6.300 Quiz #2

Spring 2024

6 questions — 110 minutes — 25% of course grade

Name: Solutions

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- Wait until we tell you to begin.
- If you have questions, come to us at the front to ask.
- If you finish the quiz with less than 15 minutes remaining, quietly remain seated and wait to upload your quiz to Gradescope until we call time.

Continuous-Time Fourier Series (CTFS)

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,\mathrm{d}t \qquad x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

Discrete-Time Fourier Series (DTFS)

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n} \qquad x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$$

Continuous-Time Fourier Transform (CTFT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega n}\mathrm{d}\Omega$$

Discrete Fourier Transform (DFT)

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \qquad x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n}$$

Problem #1: DTFT and DFT (18 points)

Discrete-Time Fourier Transform (6 points)

(a) Determine the discrete-time Fourier transform (DTFT) $X_1(\Omega)$ of the signal $x_1[n]$, defined as

$$x_1[n] = \begin{cases} 1 & -2 \le n \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

In lecture #5, we computed the DTFT of a rectangular pulse

$$p_S[n] = \begin{cases} 1 & -S \le n \le S \\ 0 & \text{otherwise} \end{cases}$$

that was symmetric about the origin. The result was

$$P_S(\Omega) = \frac{\sin(\Omega(S + \frac{1}{2}))}{\sin(\Omega/2)}.$$

Here, $x_1[n]$ is a shifted rectangular pulse. We can define a shifted intermediate signal $x_1'[n] = x_1[n+2]$ to get a rectangular pulse that is symmetric about the origin:

$$x_1'[n] = \begin{cases} 1 & -4 \le n \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

Then the DTFT of $x_1'[n]$ is

$$X_1'(\Omega) = \frac{\sin(\Omega(4 + \frac{1}{2}))}{\sin(\Omega/2)}.$$

$x_1'[n] = x_1[n+2]$ is equivalent to $x_1[n] = x_1'[n-2]$. A shift in time corresponds to a phase shift in the frequency domain. So, we get

$$X_1(\Omega) = \frac{\sin(\Omega(4 + \frac{1}{2}))}{\sin(\Omega/2)} e^{-j2\Omega}.$$

Many students applied the formula for a finite-length geometric sequence. That does, in principle, work, but any student who tried this would have been screwed if we asked for plots of the magnitude and phase of such a transform. (No student would be that good at plotting magnitude and phase based on vector diagrams!)

Discrete Fourier Transform (6 points)

(b) Determine the discrete Fourier transform (DFT) $X_1[k]$ of the signal $x_1[n]$, defined as in **(a)**. Use an analysis window of $N = 32$ samples.

There's no need for us to compute the DFT by using the analysis equation. Instead, we can take advantage of the relation between the DTFT and the DFT:

$$X[k] = \frac{1}{N} X(\Omega)\Big|_{\Omega=2\pi k/N}.$$

Making the substitution $\Omega = 2\pi k/N$ in our answer for **(a)**, we get

$$X_1[k] = \frac{1}{N} \frac{\sin(\frac{2\pi k}{N}(4+\frac{1}{2}))}{\sin(\frac{2\pi k}{N}/2)} e^{-j2\frac{2\pi k}{N}}$$

and by plugging in $N = 32$, we arrive at the final answer of

$$X_1[k] = \frac{1}{32} \frac{\sin(\frac{9\pi k}{32})}{\sin(\frac{\pi k}{32})} e^{-j\pi k/8}.$$

Even though the analysis formula for the DFT suggests that we compute a sum from $n = 0$ to $n = N - 1$, we can adjust where the analysis window is centered in order to capture all non-zero samples in $x_1[n]$, allowing us to apply the DTFT-DFT relation.

Convolved Signals (6 points)

(c) Determine the discrete-time Fourier transform (DTFT) $X_2(\Omega)$ of the signal $x_2[n] = (x_1 * x_1)[n]$.

Convolution in the time domain corresponds to multiplication in the frequency domain. So, the time-domain expression

$$x_2[n] = (x_1 * x_1)[n]$$

is equivalent to the frequency-domain expression

$$X_2(\Omega) = X_1(\Omega)X_1(\Omega) = (X_1(\Omega))^2.$$

All we need to do is square our answer from **(a)**:

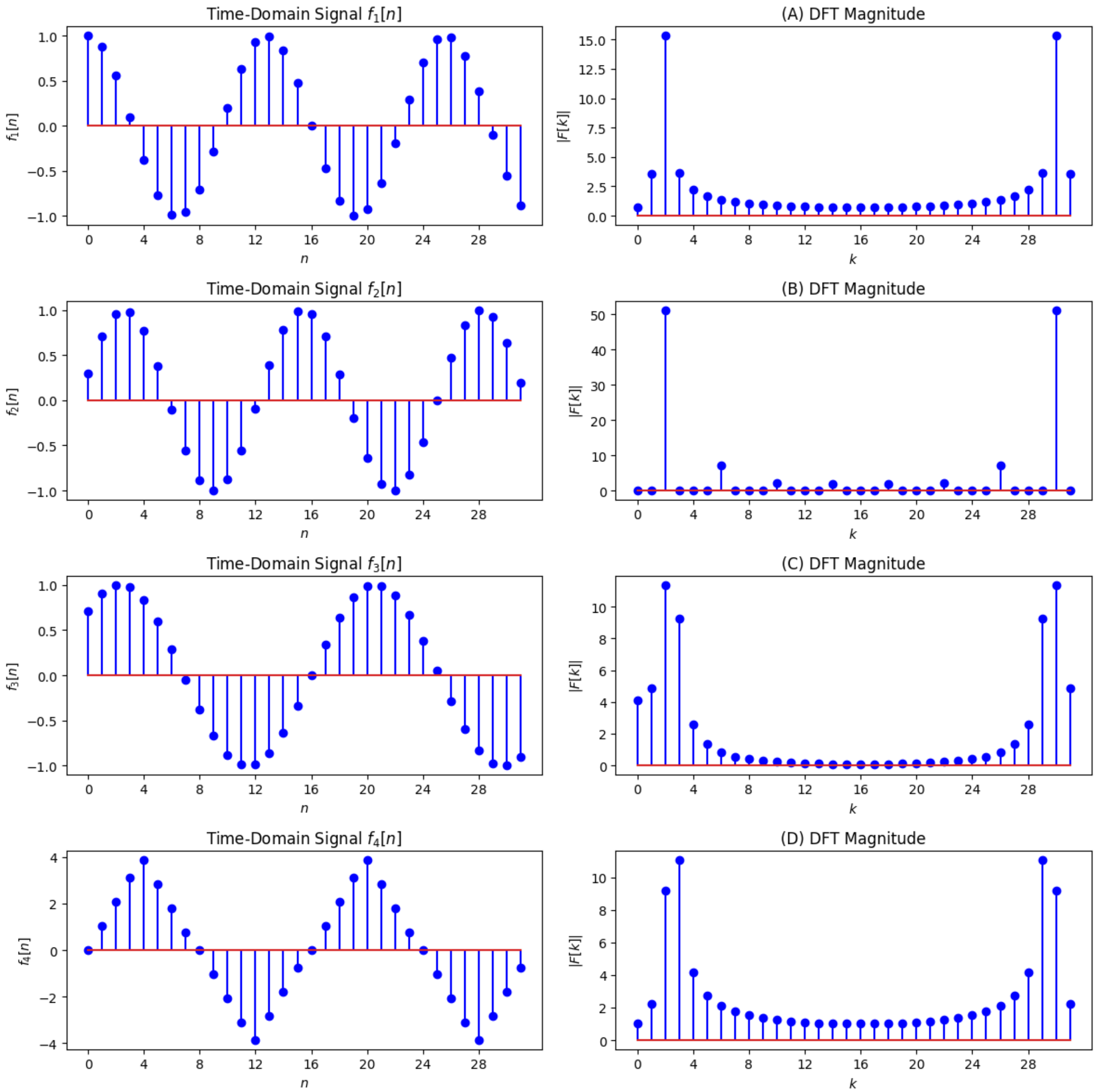
$$X_2(\Omega) = \left(\frac{\sin(\Omega(4+\frac{1}{2}))}{\sin(\Omega/2)} e^{-j2\Omega}\right)^2$$

and this could be simplified in a number of ways.

Problem #2: DFT Matching (16 points)

Match each time-domain signal — $f_1[n]$, $f_2[n]$, $f_3[n]$, and $f_4[n]$ — to the magnitude of its discrete Fourier transform — **A**, **B**, **C**, and **D**. Provide your reasoning for potential partial credit.

$f_1[n]$: **(D)**	$f_2[n]$: **(C)**	$f_3[n]$: **(A)**	$f_4[n]$: **(B)**



Problem #3: System Properties (10 points)

Determine if each system is linear and time-invariant. Briefly justify your answer.

$$x_i[n] \longrightarrow \boxed{\text{System}} \longrightarrow y_i[n]$$

System #1 (2 points)

(a) Consider a system with output $y_1[n] = n$. Is the system linear? Is the system time-invariant?

Homogeneity and additivity imply linearity. Notice that the output of this system has no direct dependence on the input. If we scaled the input, the output would not be correspondingly scaled. This system does not possess homogeneity, so this system is not linear. The output's direct dependence on the time index n is a dead ringer that this system is not time-invariant.

System #2 (2 points)

(b) Consider a system with output $y_2[n] = x_2[n] + 1$. Is the system linear? Is the system time-invariant?

This system is not linear. If we scale the input $x_2[n]$ by α, the output $y_2[n]$ is not correspondingly scaled by α. Instead of $\alpha y_2[n] = \alpha x_2[n] + \alpha$, the output will be $y_2[n] = \alpha x_2[n] + 1$. This system is time-invariant, though.

System #3 (2 points)

(c) Consider a system with output $y_3[n] = x_3[n] - x_3[n-1]$. Is the system linear? Is the system time-invariant?

This is some ambiguity here. Really, the problem should have included a statement along the lines of, "Assume that the system is at rest (i.e., $y_3[n] = 0$) until receiving an input." In that case, we can throw our hands in the air and say that, because we have a linear difference equation with constant coefficients, and all LTI systems can be represented by linear difference equations with constant coefficients, this system is both linear and time-invariant.

However, during the exam, we wrote $y[n < 0] = 0$ on the chalkboard. This constraint makes the given system neither linear nor time-invariant. Any input at time $n < 0$ would not have an impact on the output. So, we awarded points for anyone who made an attempt to justify their answers.

System #4 (2 points)

(d) Consider a system with output $y_4[n] = y_4[n-1] + x_4[n]$. Is the system linear? Is the system time-invariant?

We have a linear difference equation with constant coefficients. All LTI systems can be represented by linear difference equations with constant coefficients! This system is both linear and time-invariant.

System #5 (2 points)

(e) Consider a system with output $y_5[n] = 1$. Is the system linear? Is the system time-invariant?

The output of this system has no direct dependence on the input. Scaling the input does not correspondingly scale the output, so this system is not linear. However, this system is time-invariant: $y_5[n] = y_5[n - n_0] = 1$ for any n_0.

Problem #4: Signals and Systems (20 points)

Consider a linear, time-invariant (LTI) system

$$x[n] \longrightarrow \boxed{\text{LTI System with Unit Sample Response } h[n]} \longrightarrow y[n]$$

described by the difference equation

$$y[n] = x[n-3] + 2x[n-1] + x[n+1].$$

Unit Sample Response (4 points)

(a) Determine the system's unit sample response. That is, find an $h[n]$ such that $y[n] = (x * h)[n]$.

The unit sample response is the output of the system when the input is $\delta[n]$. So, making the substitution $x[n] = \delta[n]$, we get $h[n] = \delta[n-3] + 2\delta[n-1] + \delta[n+1]$.

Frequency Response (4 points)

(b) Determine the system's frequency response. That is, find an $H(\Omega)$ such that $Y(\Omega) = X(\Omega)H(\Omega)$.

The frequency response $H(\Omega)$ is the DTFT of the unit sample response $h[n]$:

$$H(\Omega) = e^{-j3\Omega} + 2e^{-j\Omega} + e^{j\Omega}$$

which we can write as

$$H(\Omega) = (e^{-j2\Omega} + 2 + e^{j2\Omega})e^{-j\Omega} = (2 + 2\cos(2\Omega))e^{-j\Omega}$$

which is in the form $H(\Omega) = |H(\Omega)|e^{j\angle H(\Omega)}$ for $|H(\Omega)| = 2 + 2\cos(2\Omega)$ and $\angle H(\Omega) = -\Omega$.

Frequency Response: Magnitude (4 points)

(c) Determine an expression for the magnitude of the frequency response, $|H(\Omega)|$, and plot it.

$|H'(\Omega)| = 2{+}2\cos(2\Omega)$. Note that $2{+}2\cos(2\Omega)$ is always non-negative, so this term will not contribute anything to the phase plot.

Frequency Response: Phase (4 points)

(d) Determine an expression for the phase of the frequency response, $\angle H(\Omega)$, and plot it.

$\angle H(\Omega) = -\Omega$. This is periodically extended outside a length-2π interval centered around the origin. Values of $\angle H(\Omega)$ outside the range $[-\pi, \pi]$ wrap back around in that range (e.g., after descending to a value of $-\pi$, the phase "jumps up" to a value of π).

Eigenfunctions (4 points)

(e) Suppose the input to the system is $x[n] = \cos(\frac{2\pi}{8}n)$. Determine an expression for $y[n]$ and plot it. Label all key parameters.

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\Omega n} \longrightarrow \boxed{\text{LTI System}} \longrightarrow H(\Omega)e^{j\Omega n} = |H(\Omega)|e^{j\angle H(\Omega)}e^{j\Omega n} = |H(\Omega)|e^{j(\Omega n + \angle H(\Omega))}$$

That is, if the input to an LTI system is a sinusoid, the output will be a scaled, phase-shifted sinusoid.

$$\cos(\Omega n) = \text{Re}\{e^{j\Omega n}\} \longrightarrow \boxed{\text{LTI System}} \longrightarrow |H(\Omega)|\cos(\Omega n + \angle H(\Omega))$$

Here, the output is $y[n] = (2 + 2\cos(2\frac{2\pi}{8}))\cos(\frac{2\pi}{8}n - \frac{2\pi}{8}) = (2+0)\cos(\frac{\pi}{4}n - \frac{\pi}{4}) = 2\cos(\frac{\pi}{4}(n-1))$.

The plot of $y[n]$ is nearly the same as $x[n]$, but shifted to the right by a single sample and scaled by a factor of 2.

Problem #5: Filter Design (20 points)

Suppose we sample a continuous-time (CT) signal $x(t)$ using a sampling rate $f_s = 40960$ Hz to obtain the discrete-time (DT) signal $x[n]$. We then compute the discrete Fourier transform (DFT) of $x[n]$, $X[k]$, using an analysis window of length N.

We filter $X[k]$ using an ideal band-pass filter described by

$$H[k] = \begin{cases} 1 & k_l \leq |k| \leq k_h \\ 0 & \text{otherwise} \end{cases}$$

for positive integers k_l and k_h.

Analysis Window (2 points)

(a) Suppose $x[n] = x[n + 4096]$ is periodic in 4096 samples and defined over one period as

$$x[n] = \begin{cases} n & 0 \leq n \leq 2048 \\ 4096 - n & 2049 \leq n \leq 4095. \end{cases}$$

Determine the smallest length of the DFT analysis window N that will properly capture all relevant information in the time and frequency domains.

To prevent spectral smearing, we want the length of the analysis window to be equal to the period of $x[n]$. So, we choose $N = 4096$.

Frequency Resolution (2 points)

(b) Determine the frequency resolution of $X[k]$ in Hz.

The frequency resolution is $f_s/N = 40960/8192 = 5$ Hz.

Cut-Off Frequencies (4 points)

(c) We want our band-pass filter to pass frequencies in the range $f_l \leq f \leq f_h$ and attenuate all frequencies outside that range. Determine the frequency indices k_l and k_h corresponding to the cut-off frequencies $f_l = 220$ Hz and $f_h = 1760$ Hz, respectively.

In general, the frequency f in Hz corresponding to a given frequency index k is given by the relation

$$f = k\frac{f_s}{N}$$

which we can rearrange as

$$k = \frac{f}{f_s}N.$$

So, the cut-off indices are

$$k_l = \frac{220}{40960} \cdot 8192 = 44$$

and

$$k_h = \frac{1760}{40960} \cdot 8192 = 352.$$

Band-Pass Filter Implementation in Python 3 (12 points)

(d) Implement an ideal band-pass filter in Python 3. Complete the definition of the `band_pass_filter` function below. You may not import any external libraries.

```
from lib6300.fft import fft, ifft

def band_pass_filter(x, fs, N, fl, fh):
    # x           Discrete-time signal contained in list
    # fs          Sampling rate in Hz
    # N           Length of DFT analysis window
    # fl, fh      Cut-off frequencies in Hz

    # Go to frequency domain
    X = fft(x[0:N])

    # Define cut-off indices
    kl = round(fl / fs * N)
    kh = round(fh / fs * N)

    # Create band-pass filter
    bpf = [0] * N
    for k in range(N//2):
        if (kl <= k <= kh):
            bpf[k]    = 1
            bpf[N-k]  = 1

    # Apply band-pass filter
    X_filtered = [0] * N
    for k in range(N):
        X_filtered[k] = X[k] * bpf[k]

    # Return to time domain
    x_filtered = ifft(X_filtered)
    return x_filtered
```

Problem #6: Convolution (16 points)

Each signal in the left column below — $f_0(t)$ to $f_7(t)$ — is zero outside the regions shown in the plots. Determine the result of convolving each of these signals with a periodic train of rectangular pulses given by

$$g(t) = \begin{cases} 1 & \sin(2\pi t) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Determine which waveform — **A** to **I** — shows the result of each convolution.
Provide your reasoning for potential partial credit.