# 6.300 Quiz 2

Spring 2023

Name: **Answers**

Kerberos/Athena Username:

5 questions 1 hour and 50 minutes

- Please **write your name and your Athena username in the box above**, and please do not write your name on any of the other pages of the exam.
- Please **WAIT** until we tell you to begin.
- This quiz is closed-book, but you may use two 8.5 $\times$ 11 sheets of paper (both sides) as a reference.
- You may **NOT** use any electronic devices (including computers, calculators, phones, etc.).
- If you have questions, please **come to us at the front** to ask them.
- Enter all answers in the boxes provided. Work on other pages with QR codes may be taken into account when assigning partial credit. **Please do not write on the QR codes.**
- If you finish the exam more than 10 minutes before the end time, please quietly bring your exam to us at the front of the room. If you finish within 10 minutes of the end time, please remain seated so as not to disturb those who are still finishing their quizzes.
- You may not discuss the details of the quiz with anyone other than course staff until final quiz grades have been assigned and released.

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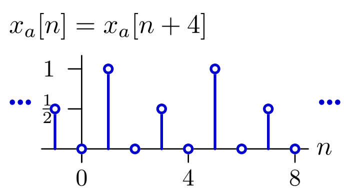
# 1 Summing Signals

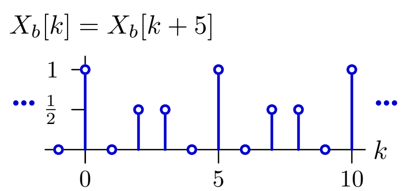
Let $x[n]$ represent the sum of two discrete-time signals:

$$x[n] = x_a[n] + x_b[n]$$

The signal $x_a[n]$, shown in the graph below, is periodic with a period of $4$:




The signal $x_b[n]$ is represented by its DTFS coefficients $X_b[k]$, which are periodic with a period of $5$ and which are shown in the graph below:




Find $x[-17]$.

We know that $x[-17] = x_a[-17] + x_b[-17]$, so we can find each of those pieces separately.

Since $x_a[\cdot]$ is periodic in 4, we know that $x[-17] = x[-13] = x[-9] = x[-5] = x[-1] = x[3] = \frac{1}{2}$.

Finding $x_b[-17]$ requires a little bit more work, but we can approach this by first applying the DTFS synthesis equation to find $x_b[n] = 1 + \cos\left(\frac{4\pi}{5}n\right)$. So $x_b[-17] = 1 + \cos\left(\frac{-17 \times 4\pi}{5}\right) = 1 + \cos\left(\frac{8\pi}{5}\right)$.

Putting those together, we find the answer below.

$x[-17] =$ $\frac{3}{2} + \cos\left(\frac{8\pi}{5}\right)$

*Worksheet (intentionally blank)*

## 2 Sampling

Consider taking a signal $x(\cdot)$ that is periodic in $T = 1$ second and sampling at a sampling rate of 6 samples per second to obtain DT signal $x[\cdot]$ that is periodic in $N = 6$. Analyzing the resulting DT signal, you find that:

- $x[\cdot]$ is a symmetric function of $n$.
- $x[n]$ is positive for all values of $n$.
- $x[0] + x[1] + x[2] + x[3] + x[4] + x[5] = 3$.
- $x[0] - x[1] + x[2] - x[3] + x[4] - x[5] = 1$.
- most of the Fourier series coefficients $X[\cdot]$ are 0; only two out of every 6 coefficients are nonzero.

What are two distinct CT functions $x(\cdot)$ that could have produced the results shown above?

We can start by thinking about the DTFS coefficients (with $N = 6$) of the sampled signal. The fact that $\sum_{n=\langle N\rangle} x[n] = 3$ tells us that the DC component is:

$$X[0] = \frac{1}{6}\left(\sum_{n=\langle 6\rangle} x[n]\right) = \frac{1}{6}(3) = \frac{1}{2}$$

The other important fact has to do with the alternating sum. We are given that $\sum_{n=\langle 6\rangle} x[n](-1)^n = 1$ This tells us about the component at $k = 3$:

$$X[3] = \frac{1}{6}\sum_{n=\langle 6\rangle} x[n]e^{-j\pi n} = \frac{1}{6}\left(\sum_{n=\langle 6\rangle} x[n](-1)^n\right) = \frac{1}{6}(1) = \frac{1}{6}$$

Since those are the only two non-zero DTFS coefficients, we can determine an expression for the discretized signal:

$$x[n] = \frac{1}{2} + \frac{1}{6}\cos(\pi n)$$

Now we need to determine a CT signal that, when sampled at a rate of 6 samples per second, gives us the expression above.

The DC component is unaffected by sampling, so we just need to find a CT cosine that, when sampled appropriately, gives us $\cos(\pi n)$.

We want a DT cosine where $\Omega = \pi$ (rad/sample), and we are given that $f_s = 6$ (samples/second). So we need $\omega$ (rad/sec) = $\Omega f_s = 6\pi$.

Thus, one CT signal that has these properties is given by $x(t) = \frac{1}{2} + \frac{1}{6}\cos(6\pi t)$.

We can find another CT signal that aliases down to the proper value by adding $2\pi$ to $\Omega$. In order to make $\Omega = 3\pi$, we need $\omega = 18\pi$. Thus, another CT signal that has the given properties is $x(t) = \frac{1}{2} + \frac{1}{6}\cos(18\pi t)$

$x(t) =$ $\frac{1}{2} + \frac{1}{6}\cos(6\pi t)$

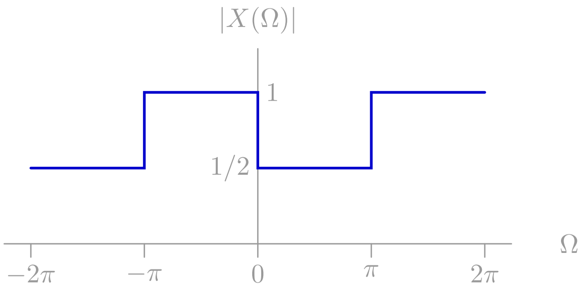
or

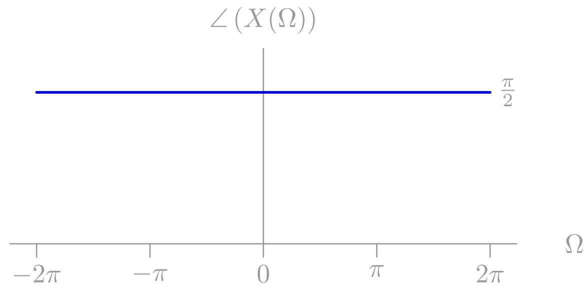
$x(t) =$ $\frac{1}{2} + \frac{1}{6}\cos(18\pi t)$

# 3 Filtering

**Part A**

Consider a signal $x[n]$, whose DTFT magnitude and phase are shown below:



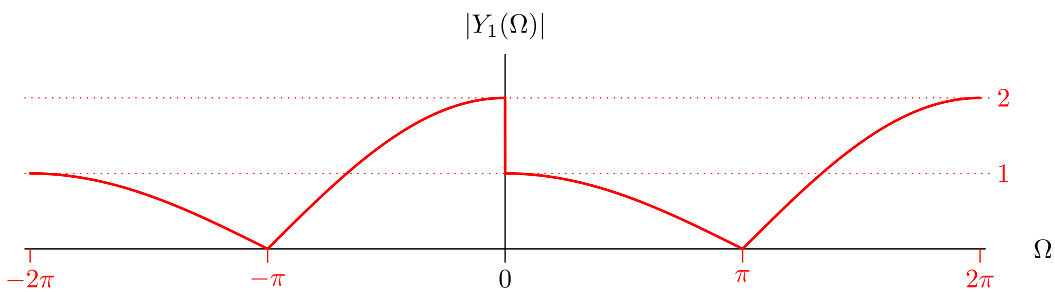

Assume that when $x[n]$ is the input to an LTI system whose frequency response is $H(\Omega) = e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}\right)$, the output is a new signal $y_1[n]$:

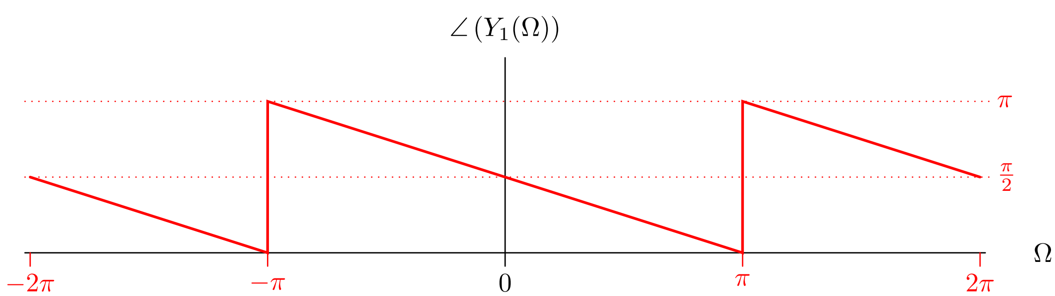
$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow y_1[n]$$

Sketch the magnitude and phase of $Y_1(\Omega)$, the DTFT of $y_1[n]$, on the axes below, and label all key points:

$Y_1(\Omega) = X(\Omega)H(\Omega)$, so $|Y_1(\Omega)| = |X(\Omega)| \times |H(\Omega)|$ and $\angle(Y_1(\Omega)) = \angle(X(\Omega)) + \angle(Y(\Omega))$.

Rearranging, we have $H(\Omega) = 2e^{-j\frac{\Omega}{2}}\cos\left(\frac{\Omega}{2}\right)$, which adds $\angle(\cos(\Omega/2)) - \frac{\Omega}{2}$ to the angle everywhere and scales the magnitude by $|2\cos(\Omega/2)|$. Putting those together, we find:




Note that the jumps of $\angle(Y_1(\Omega))$ from 0 to $\pi$ come from the underlying cosine flipping sign.

**Part B**

Now consider feeding $y_1[n]$ (the result from the last part) through the same system to produce a new result $y_2[n]$:

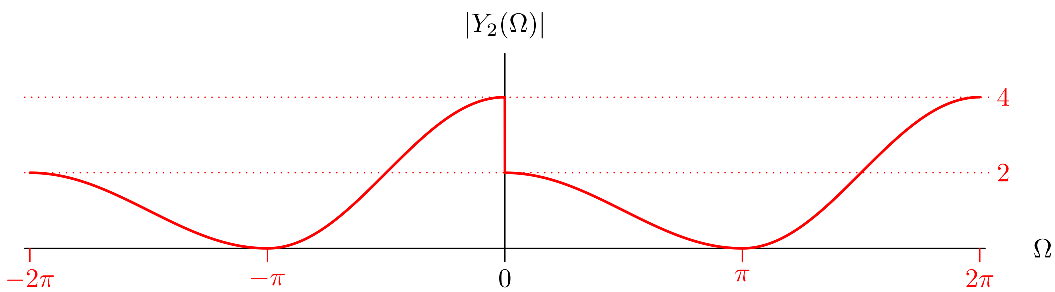

Sketch the magnitude and phase of $Y_2(\Omega)$, the DTFT of $y_2[n]$, on the axes below, and label all key points:

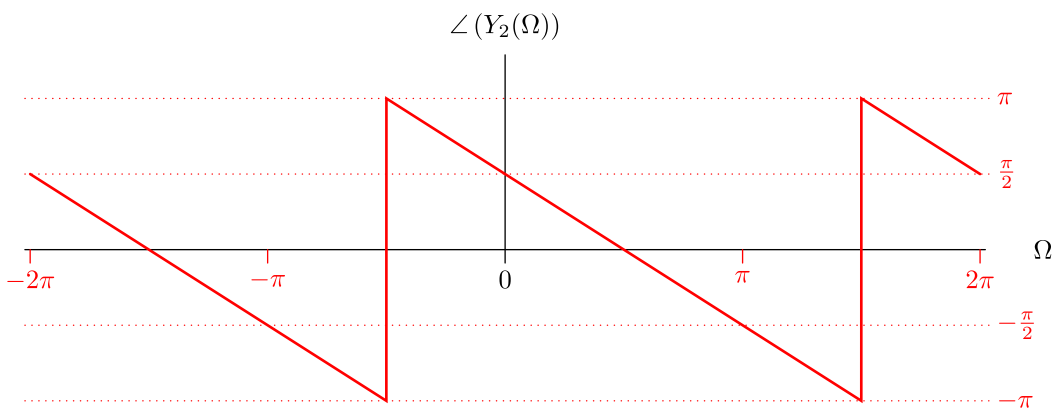
There are multiple approaches we could take here, but we can think of this as a single operation on $x$, where $Y_2(\Omega) = X(\Omega)H^2(\Omega)$.

Solving, $H^2(\Omega) = 2e^{-j\Omega}\left(1 + \cos(\Omega)\right)$.

So we have $Y_2(\Omega) = X(\Omega)H^2(\Omega)$, so $|Y_2(\Omega)| = |X(\Omega)| \times |2 + 2\cos(\Omega)|$.

The angle is actually easier to think about here than it was in part A. We have:
$\angle(Y_2(\Omega)) = \underbrace{\angle(X(\Omega))}_{\frac{\pi}{2}} + \underbrace{\angle(2+2\cos(\Omega))}_{0} + \underbrace{\angle(e^{-j\Omega})}_{-\Omega} = \frac{\pi}{2} - \Omega$




Note that the jumps of $\angle(Y_2(\Omega))$ from $-\pi$ to $\pi$ are just an artifact of the plotting here (since $-\pi$ and $\pi$ are the same number) and not actually an abrupt change in phase like the ones we saw with $\angle(Y_1(\Omega))$.

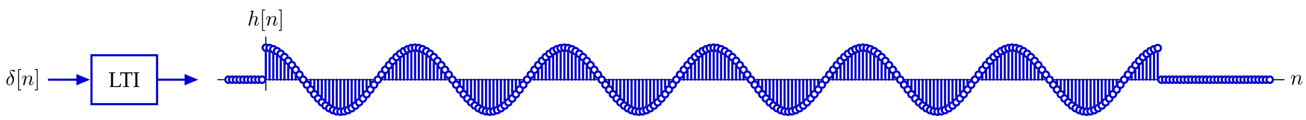
# 4 Sinusoidal Pulse

**Part A**

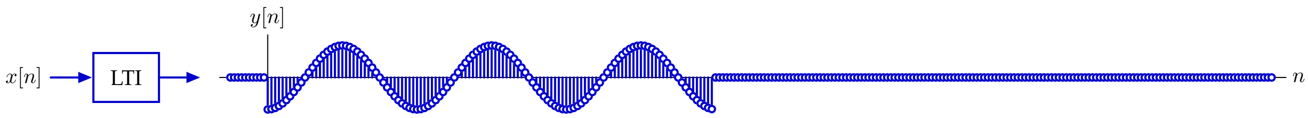
Consider an LTI system whose unit-sample response is given by:

$$h[n] = \begin{cases} \cos\left(\frac{\pi}{20}n\right) & \text{if } 0 \le n < 6N \\ 0 & \text{otherwise} \end{cases}$$

where $N$ is the fundamental period of the cosine wave, so that exactly six full periods of the cosine can be seen in the unit sample response:




For some other input $x[n]$, this system's output will consist of **three** full periods of an **upside-down** cosine at the same frequency, followed by zeros for all time:




Determine this signal $x[n]$. In the boxes below, enter both the value of $n$ and the associated value of $x[n]$ for the first 10 nonzero values of $x[n]$. The boxes below should contain **only numbers**. If there are fewer than 10 nonzero values in $x[n]$, write `None` in the remaining boxes.

| $n$ | $x[n]$ |
|---|---|
| 0 | -1 |
| 120 | 1 |
| 240 | -1 |
| 360 | 1 |
| 480 | -1 |
| 600 | 1 |
| 720 | -1 |
| 840 | 1 |
| 960 | -1 |
| 1080 | 1 |

(see explanation on following page)

This problem is similar to the "echo cancellation" lab, and so we can take a similar approach here to how we approached that problem. We can start by thinking about $x[n] = -\delta[n]$. In this case, $y[n] = -h[n]$, i.e., an upside-down cosine that goes through four six periods.

Adding another delta at $n = 3N = 120$ would cancel out the last three periods, but it would leave an additional three periods of the wave starting at $n = 240$. We can add another upside-down delta at $n = 240$ to cancel those out, but that leave *another* three periods off at $n = 360$.

In order to fully cancel things out, we need an infinite sequence of deltas, spaced out by 120 samples and alternating in sign.

**Part B**

We could compute this response by multiplying the DTFTs of these two signals, $X(\Omega)$ and $H(\Omega)$. Find closed-form expressions for $X(\Omega)$ and $H(\Omega)$ and enter them in the boxes below.

$$x[n] = -\sum_{m=0}^{\infty}(-1)^m\delta[n-120m] \quad\Rightarrow\quad X(\Omega) = -\sum_{m=0}^{\infty}(-1)^m e^{-j\Omega(120m)} = -\sum_{m=0}^{\infty}\left(-e^{-j120\Omega}\right)^m$$

Closing the sum gives:

$$X(\Omega) = \frac{-1}{1+e^{-j120\Omega}}$$

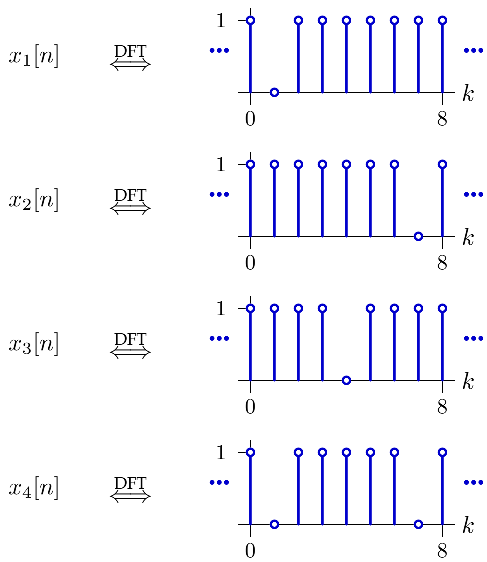
$$\begin{aligned}
H(\Omega) &= \sum_{n=0}^{239}\cos\left(\frac{\pi}{20}n\right)e^{-j\Omega n} \\
&= \sum_{n=0}^{239}\frac{1}{2}\left(e^{j\frac{\pi}{20}n}+e^{-j\frac{\pi}{20}n}\right)e^{-j\Omega n} \\
&= \sum_{n=0}^{239}\frac{1}{2}\left(e^{j\left(\frac{\pi}{20}-\Omega\right)n}+e^{-j\left(\frac{\pi}{20}+\Omega\right)n}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{239}\left(e^{j\left(\frac{\pi}{20}-\Omega\right)n}\right)+\sum_{n=0}^{239}\left(e^{-j\left(\frac{\pi}{20}+\Omega\right)n}\right)\right)
\end{aligned}$$

Simplifying using the hint from above, we find:

$$H(\Omega) = \frac{1}{2}\left(\frac{1-e^{j\left(\frac{\pi}{20}-\Omega\right)240}}{1-e^{j\left(\frac{\pi}{20}-\Omega\right)}}+\frac{1-e^{-j\left(\frac{\pi}{20}+\Omega\right)240}}{1-e^{-j\left(\frac{\pi}{20}+\Omega\right)}}\right)$$

# 5 Missing Components

Consider a set of discrete-time signals $x_i[n]$ represented by the DFT coefficients (computed with $N = 8$) shown below:




Fill in the table below by writing a single letter indicating the plot on the facing page (page 13) that matches the shape of the indicated signal. Note that the plots are not necessarily all on the same vertical scale.

| Signal | Real Part | Imaginary Part | Magnitude | Phase |
|---|---|---|---|---|
| $x_1[n]$ | D | K | C | O |
| $x_2[n]$ | D | L | C | N |
| $x_3[n]$ | B | A | C | M |
| $x_4[n]$ | E | A | F | J |

(see explanations on following page)

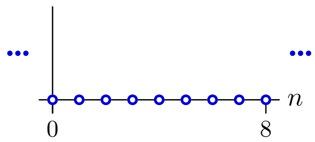
Looking at the $X_1[k]$ values we're given, we *could* find $x_1[n]$ by running a seven-point sum; but there is an easier way, by decomposing $X_1[k]$ into simpler signals. Specifically, we can say $X_1[k] = 1 - \delta[k-1]$. Thus, by linearity, $x_1[n] = 8\delta[n] - e^{j\frac{\pi}{4}n} = \left(8\delta[n] - \cos\left(\frac{\pi}{4}n\right)\right) - j\sin\left(\frac{\pi}{4}n\right)$. So the real part looks like $-\cos(\pi n/4)$ except at $n \equiv 0 \pmod 8$ where it has a value of 7, which is graph **D**. The imaginary part is $-\sin(\pi n/4)$, which is graph **K**. The magintude of this signal is 7 at $n \equiv 0 \pmod 8$ and 1 for the other values, which is graph **C**. For the phase, we know that the $x[0] = 7$, which has an angle of 0; the other values are just $-e^{j\pi n/4}$, which have angles of $\pi + \pi n/4$, which is graph **O**.

We can think about $X_2[k]$ in a similar way. It looks like a constant minus a delta, but now the delta has been shifted in the other direction. Thus, $x_2[n] = 8\delta[n] - e^{-j\pi n/4}$. This will have the same real part and magnitudes as the first signal (**D** and **C**, respectively); but its imaginary part and phase will have their signs flipped (except at $n \equiv 0 \pmod 8$), which gives us graph **L** for the imaginary part and graph **N** for the phase.

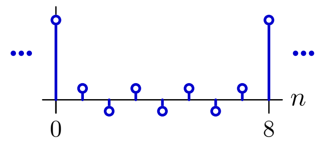
$X_3[k]$ looks like a shifted version of $X_1[k]$ or $X_2[k]$, so we know that its magnitude must be the same as those signals (graph **C**). But to find its real part, imaginary part, and phase, we need to look a little more closely. Here, we can again think of $X_3[k] = 1 - \delta[k-4]$, so by linearity, $x_3[n] = 8\delta[n] - e^{j\pi n} = 8\delta[n] - (-1)^n$, which within one period will look like $\{7, 1, -1, 1, -1, 1, -1, 1\}$. These numbers are all real, so the imaginary part must be 0 (graph **A**), and the real part will match that shape (graph **B**). Because we're dealing with purely-real numbers, its phase will be 0 when the signal is positive and $\pi$ when it is negative; so the phase is represented by graph **M**.

Finally, $X_4[k] = 1 - (\delta[k-1] + \delta[k+1])$, so $x_4[n] = 8\delta[n] - 2\cos(\pi n/4)$. So the real part will look similar to that from $x_1$ (a delta minus a cosine with a period of 8), but they are different in that the amplitude of the cosine is doubled while the height of the delta went down to 6 (instead of 7); so the real part must be graph **E**. Like $x_3[n]$, $x_4[n]$ is purely real, so it has no imaginary part (which corresponds to graph **A**). Since there is no imaginary part, the magnitude is just the absolute value of the real part, which looks like taking the parts of graph E that are below the horizontal axis and flipping them above the axis instead; thus, the magnitude is shown in plot **F**. And we can think about the phase in a similar way: anywhere graph E is negative, the phase will be $\pi$, and anywhere it is positive, the phase will be 0; thus, the phase is represented by graph **J**.

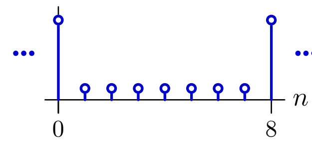
**Plot A**



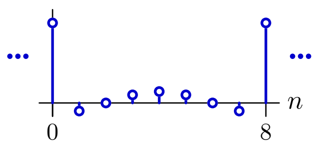

**Plot B**



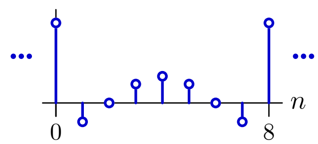

**Plot C**



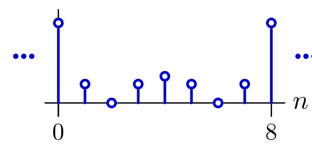

**Plot D**



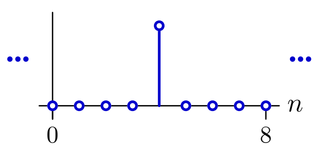

**Plot E**



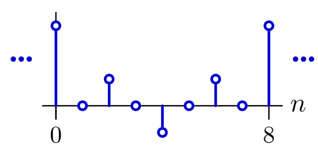

**Plot F**



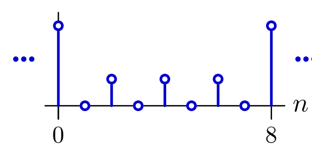

**Plot G**



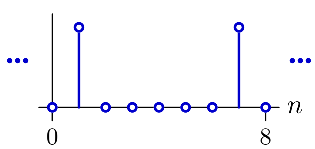

**Plot H**



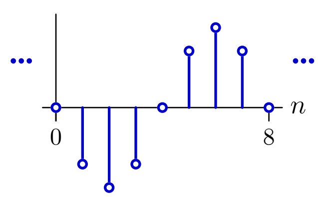

**Plot I**



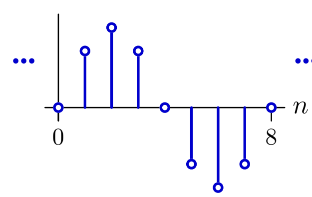

**Plot J**



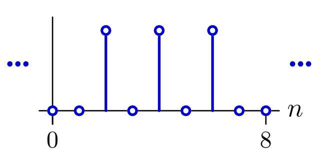

**Plot K**



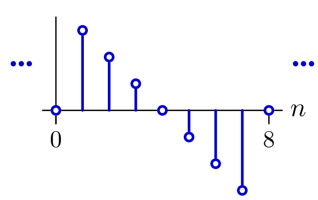

**Plot L**



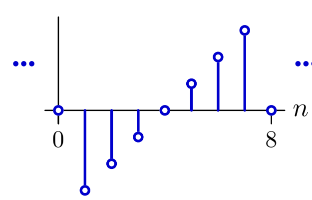

**Plot M**




**Plot N**




**Plot O**



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