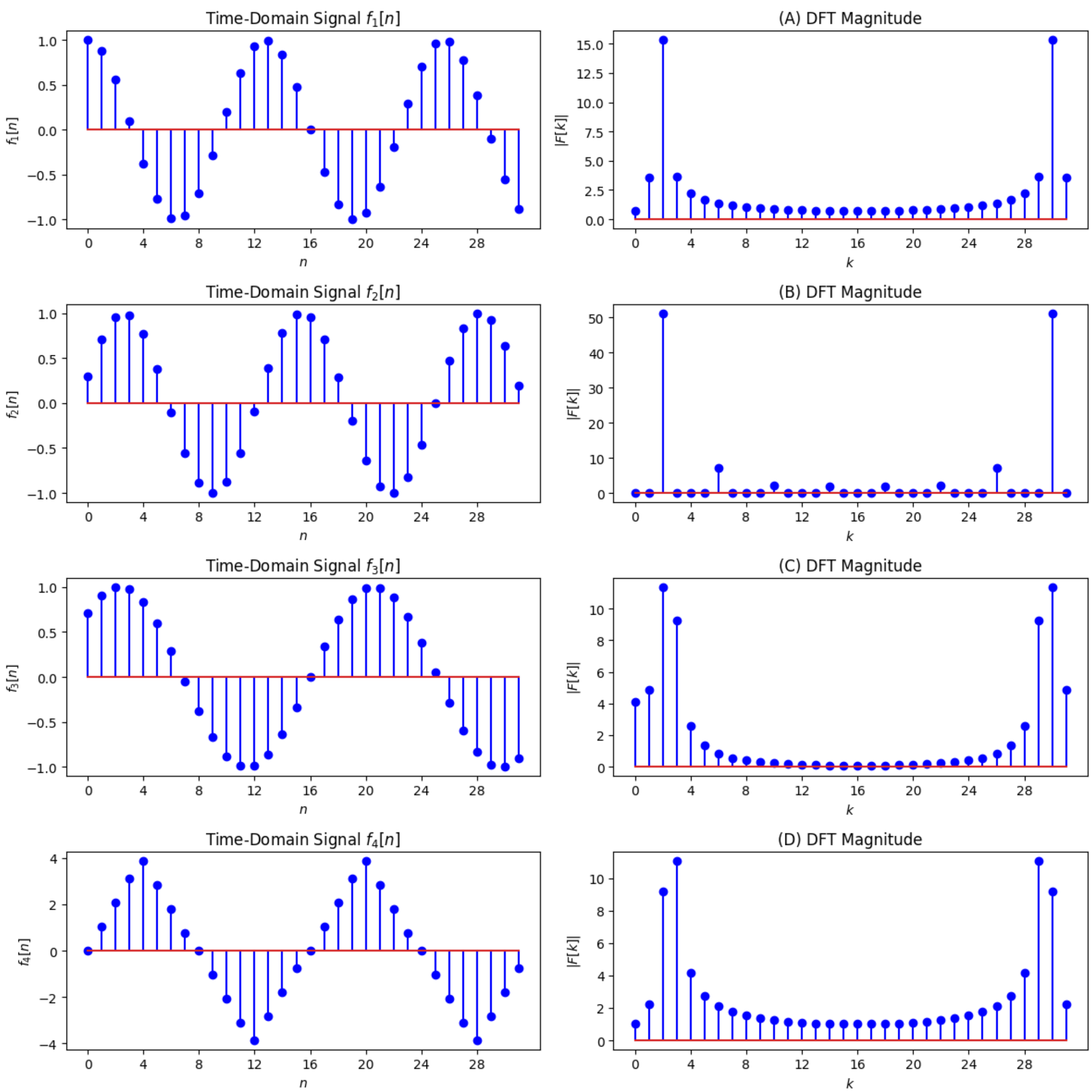
Convolved Signals (6 points)

(c) Determine the discrete-time Fourier transform (DTFT) $X_2(\Omega)$ of the signal $x_2[n] = (x_1 * x_1)[n]$.

Problem #2: DFT Matching (16 points)

Match each time-domain signal — $f_1[n]$, $f_2[n]$, $f_3[n]$, and $f_4[n]$ — to the magnitude of its discrete Fourier transform — **A**, **B**, **C**, and **D**. Provide your reasoning for potential partial credit.

$f_1[n]$:	$f_2[n]$:	$f_3[n]$:	$f_4[n]$:



Problem #3: System Properties (10 points)

Determine if each system is linear and time-invariant. Briefly justify your answer.

$$x_i[n] \longrightarrow \boxed{\text{System}} \longrightarrow y_i[n]$$

System #1 (2 points)

(a) Consider a system with output $y_1[n] = n$. Is the system linear? Is the system time-invariant?

System #2 (2 points)

(b) Consider a system with output $y_2[n] = x_2[n] + 1$. Is the system linear? Is the system time-invariant?

System #3 (2 points)

(c) Consider a system with output $y_3[n] = x_3[n] - x_3[n-1]$. Assume that the system is "at rest" (i.e., $y_3[n] = 0$) until receiving an input. Is the system linear? Is the system time-invariant?

System #4 (2 points)

(d) Consider a system with output $y_4[n] = y_4[n-1] + x_4[n]$. Is the system linear? Is the system time-invariant?

System #5 (2 points)

(e) Consider a system with output $y_5[n] = 1$. Is the system linear? Is the system time-invariant?

Problem #4: Signals and Systems (20 points)

Consider a linear, time-invariant (LTI) system

$$x[n] \longrightarrow \boxed{\text{LTI System with Unit Sample Response } h[n]} \longrightarrow y[n]$$

described by the difference equation

$$y[n] = x[n-3] + 2x[n-1] + x[n+1].$$

Unit Sample Response (4 points)

(a) Determine the system's unit sample response. That is, find an $h[n]$ such that $y[n] = (x * h)[n]$.

Frequency Response (4 points)

(b) Determine the system's frequency response. That is, find an $H(\Omega)$ such that $Y(\Omega) = X(\Omega)H(\Omega)$.

Frequency Response: Magnitude (4 points)

(c) Determine an expression for the magnitude of the frequency response, $|H(\Omega)|$, and plot it.

Frequency Response: Phase (4 points)

(d) Determine an expression for the phase of the frequency response, $\angle H(\Omega)$, and plot it.

Eigenfunctions (4 points)

(e) Suppose the input to the system is $x[n] = \cos(\frac{2\pi}{8}n)$. Determine an expression for $y[n]$ and plot it. Label all key parameters.

Problem #5: Filter Design (20 points)

Suppose we sample a continuous-time (CT) signal $x(t)$ using a sampling rate $f_s = 40960$ Hz to obtain the discrete-time (DT) signal $x[n]$. We then compute the discrete Fourier transform (DFT) of $x[n]$, $X[k]$, using an analysis window of length N.

We filter $X[k]$ using an ideal band-pass filter described by

$$H[k] = \begin{cases} 1 & k_l \leq |k| \leq k_h \\ 0 & \text{otherwise} \end{cases}$$

for positive integers k_l and k_h.

Analysis Window (2 points)

(a) Suppose $x[n] = x[n + 4096]$ is periodic in 4096 samples and defined over one period as

$$x[n] = \begin{cases} n & 0 \leq n \leq 2048 \\ 4096 - n & 2049 \leq n \leq 4095. \end{cases}$$

Determine the smallest length of the DFT analysis window N that will properly capture all relevant information in the time and frequency domains.

Frequency Resolution (2 points)

(b) Suppose that $N = 8192$ is used for the length of the analysis window. Determine the frequency resolution of $X[k]$ in Hz.

Cut-Off Frequencies (4 points)

(c) We want our band-pass filter to pass frequencies in the range $f_l \leq f \leq f_h$ and attenuate all frequencies outside that range. Suppose that $N = 8192$ is used as the length of the analysis window. Determine the frequency indices k_l and k_h corresponding to the cut-off frequencies $f_l = 220$ Hz and $f_h = 1760$ Hz, respectively.

Band-Pass Filter Implementation in Python 3 (12 points)

(d) Implement an ideal band-pass filter in Python 3. Complete the definition of the `band_pass_filter` function below. This function has five input parameters, as described below. It takes the original signal `x` and returns the band-pass filtered result `x_filtered` at the end. You may not import any external libraries.

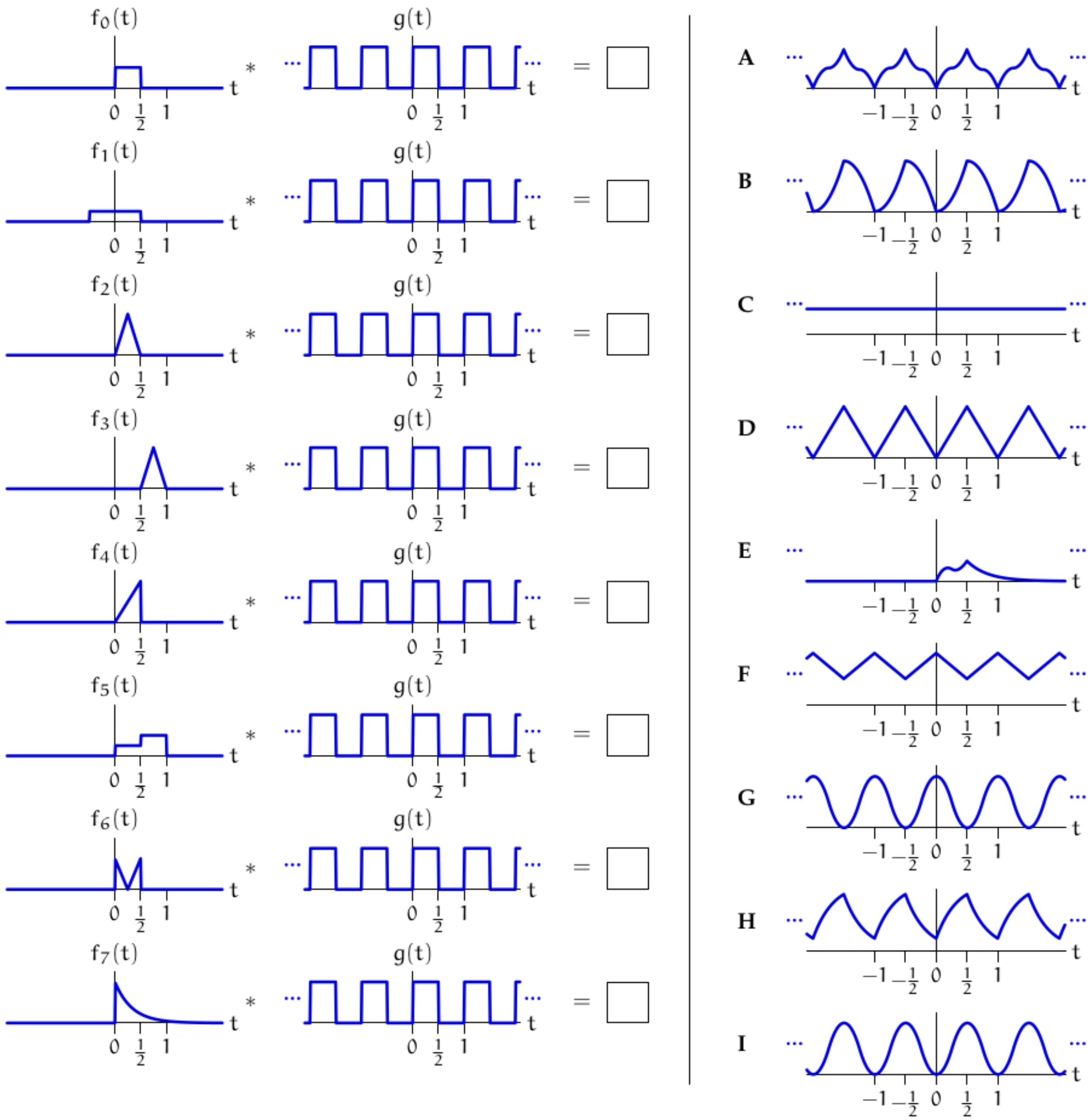
```
from lib6300.fft import fft, ifft

def band_pass_filter(x, fs, N, fl, fh):
    # x          Discrete-time signal contained in list
    # fs         Sampling rate in Hz
    # N          Length of DFT analysis window
    # fl, fh     Cut-off frequencies in Hz

    # Go to frequency domain
    X = fft(x[0:N])

    # YOUR CODE HERE













    # Return to time domain
    x_filtered = ifft(X_filtered)
    return x_filtered
```

Problem #6: Convolution (16 points)

Each signal in the left column below — $f_0(t)$ to $f_7(t)$ — is zero outside the regions shown in the plots. Determine the result of convolving each of these signals with a periodic train of rectangular pulses given by

$$g(t) = \begin{cases} 1 & \sin(2\pi t) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Determine which waveform — **A** to **I** — shows the result of each convolution.
Provide your reasoning for potential partial credit.



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