6.300: Signal Processing

Fall 2024
Quiz #2

4 problems • 100 minutes • 25% of course grade

Name: **Solutions**

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- Wait until we tell you to begin.
- If we can't read it, we can't grade it.
- If you have questions, come to us at the front to ask.
- If you finish the quiz with less than 10 minutes remaining, quietly remain seated until we call time.

Problem #1	25 points	**(a)** 5, **(b)** 3, **(c)** 3, **(d)** 5, **(e)** 4, **(f)** 5
Problem #2	16 points	**(a)** 8, **(b)** 8
Problem #3	25 points	**(a)** 5, **(b)** 4, **(c)** 5, **(d)** 6, **(e)** 5
Problem #4	34 points	**(a)** 4, **(b)** 4, **(c)** 5, **(d)** 4, **(e)**4, **(f)** 4, **(g)** 4, **(h)** 5

Problem #1: A Cappella Transforms

Note: (a), (b), and (c) may be completed independently of (d), (e), (f).

Your TA, John (who has lost his signal processing memory), needs your help. The Chorallaries have come across a signal that is said to be the answer to the ultimate a cappella sound. Unfortunately, they only have the frequency response. Knowing that you're an expert in signal synthesis, John has come to you asking for your help finding the original signal from this frequency response:

$$X(\omega) = \begin{cases} e^{j\omega} & -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

(a) The Ultimate Signal

Help the Chorallaries find the ultimate signal by determining a closed-form expression for $x(t)$

$$\boxed{x(t) = \frac{\sin(\pi(t+1))}{\pi(t+1)} \text{ or } -\frac{\sin(\pi t)}{\pi(t+1)}}$$

The Chorallaries thank you so much for your hard work, but they started messing with things and messed up their frequency response — several times! They have a hunch that the following signals can be represented in terms of the original signal $x(t)$, but they don't know what to do. For the signal $x_b(t)$ and $x_c(t)$ below, could you help them to express in terms of $x(t)$?

(b) Encore!

$$X_b(\omega) = \begin{cases} 1 + e^{-2j\omega} & -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{x_b(t) = x(t-1) + x(t-3) \text{ or } x(t-1) + \tfrac{1}{2}x(-\tfrac{t}{2})}$$

(c) Here We Go Again

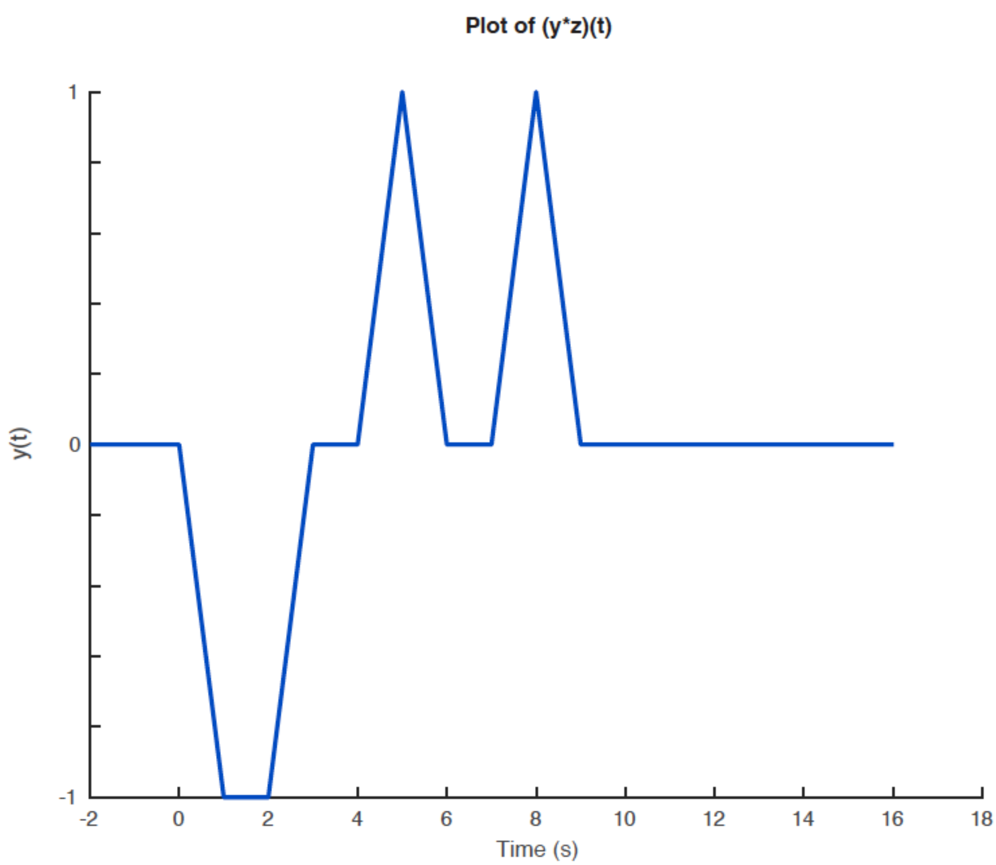
$$X_c(\omega) = \begin{cases} -\omega^2 e^{-j\omega} + j\omega^3 e^{-j\omega} & -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{x_c(t) = x''(-t) - x'''(-t) \text{ or } x''(t-2) - x'''(t-2)}$$

(d) The Next Great Signal

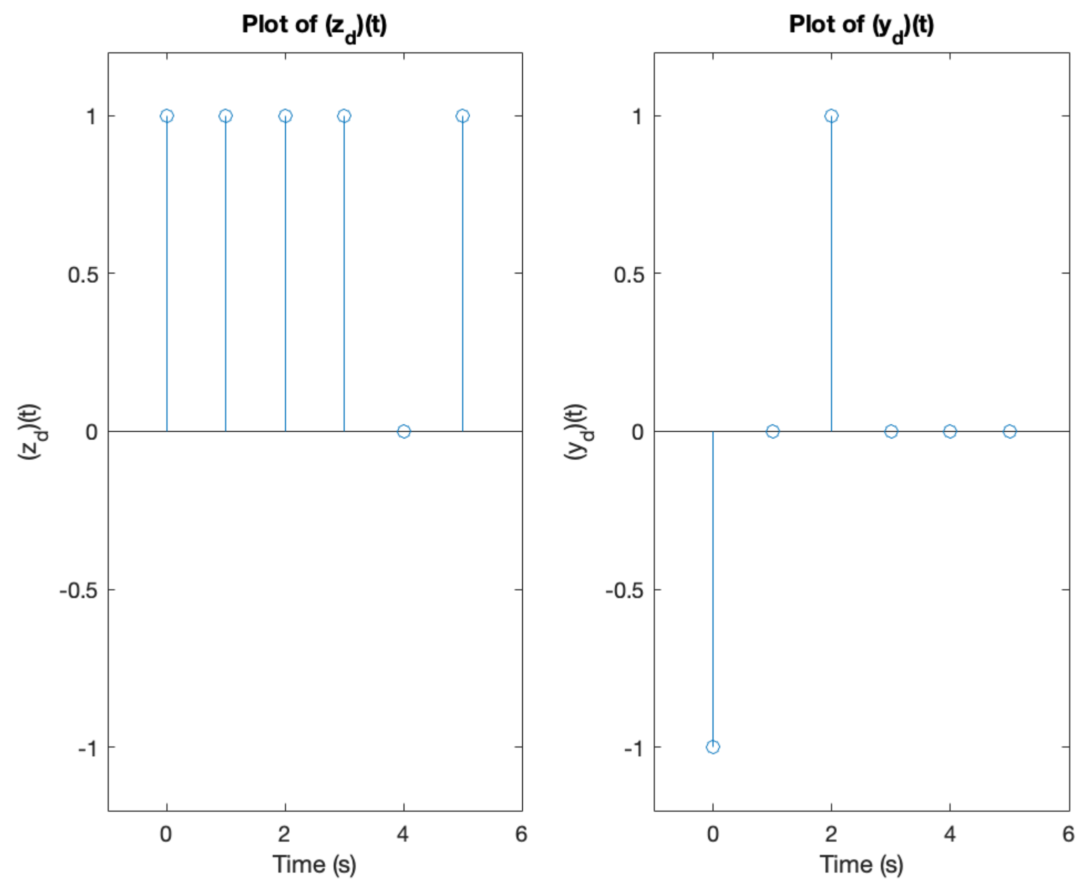
The Chorallaries now discover a new signal $y(t)$ that can make any other signal sound beautiful! They want to improve a boring signal $z(t)$ by convolving it with $y(t)$. Plot the convolution $(y*z)(t)$. Label all key parameters, including zero-crossings, maxima, minima, and where those extrema occur.

$$y(t) = \begin{cases} -1 & 0 \leq t < 1 \\ 1 & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases} \qquad z(t) = \begin{cases} 1 & 0 \leq t < 4 \\ 1 & 5 \leq t < 6 \\ 0 & \text{otherwise} \end{cases}$$



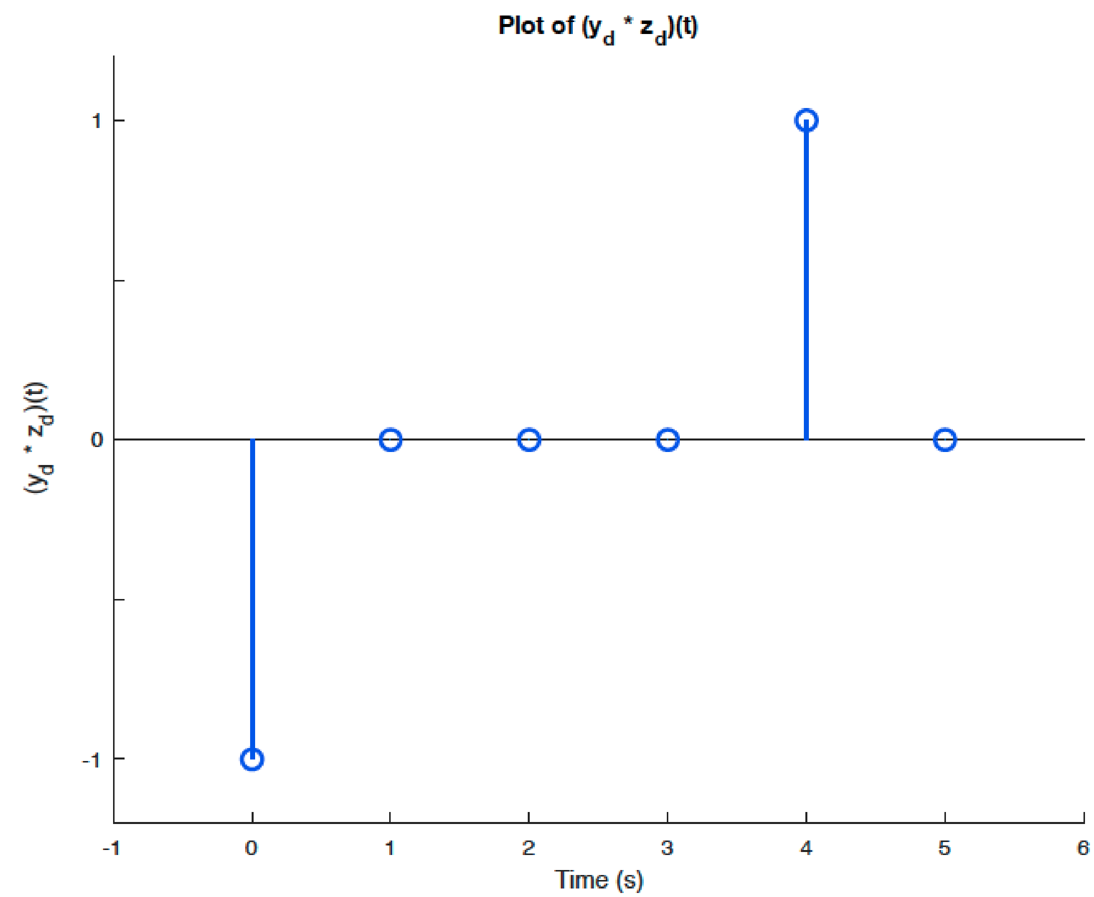
(e) Sampling the Signal on a Slow Computer

Now they want to use an old computer to analyze these signals. They take a discrete version of the signals by sampling at 1 Hz with sample $n = 0$ taken at $t = 0$, to get $z_d[n]$ and $y_d[n]$. Plot $z_d[n]$ and $y_d[n]$ with all important elements labeled.



(f) Making a New Sampled Signal

Their computer is slow, so they decide to analyze just $N = 6$ samples of this signal, from $n = 0$ to $n = 5$. What is the circular convolution of $z_d[n]$ with $y_d[n]$, analyzed with $N = 6$? Plot with all important elements labeled.



Problem #2: DFT Matching

Eight signals $g_1[n]$ to $g_8[n]$ are derived from $f_1[n]$, $f_2[n]$, $f_3[n]$, and $f_4[n]$. Match each time-domain signal $g_i[n]$ to the magnitude plot of its DFT computed with window size $N = 24$. For partial credit, briefly explain why the graph you chose is correct (a few words/expression is sufficient). Magnitude plots (**A-P**) are on the next two pages. The same plot may be used more than once.

(a) Periodic in DFT Window $N = 24$

Consider time-domain signals $f_1[n]$ and $f_2[n]$.

$$f_1[n] = \cos\left(\frac{5\pi n}{12}\right) \qquad f_2[n] = \cos\left(\frac{7\pi n}{12}\right)$$

	Answer	Explanation
1. $g_1[n] = f_1[n-3]$	C	
2. $g_2[n] = f_1[n] + f_2[n]$	H	
3. $g_3[n] = f_1[n] \cdot f_2[n]$	F	
4. $g_4[n] = (f_1 \circledast f_2)[n]$	A	

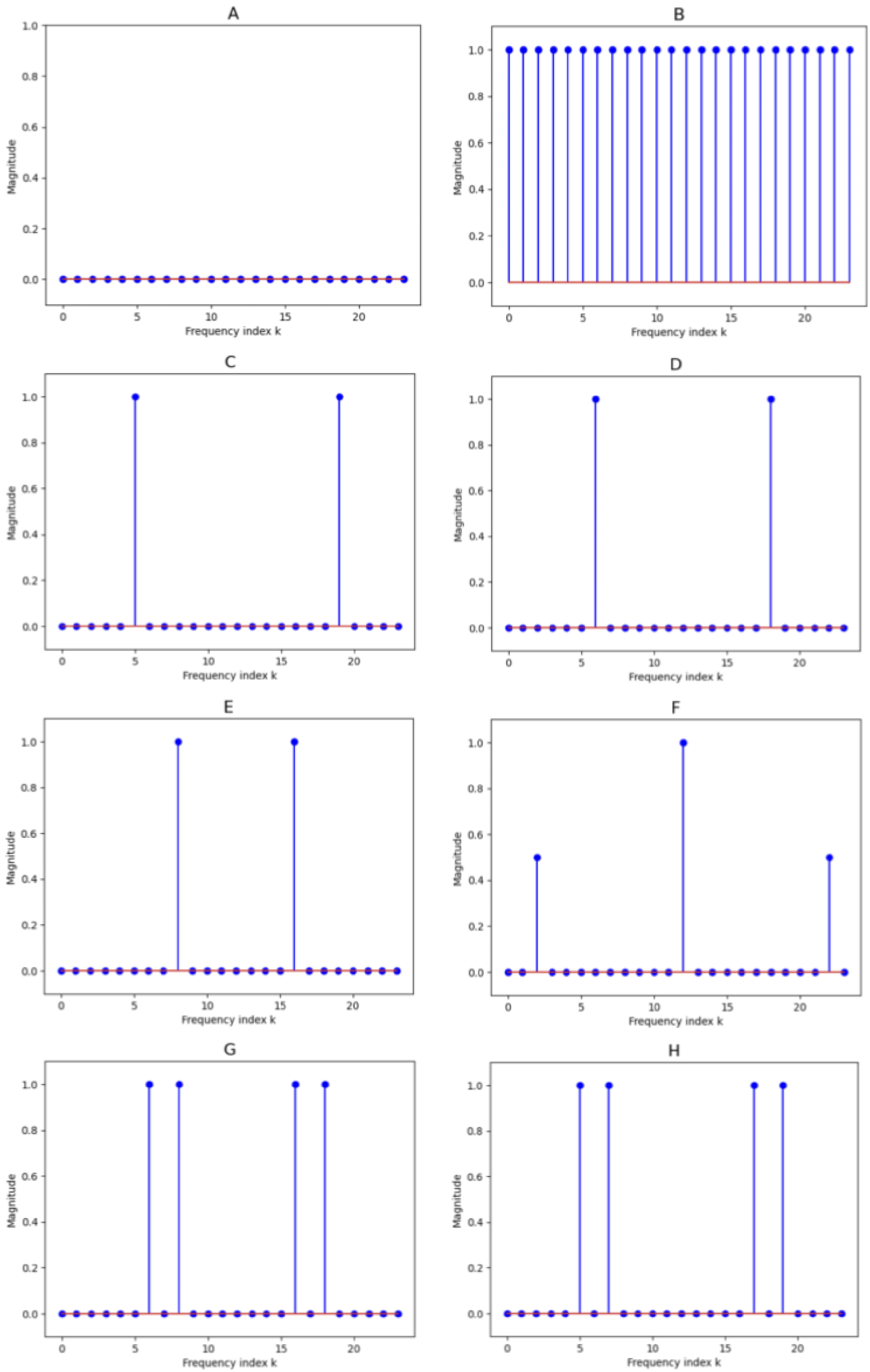
(b) Aperiodic in DFT Window $N = 24$

Consider time-domain signals $f_3[n]$ and $f_4[n]$.

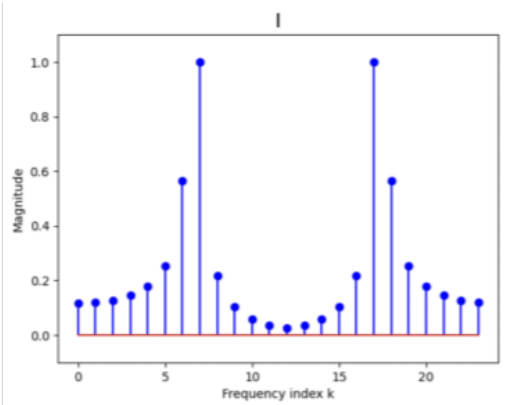
$$f_3[n] = \cos\left(\frac{5\pi n}{9}\right) \qquad f_4[n] = \cos\left(\frac{7\pi n}{9}\right)$$

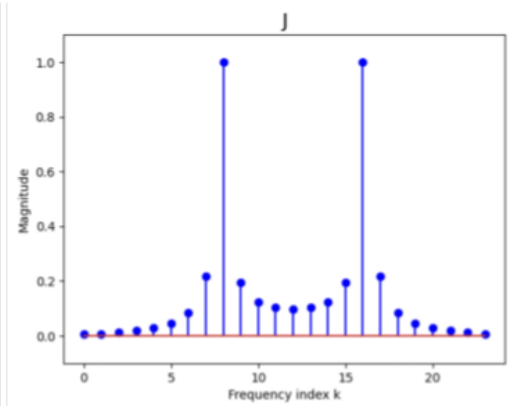
	Answer	Explanation
1. $g_5[n] = f_3[n-6]$	I	
2. $g_6[n] = f_4[n+4]$	L	
3. $g_7[n] = f_3[n] + f_4[n]$	M	
4. $g_8[n] = \begin{cases} f_3[\frac{n}{2}] + f_4[\frac{n}{2}] & \text{if } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$	O	

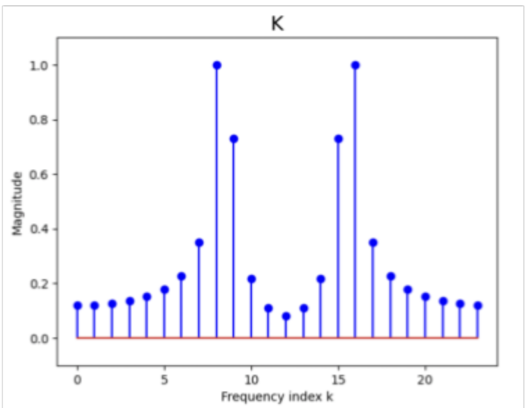
Note: Magnitude plots are normalized such that the largest magnitude is 1.

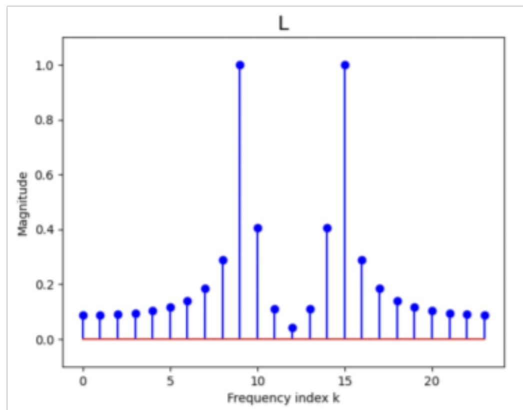


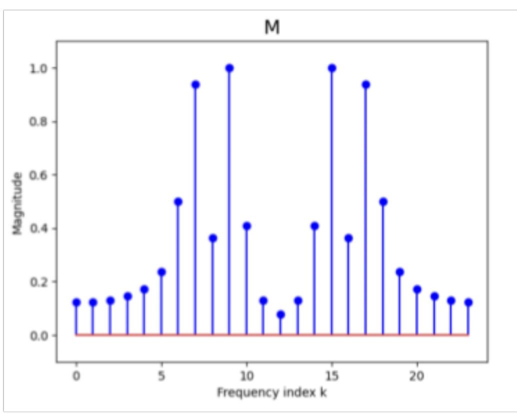
Note: Magnitude plots are normalized such that the largest magnitude is 1.

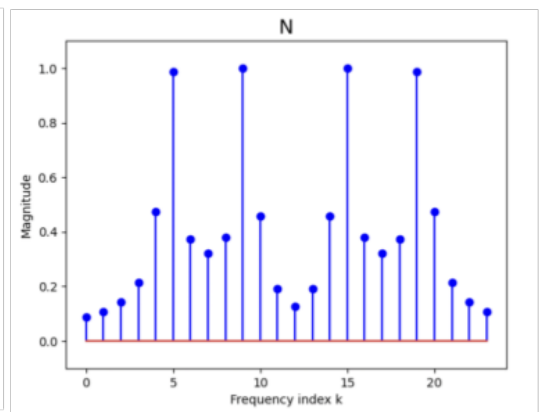


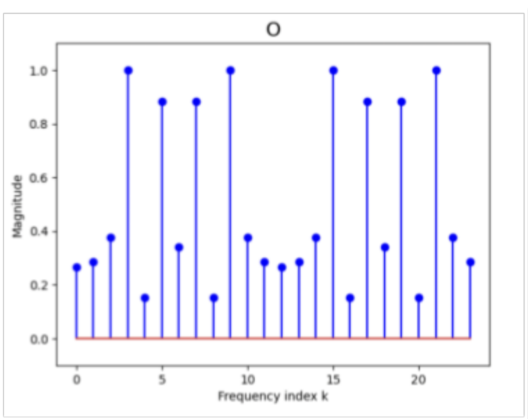


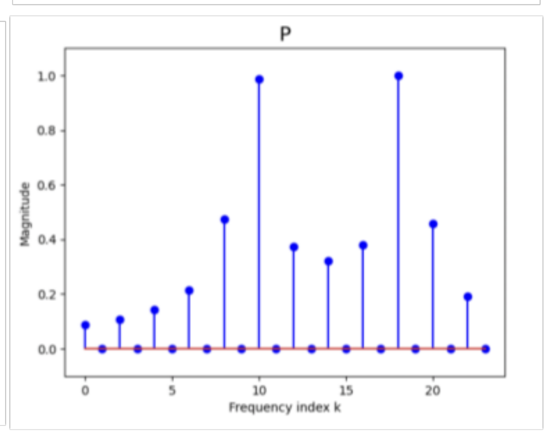












Problem #3: System Responses

(a) Consider the following linear, time-invariant system.

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$$

When the input is

$$x_1[n] = \delta[n] + \delta[n-1],$$

the output is

$$y_1[n] = 2\delta[n] + 2\delta[n-1] + \delta[n+1] + \delta[n-2].$$

Determine a simplified expression for the system's unit sample response $h_1[n]$.

Solution: Re-write $y[n]$ as

$$y[n] = (\delta[n] + \delta[n+1]) + (\delta[n] + \delta[n-1]) + (\delta[n-1] + \delta[n-2]).$$

We can see that if the input is $x_1[n] = \delta[n] + \delta[n-1]$, then to get the output $y[n]$, the convolution with $h_1[n]$ takes one shift to the left, one shift to the right, one without shift; all three have scaling factor of 1. Therefore we can conclude $h_1[n] = \delta[n+1] + \delta[n] + \delta[n-1]$.

(b) Suppose that the input is $x_2[n] = 2\delta[n-3]$. Determine the system's output $y_2[n]$.

Solution: $y_2[n]$=$(x_2 * h_1)[n] = 2\delta[n-2] + 2\delta[n-3] + 2\delta[n-4]$.

(c) Suppose that the input is $x_3[n] = \sin(\frac{2\pi}{3}n)$. Determine the system's output $y_3[n]$.

Solution: $h_1[n] = \delta[n+1] + \delta[n] + \delta[n-1] \leftrightarrow H_1(\Omega) = 1 + 2\cos(\Omega)$

$$x_3[n] = \sin\left(\frac{2\pi}{3}n\right) = \frac{1}{2j}(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n})$$

$$y_3[n] = H_1\left(\frac{2\pi}{3}\right)\left(\frac{1}{2j}(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n})\right) = 0$$

(d) Suppose that the input is $x_4[n] = \cos(\frac{\pi}{4}n)$. Determine $Y_4(\Omega)$, the discrete-time Fourier transform (DTFT) of the system's output.

Solution: $H_1(\Omega) = 1 + 2\cos(\Omega)$.

$$x_4[n] = \cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n})$$

$$X_4(\Omega) = \pi\delta\left(\Omega - \frac{\pi}{4}\right) + \pi\delta\left(\Omega + \frac{\pi}{4}\right) \text{ for } \Omega \in [-\pi, \pi]$$

$$Y_4(\Omega) = H_1(\Omega)X_4(\Omega) = \left(1 + 2\cos\left(\frac{\pi}{4}\right)\right)X_4\left(\frac{\pi}{4}\right) = \pi(1 + \sqrt{2})$$

(e) Consider a new linear, time-invariant system.

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow w[n] \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$$

Determine the cascaded system's unit sample response $h_2[n]$, such that $y[n] = (x * h_2)[n]$.

Solution: $h_1[n] = \delta[n+1] + \delta[n] + \delta[n-1]$.

$$h_2[n] = (h_1 * h_1)[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2].$$

Problem #4: Discrete-Time Systems

(a) Unit-Sample Response of System $\mathcal{S}_1$

Determine an expression for the unit-sample response $h[n]$ of the linear, time-invariant system $\mathcal{S}_1$ defined by the following linear, constant-coefficient difference equation, where $0 < \alpha < 1$.

$$y[n] + \alpha y[n-1] = x[n]$$

Assume that, before time $n = 0$, the system is at rest. That is, $x[n] = 0$ and $y[n] = 0$ for $n < 0$.

Solution: It may be more insightful to re-write the difference equation as

$$y[n] = x[n] - \alpha y[n-1].$$

To determine the unit-sample response, set $x[n] = \delta[n]$.

$$y[0] = x[0] - \alpha y[-1] = 1 - \alpha(0) = 1 = h[0]$$

$$y[1] = x[1] - \alpha y[0] = 0 - \alpha(1) = -\alpha = h[1]$$

$$y[2] = x[2] - \alpha y[1] = 0 - \alpha(-\alpha) = \alpha^2 = h[2]$$

$$y[3] = x[3] - \alpha y[2] = 0 - \alpha(\alpha^2) = -\alpha^3 = h[3]$$

$$\cdots$$

The unit-sample response $h[n]$ is a geometric sequence.

$$h[n] = (-\alpha)^n u[n] = \delta[n] - \alpha\delta[n-1] + \alpha^2\delta[n-2] - \alpha^3\delta[n-3] + \cdots$$

(b) Frequency Response of System $\mathcal{S}_1$

Determine an expression for the frequency response $H(\Omega)$ of the aforementioned system.

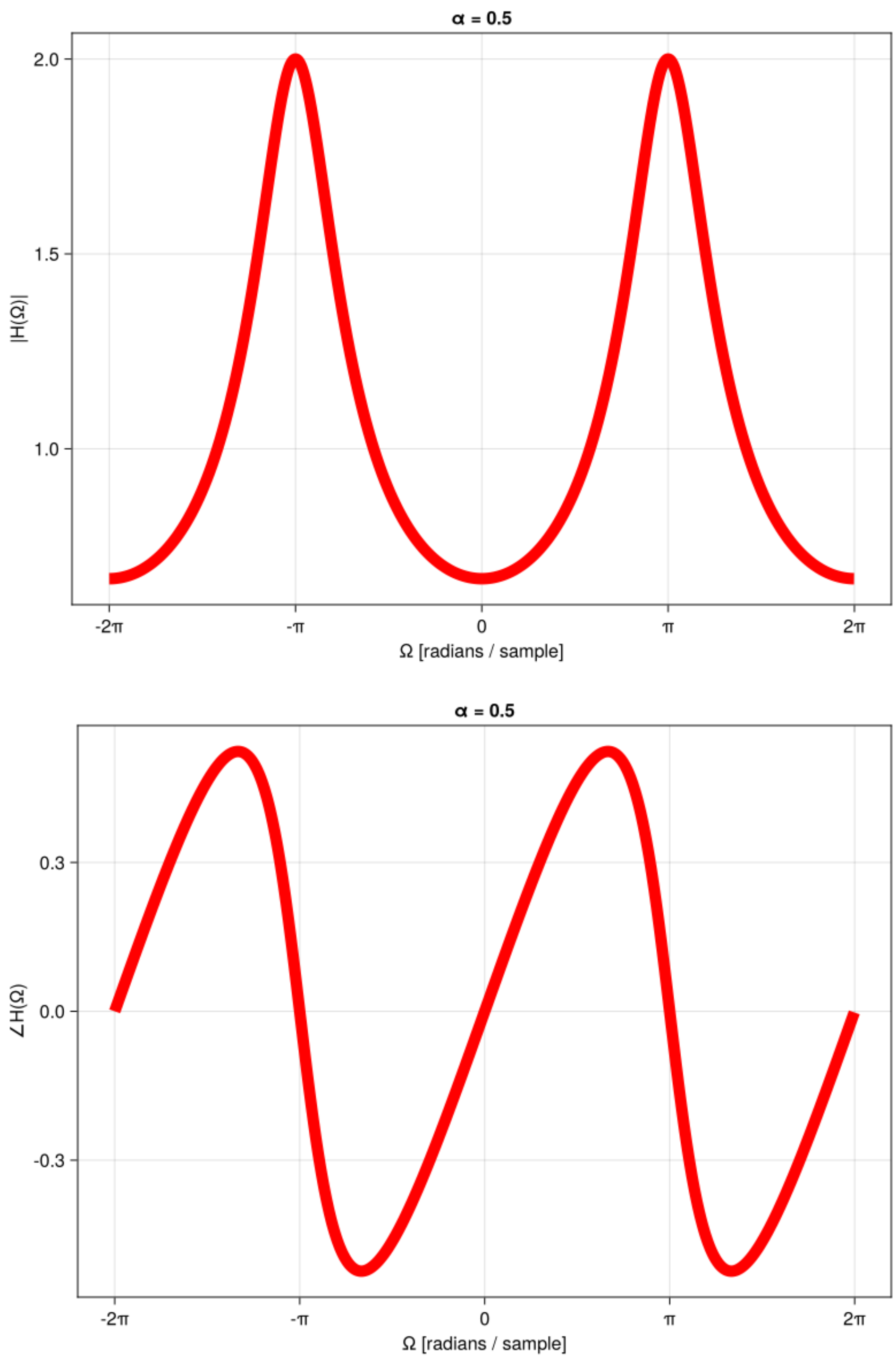
Solution: Apply properties of the DTFT to transform the time-domain difference equation into a frequency-domain algebraic equation. The frequency response $H(\Omega)$ is the ratio of $Y(\Omega)$ to $X(\Omega)$.

$$\left(1 + \alpha e^{-j\Omega}\right) Y(\Omega) = X(\Omega)$$

$$\frac{Y(\Omega)}{X(\Omega)} = H(\Omega) = \frac{1}{1 + \alpha e^{-j\Omega}}$$

(c) Plot the Frequency Response of System $\mathcal{S}_1$

Sketch the frequency response $H(\Omega)$ of the aforementioned system. Label all key parameters, including maxima, minima, and the points at which those extrema occur.



(d) Name that Filter (System $\mathcal{S}_1$)

Is the frequency response $H(\Omega)$ best described as a low-pass filter, high-pass filter, band-pass filter, or band-stop filter? Recall that $0 < \alpha < 1$.

No credit will be awarded for answers without justification.

low-pass filter **high-pass filter** **band-pass filter** **band-stop filter**

Solution: All DTFTs are periodic in $\Omega = 2\pi$. A low-pass filter has maximum gain near $\Omega = 0$ and attenuates frequencies near $\Omega = \pm\pi$, while a high-pass filter attenuates frequencies near $\Omega = 0$ and has maximum gain near $\Omega = \pm\pi$.

$$H(0) = \frac{1}{1+\alpha} < H(\pm\pi) = \frac{1}{1-\alpha} \implies \text{high-pass filter}$$

(e) Unit-Sample Response of System $\mathcal{S}_2$

Consider another linear, time-invariant system $\mathcal{S}_2$ with frequency response

$$G(\Omega) = 1 + \alpha e^{-j\Omega}$$

where, as before, $0 < \alpha < 1$. Determine the system's unit-sample response $g[n]$.

Solution: Compute the inverse DTFT of $G(\Omega)$.

$$g[n] = \frac{1}{2\pi}\int_{2\pi}\left(1 + \alpha e^{-j\Omega}\right)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} e^{j\Omega n}d\Omega + \frac{\alpha}{2\pi}\int_{2\pi} e^{j\Omega(n-1)}d\Omega = \delta[n] + \alpha\delta[n-1]$$

(f) Difference Equation for System $\mathcal{S}_2$

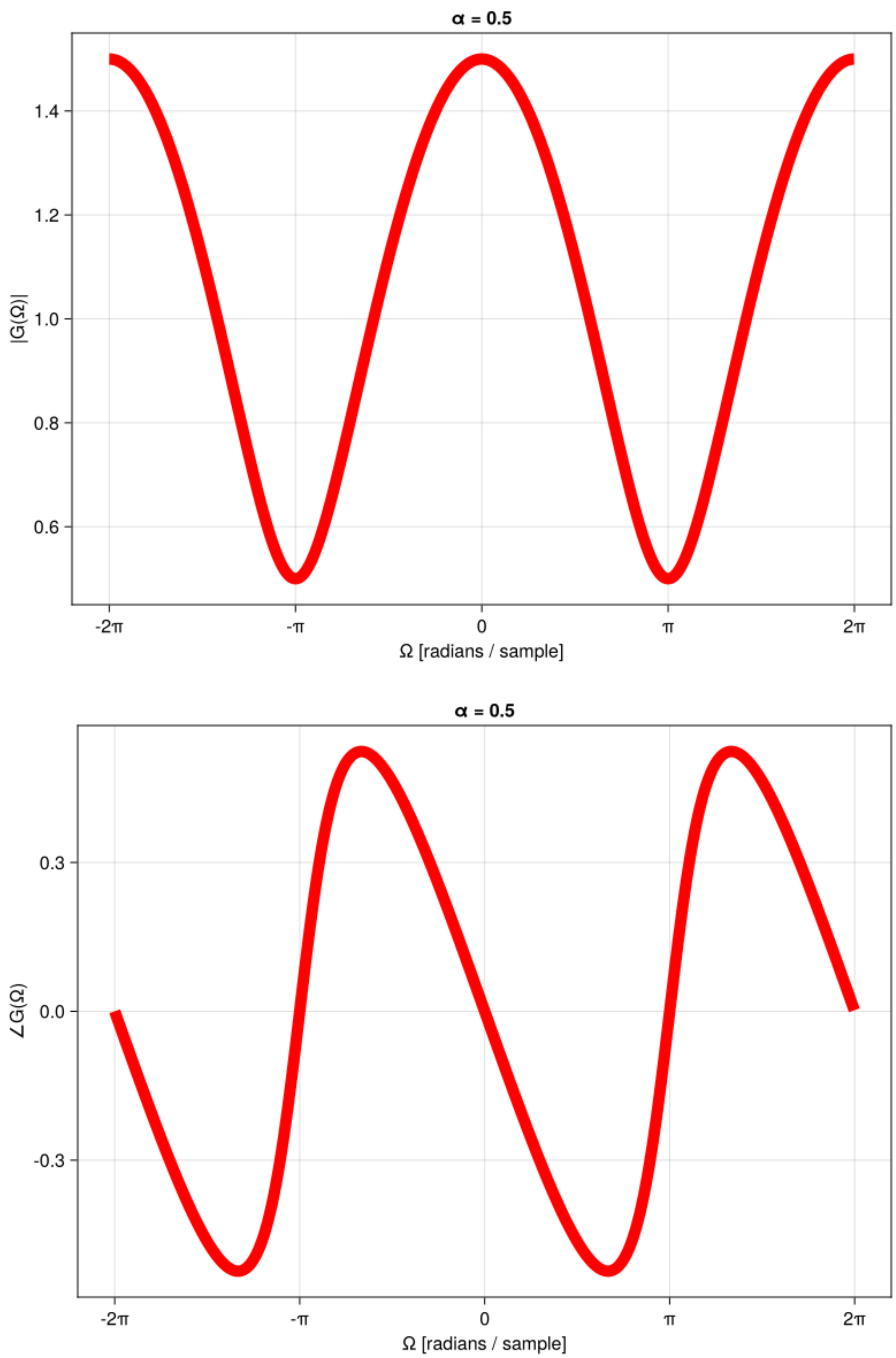
For system $\mathcal{S}_2$, determine a difference equation relating the input $x[n]$ to the output $y[n]$. Again, assume that, before time $n = 0$, the system is at rest — that is, $x[n] = 0$ and $y[n] = 0$ for $n < 0$.

Solution: Recall that the frequency response $G(\Omega)$ is the ratio of $Y(\Omega)$ to $X(\Omega)$.

$$G(\Omega) = 1 + \alpha e^{-j\Omega} = \frac{Y(\Omega)}{X(\Omega)} \implies Y(\Omega) = \left(1 + \alpha e^{-j\Omega}\right)X(\Omega) \implies y[n] = x[n] + \alpha x[n-1]$$

(g) Plot the Frequency Response of System $\mathcal{S}_2$

Sketch the frequency response $G(\Omega)$ of system $\mathcal{S}_2$. Label all key parameters, including maxima, minima, and the points at which those extrema occur.



(h) Name that Filter (System $\mathcal{S}_2$)

Is the frequency response $G(\Omega)$ best described as a low-pass filter, high-pass filter, band-pass filter, or band-stop filter? Recall that $0 < \alpha < 1$.

No credit will be awarded for answers without justification.

low-pass filter **high-pass filter** **band-pass filter** **band-stop filter**

Solution: $G(\Omega) = 1/H(\Omega)$. The inverse of a high-pass filter is a low-pass filter.

$$G(\Omega) = \frac{1}{H(\Omega)} \implies G(0) > G(\pm\pi) \implies \text{low-pass filter}$$