6.3000 Quiz 2 Fall 2023

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use two 8.5×11 sheets of notes (four sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us** at the front of the room to ask.

Please enter all solutions in the boxes provided.

Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam.

Please do not write on the QR codes at the bottom of each page.

We use those codes to identify which pages belong to each student.

Trigonometric Identities Reference

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$

$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$

$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$

$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$

$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$

$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$

$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$

$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Relating Transforms (18 points)

Let $F(\Omega)$ represent the discrete-time Fourier transform of the following discrete-time signal:

$$f[n] = \begin{cases} 1 & \text{if } 5 \leqslant n \leqslant 9 \\ 0 & \text{otherwise} \end{cases}$$

Part a. Determine numerical values (no sums or integrals) for $F(0)$, $F(\pi/2)$, and $F(\pi)$ and enter those values in the boxes below.

$F(0) =$	5
$F(\pi/2) =$	$-j$
$F(\pi) =$	-1

Briefly explain your reasoning in the box below.

Evaluate the DTFT analysis equation for $f[n]$ when $\Omega = 0, \pi/2$, and π.

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=5}^{9} e^{-j\Omega n}$$

$$F(0) = \sum_{n=5}^{9} 1 = 5$$

$$F(\pi/2) = \sum_{n=5}^{9} e^{-j\pi n/2} = (-j) + (-1) + (j) + (1) + (-j) = -j$$

$$F(\pi) = \sum_{n=5}^{9} e^{-j\pi n} = (-1) + 1 + (-1) + 1 + (-1) = -1$$

Part b. Let $F_{15}[k]$ represent the DFT of $f[n]$ computed with an analysis window $N = 15$. Let $g_{15}[n]$ represent the signal whose DFT is $F_{15}^2[k] = F_{15}[k] \times F_{15}[k]$. Determine the first five samples of $g_{15}[n]$ and enter those values in the boxes below (no sums or integrals).

$g_{15}[0] =$ 4/15

$g_{15}[1] =$ 3/15

$g_{15}[2] =$ 2/15

$g_{15}[3] =$ 1/15

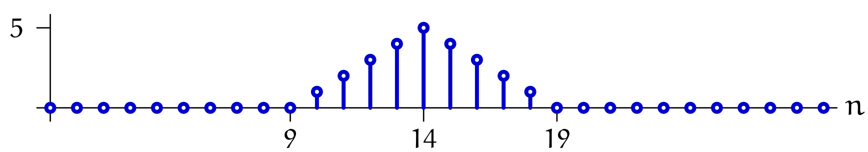
$g_{15}[4] =$ 0

Briefly explain your reasoning in the box below.

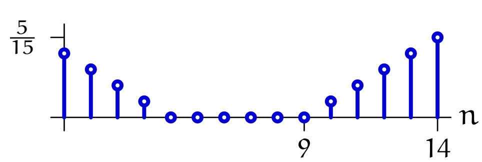
Convolve $f[n]$ with itself, then alias non-zero samples outside $n = 0$ to $n = 14$ into that range.

If the DFT of $g_{15}[n]$ is $F_{15}^2[k]$ then $g_{15}[n]$ must be equal to $\frac{1}{N}(f \circledast f)[n]$ where $\circledast$ represents circular convolution with $N = 15$.

Conventional convolution of $f[n]$ with itself yields the following signal.



The parts of this result that are outside the range $n = 0$ to 14 alias back into the region $n = 0$ to 14, resulting in the following:



Part c. Let $F_{12}[k]$ represent the DFT of $f[n]$ computed with an analysis window $N = 12$. Let $g_{12}[n]$ represent the signal whose DFT is $F_{12}^2[k] = F_{12}[k] \times F_{12}[k]$. Determine the first five samples of $g_{12}[n]$ and enter those values in the boxes below (no sums or integrals).

$g_{12}[0] =$ 3/12

$g_{12}[1] =$ 4/12

$g_{12}[2] =$ 5/12

$g_{12}[3] =$ 4/12

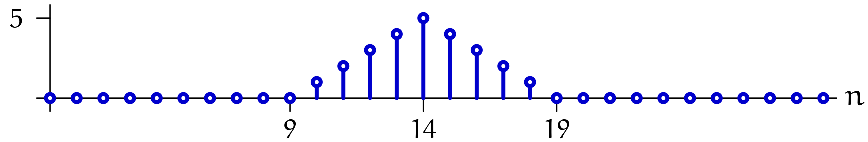
$g_{12}[4] =$ 3/12

Briefly explain your reasoning in the box below.

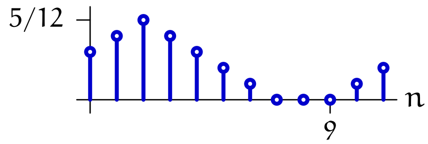
Convolve $f[n]$ with itself, then alias non-zero samples outside $n = 0$ to $n = 11$ into that range.

If the DFT of $g_{12}[n]$ is $F_{12}^2[k]$ then $g_{12}[n]$ must be equal to $\frac{1}{N}(f \circledast f)[n]$ where $\circledast$ represents circular convolution with $N = 12$.

Conventional convolution of $f[n]$ with itself yields the following signal.



The parts of this results that are outside the range $n = 0$ to 11 alias back into the region $n = 0$ to 11, resulting in the following:



Part d. Let $F_{10}[k]$ represent the DFT of $f[n]$ computed with an analysis window $N = 10$. Let $g_{10}[n]$ represent the signal whose DFT is $F_{10}^2[k] = F_{10}[k] \times F_{10}[k]$. Determine the first five samples of $g_{10}[n]$ and enter those values in the boxes below (no sums or integrals).

$g_{10}[0] =$ 1/10

$g_{10}[1] =$ 2/10

$g_{10}[2] =$ 3/10

$g_{10}[3] =$ 4/10

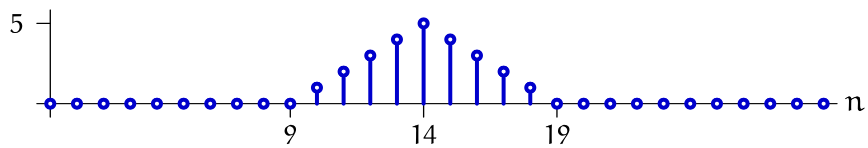
$g_{10}[4] =$ 5/10

Briefly explain your reasoning in the box below.

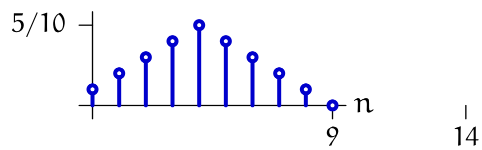
Convolve $f[n]$ with itself, then alias non-zero samples outside $n = 0$ to $n = 9$ into that range.

If the DFT of $g_{10}[n]$ is $F_{10}^2[k]$ then $g_{10}[n]$ must be equal to $\frac{1}{N}(f \circledast f)[n]$ where $\circledast$ represents circular convolution with $N = 10$.

Conventional convolution of $f[n]$ with itself yields the following signal.

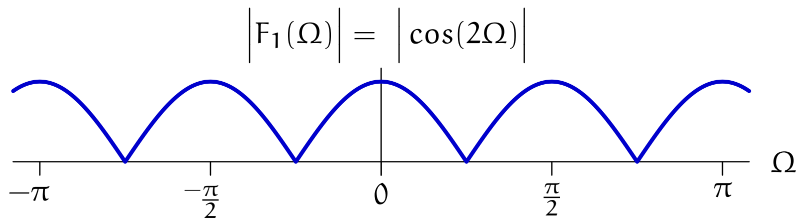


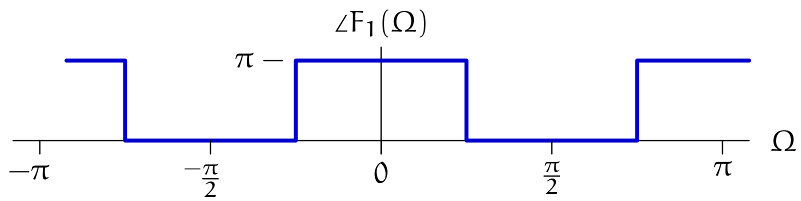
The parts of this results that are outside the range $n = 0$ to 9 alias back into the region $n = 0$ to 9, resulting in the following:



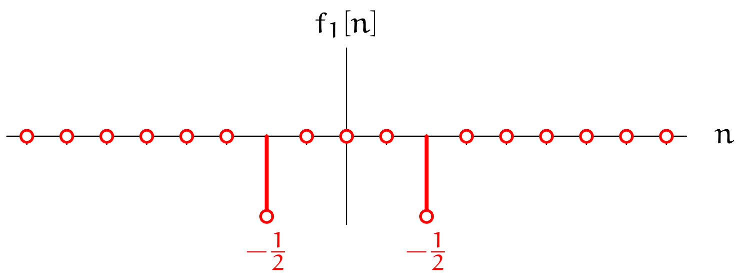
2 Finding Time (18 points)

Part a. Let $f_1[n]$ represent a discrete-time signal whose DTFT has the magnitude and angle shown below.





Determine $f_1[n]$ and plot it on the axes below. Label the values of each non-zero sample.



Shifting the phase of a DTFT by π is equivalent to multiplying its value by -1. Thus $F_1(\Omega)$ is given by

$$F_1(\Omega) = -\cos(2\Omega)$$

We can use Euler's formula to express the cosine function in terms of complex exponentials:

$$F_1(\Omega) = -\frac{1}{2}e^{j2\Omega} - \frac{1}{2}e^{-j2\Omega}$$

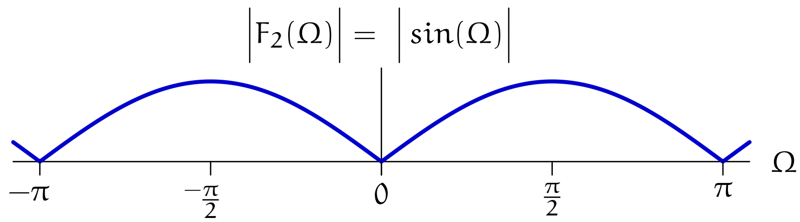
These terms have the same form as the $n = -2$ and $n = 2$ terms of the analysis formula.

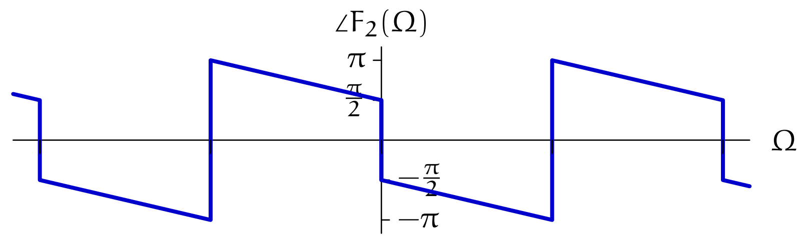
$$F_1(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n}$$

Therefore

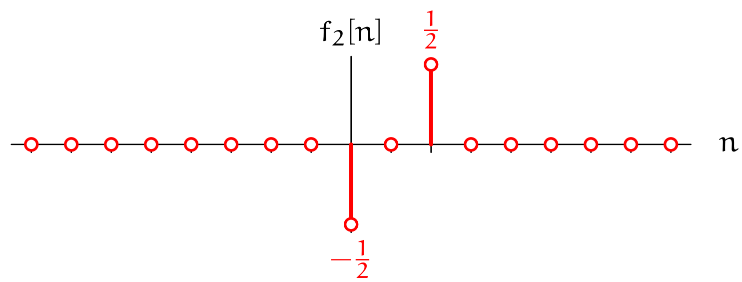
$$f_1[n] = -\frac{1}{2}\delta[n+2] - \frac{1}{2}\delta[n-2]$$

Part b. Let $f_2[n]$ represent a discrete-time signal whose DTFT has the magnitude and angle shown below.

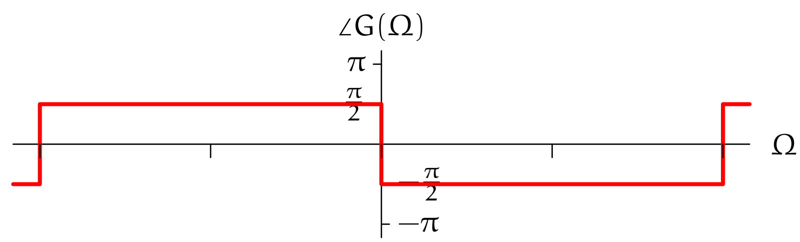




Determine $f_2[n]$ and plot it on the axes below. Label the values of each non-zero sample.



Except for Ω at integer multiples of $\pi/2$, the angle of $F_2(\Omega)$ has a slope of $-\Omega$. This downward sloping phase is equivalent to a delay of 1 in time. Therefore we can write $f_2[n]$ and a new function $g[n] = f_2[n+1]$ where the magnitude of $G(\Omega)$ is the same as that for $F(\omega)$ but the angle of $G(\Omega)$ is simpler, as shown below.



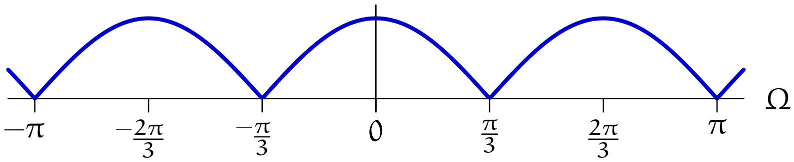
The phase shift of $-\pi/2$ is equivalent to multiplication by $-j$ and the phase shift of $\pi/2$ is equivalent to multiplication by j. Therefore $G(\Omega) = -j\sin(2\Omega)$ and $g[n]$ is given by the following.

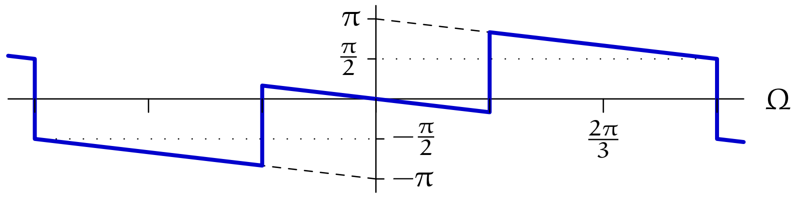
$$g[n] = \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+1]$$

The corresponding $f_2[n]$ is given by the following expression.

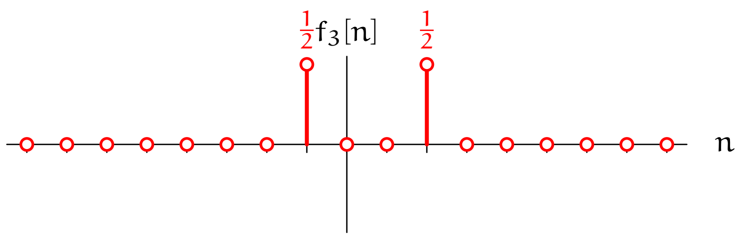
$$f_2[n] = \frac{1}{2}\delta[n-2] - \frac{1}{2}\delta[n]$$

Part c. Let $f_3[n]$ represent a discrete-time signal whose DTFT has the magnitude and angle shown below.





Determine $f_3[n]$ and plot it on the axes below. Label the values of each non-zero sample.



The complex amplitude of $F_3(\Omega)$ is the product of its magnitude $|F_3(\Omega)|$ and $e^{j\angle F_3(\Omega)}$. Notice that the phase jumps discontinuously by π at $\Omega = \pm\frac{\pi}{3}$ and $\pm\pi$. These jumps in phase coincide with discontinuities in the slope of the magnitude function, and correspond to changes in the sign of $\cos\left(\frac{3}{2}\Omega\right)$. Therefore we can combine the magnitude and phase to derive an equivalent expression for $F_3(\Omega)$ as follows:

$$F_3(\Omega) = \cos\left(\frac{3}{2}\Omega\right)\, e^{-j\frac{1}{2}\Omega}$$

where the phase term is linear (with a slope of -1/2) after the discontinuities have been removed.

Next, we can express the cosine term as complex exponentials to get

$$F_3(\Omega) = \frac{1}{2}\left(e^{j\frac{3}{2}\Omega} + e^{-j\frac{3}{2}\Omega}\right) e^{-j\frac{1}{2}\Omega} = \frac{1}{2}e^{j\Omega} + \frac{1}{2}e^{-j2\Omega}$$

The inverse DTFT of this expression is

$$f_3[n] = \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-2]$$

so there are only two nonzero samples: $f_3[-1] = \frac{1}{2}$ and $f[2] = \frac{1}{2}$.

Worksheet (intentionally blank)

3 Find the System (18 points)

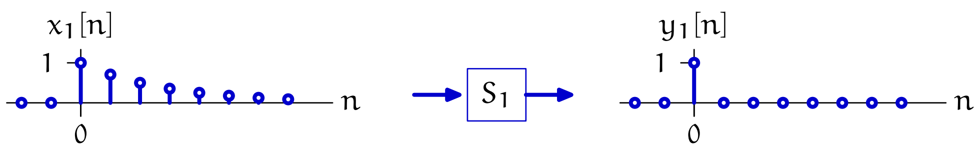
Part a. Let S_1 represent a discrete-time system that is linear and time invariant. When the input to S_1 is

$$x_1[n] = \begin{cases} 2^{-n/2} & \text{if } n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

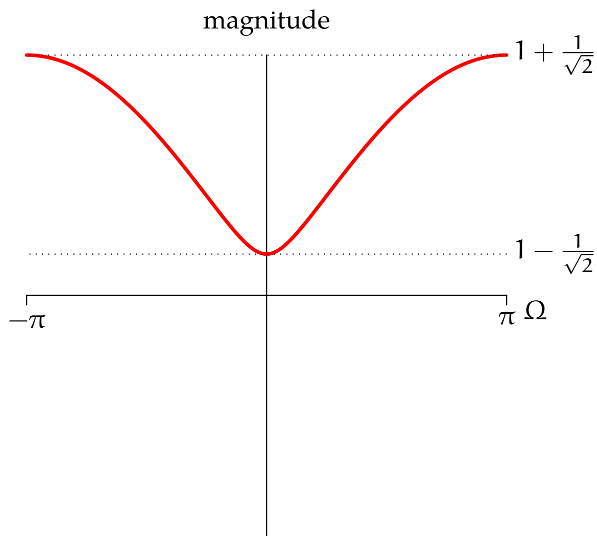
the output of the system is the unit-sample signal

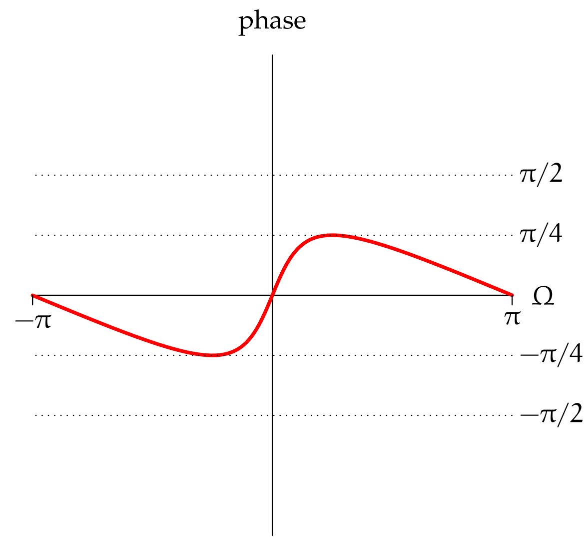
$$y_1[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

as illustrated below.



On the axes below, plot the magnitude and phase of the frequency response of system S_1. Label the important values in each sketch.





The frequency response of system S_1 is

$$H_1(\Omega) = \frac{Y_1(\Omega)}{X_1(\Omega)}$$

where $X_1(\Omega)$ is the Fourier transform of $x_1[n]$

$$X_1(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} 2^{-n/2}e^{-j\Omega n} = \frac{1}{1-\frac{1}{\sqrt{2}}e^{-j\Omega}}$$

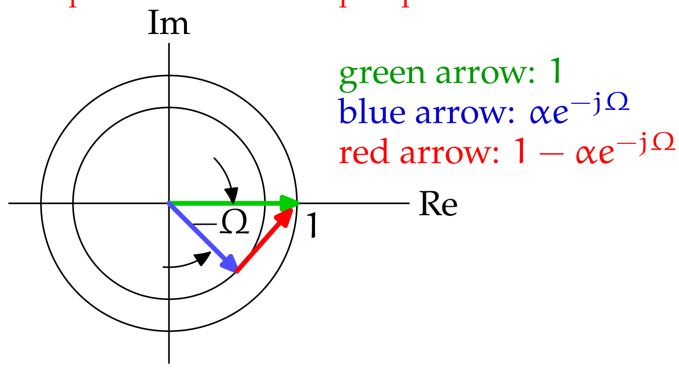
and $Y_1(\Omega)$ is the Fourier transform of $y_1[n]$.

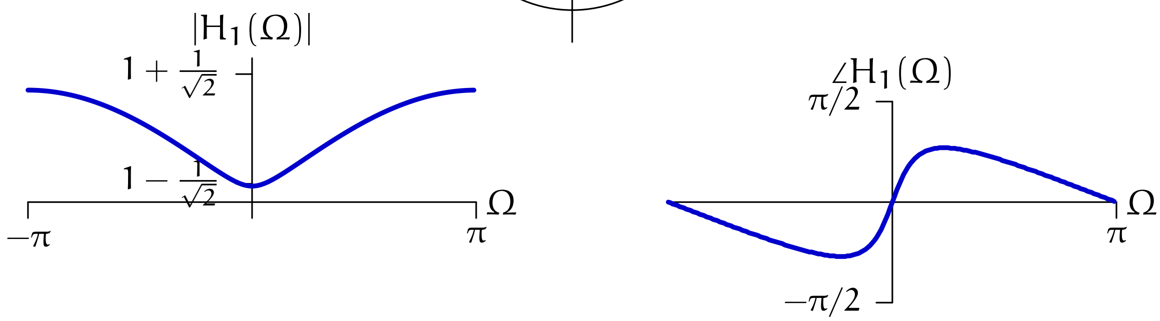
$$Y_1(\Omega) = \sum_{n=-\infty}^{\infty} y_1[n]e^{-j\Omega n} = 1$$

Therefore

$$H_1(\Omega) = 1 - \frac{1}{\sqrt{2}}e^{-j\Omega}$$

which is the sum of two numbers that can be represented in the complex plane.





Key features of magnitude plot:

- Maximum values of $|H_1(\Omega)|$ at $\Omega = \pm\pi$.
- Minimum value of $|H_1(\Omega)|$ at $\Omega = 0$.
- Maximum value of $1+\frac{1}{\sqrt{2}}$
- Minimum value of $1-\frac{1}{\sqrt{2}}$.
- Periodic in 2π.

Key features of angle plot:

- $\angle H(\Omega) = 0$ at Ω = integer multiples of π.
- Maximum $\angle H_2(\Omega)$ approximately $\pi/4$.
- Minimum $\angle H_2(\Omega)$ approximately $-\pi/4$.
- Steep slope at $\Omega = 0$; shallower negative slope at $\Omega = \pi$.
- Periodic in 2π.

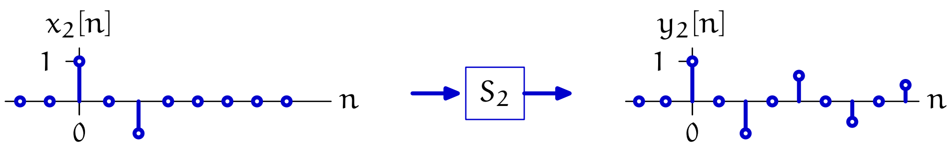
Part b. Let S_2 represent a discrete-time system that is linear and time invariant. When the input to S_2 is

$$x_2[n] = \begin{cases} 1 & \text{if } n = 0 \\ -0.8 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

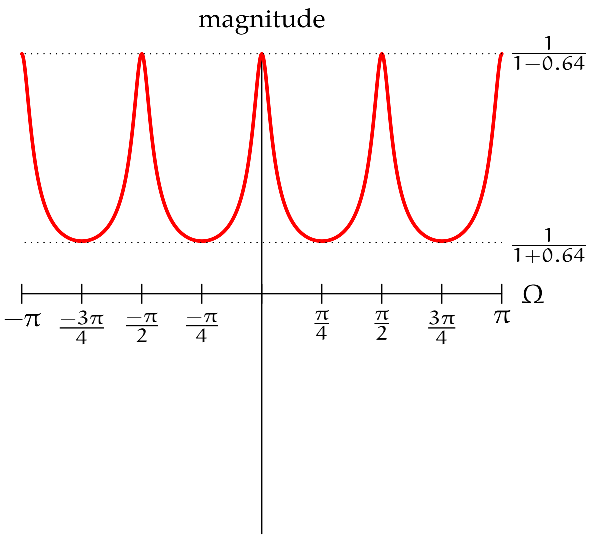
the output of the system is

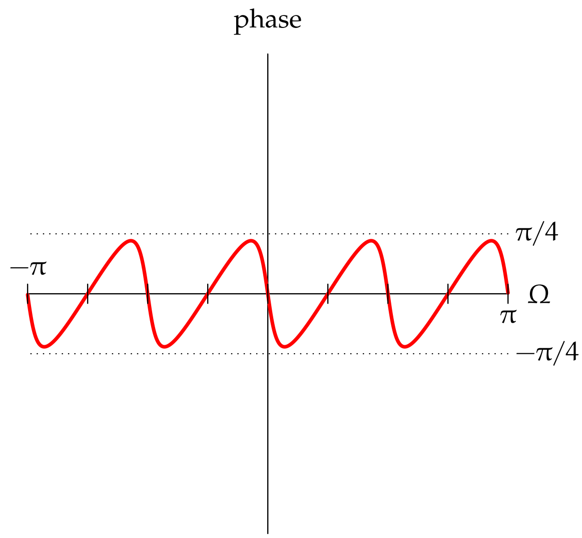
$$y_2[n] = \begin{cases} (-0.8)^{n/2} & \text{if } n \text{ is even and } n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

as illustrated below.



On the axes below, plot the magnitude and phase of the frequency response of system S_2. Label the important values in each sketch.





The frequency response of system S_2 is

$$H_2(\Omega) = \frac{Y_2(\Omega)}{X_2(\Omega)}$$

where $X_2(\Omega)$ is the Fourier transform of $x_2[n]$

$$X_2(\Omega) = \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\Omega n} = 1 - 0.8e^{-j2\Omega}$$

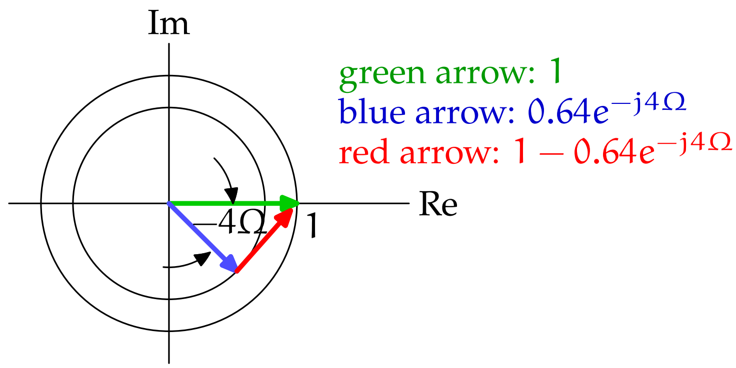
and $Y_2(\Omega)$ is the Fourier transform of $y_2[n]$.

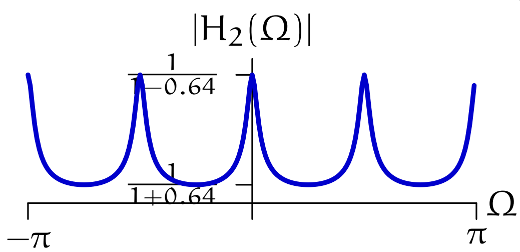
$$Y_2(\Omega) = \sum_{n=-\infty}^{\infty} y_2[n]e^{-j\Omega n} = \sum_{i=0}^{\infty}(-0.8)^i e^{-j2\Omega i} = \frac{1}{1+0.8e^{-j2\Omega}}$$

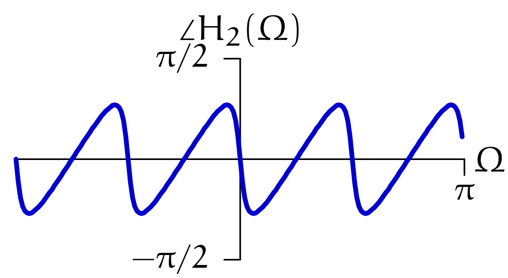
Therefore

$$H_2(\Omega) = \frac{1}{(1+0.8e^{-j2\Omega})(1-0.8e^{-j2\Omega})} = \frac{1}{1-0.8^2e^{-j4\Omega}}$$

The frequency response is the INVERSE of the difference of two numbers, which can be represented in the complex plane.







Key features of magnitude plot:

- Maximum values of $|H_2(\Omega)|$ at $\Omega =$ integer multiples of $\pi/2$.
- Minimum values of $|H_2(\Omega)|$ at $\Omega = \pi/4, 3\pi/4$.
- Maximum value approximately 2.78, minimum value approximately 0.61.
- Smooth curve with cusps.
- Periodic in $\pi/2$.

Key features of angle plot:

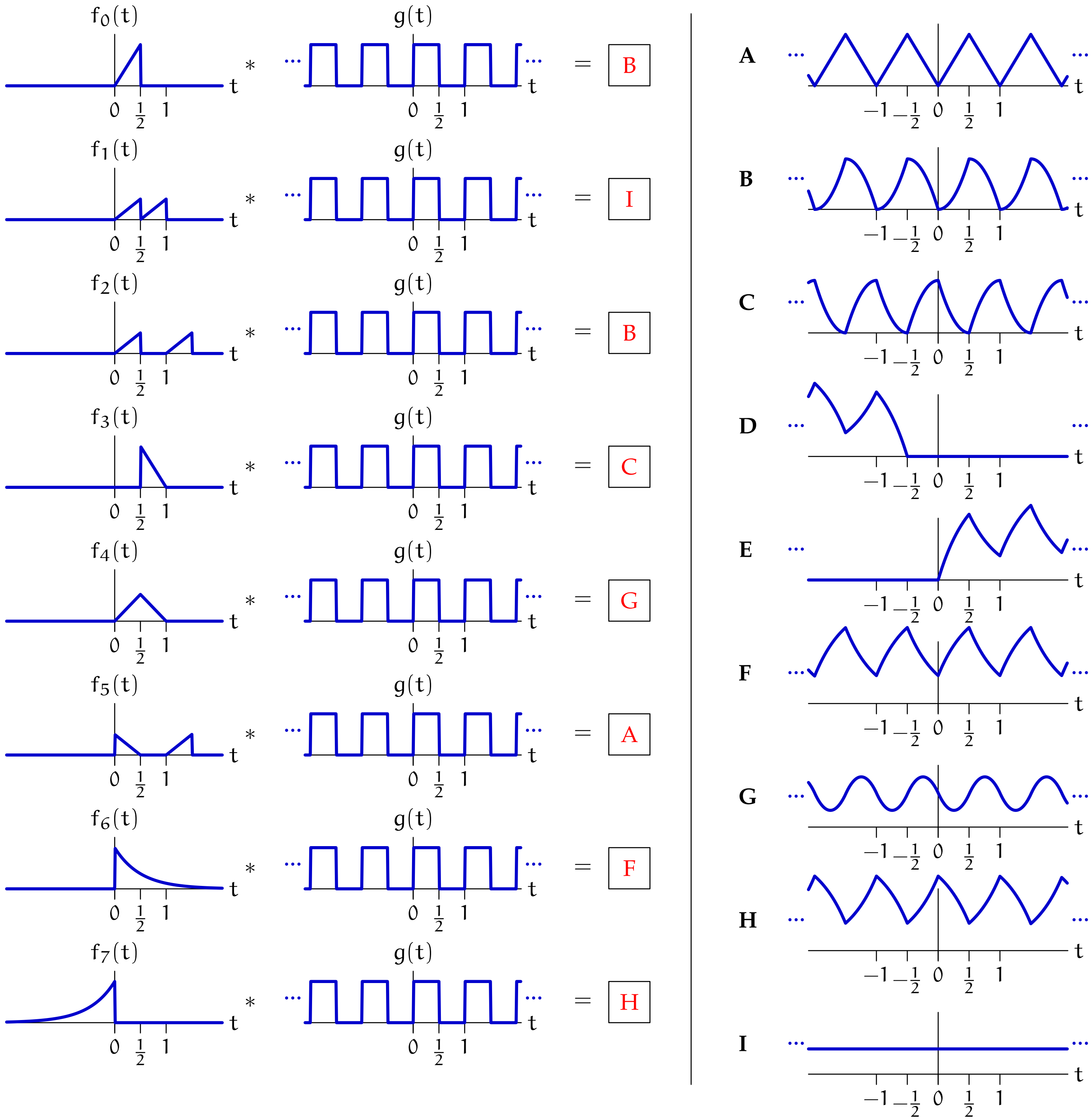
- $\angle H(\Omega) = 0$ at $\Omega =$ integer multiples of $\pi/4$.
- $\angle H_2(\Omega)$ never quite reaches $\pi/4$ or $-\pi/4$.
- shape is that of a smoothed sawtooth.
- Periodic in $\pi/2$.

4 Mixed Transformations (16 points)

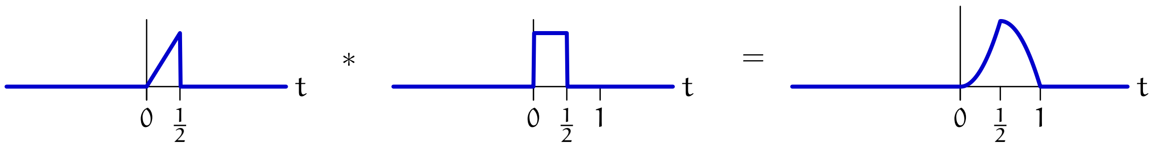
Each signal in the left column below ($f_0(t)$ to $f_7(t)$) goes to zero outside the regions shown in the plots. Determine the result of convolving each of these signals with a periodic train of rectangular pulses given by

$$g(t) = \begin{cases} 1 & \text{if } \sin(2\pi t) \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine which waveform (**A** to **I**) shows the result of each convolution and enter its label in the box provided.

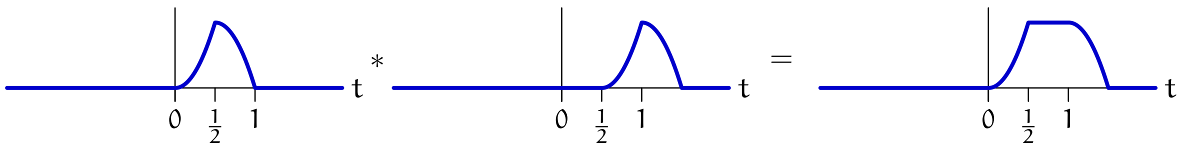


Part a. The square wave function $g(t)$ can be thought of as a sequence of rectangular pulses. Because convolution is both linear and time-invariant, we can compute the response to the sequence of rectangle pulses as the sum of responses to each rectangular pulse taken one at a time. Start with the pulse that is 1 for $0 \leqslant t \leqslant \frac{1}{2}$. Flip that pulse about $t = 0$ and multiply by $f_0(t)$ to get zero. Then as the pulse is shifted to the right, the output is the integral of $f_0(t)$ until the shift reaches $t = \frac{1}{2}$ at which point overlap with $f_0(t)$ shrinks. This is illustrated below.



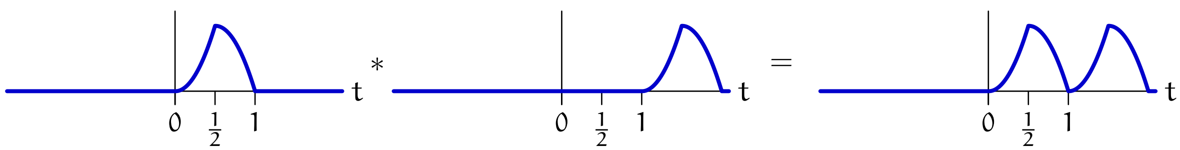
The additional rectangular pulses in $g(t)$ add integer shifts of the response above, so the answer is **B**.

Part b. The response of a single rectangular pulse in $g(t)$ to the double triangle in $f_1(t)$ can be computed as the sum of the response to one pulse and the response to a pulse shifted to the right by $t = \frac{1}{2}$.



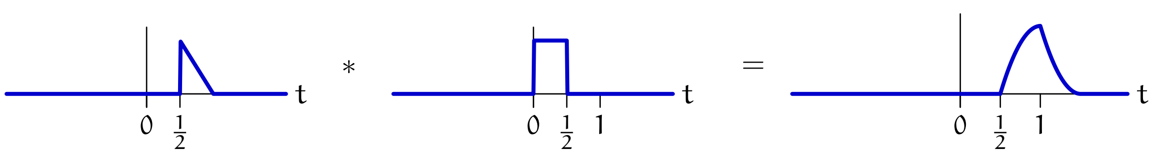
Notice that the downward part of the first signal (for $\frac{1}{2} \leqslant t < 1$) adds to the upward part of the second signal (for $\frac{1}{2} \leqslant t < 1$) to generate a constant over that region. The additional rectangular pulses in $g(t)$ add integer shifts of the response above, with the result being a constant function of time. Thus the answer is **I**.

Part c. The response of a single rectangular pulse in $g(t)$ to the double triangle in $f_2(t)$ can be computed as the sum of the response to one pulse and the response to a pulse shifted to the right by $t = 1$.



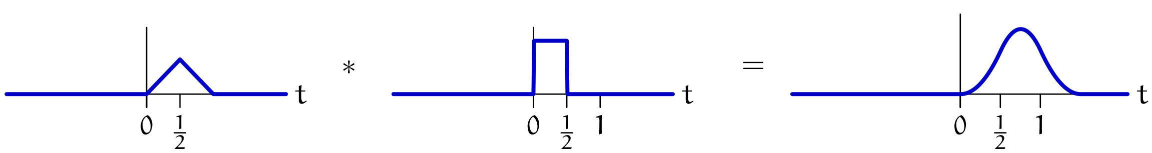
Adding shifted versions of this response produces the same waveform as that for part a. The amplitude of the result is the same as that in part a because there are twice as many contributions to the sum where each is only half as big as those in part a. Thus the answer is **B**.

Part d. Convolving $f_3(t)$ with the rectangular pulse that starts at $t = 0$ yields the following response.



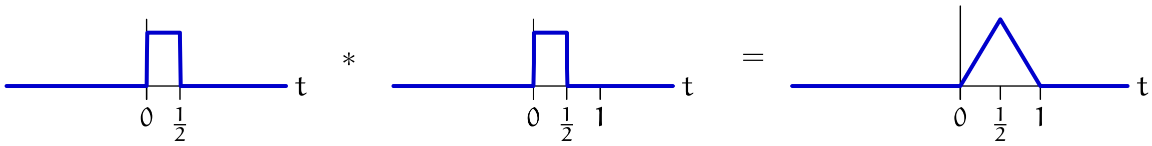
Summing shifted versions of this response yields the answer **C**.

Part e. Convolving $f_4(t)$ with the rectangular pulse that starts at $t = 0$ yields the following response.



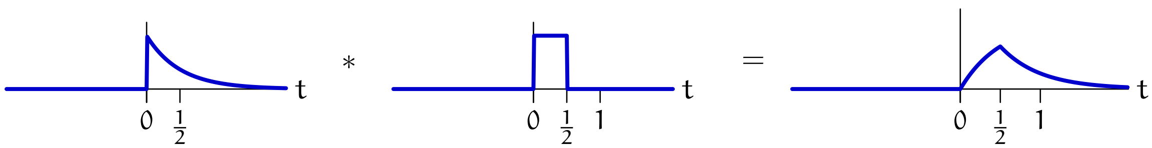
Shifted versions of this response overlap with each other since the response to each extends over a 1.5 second interval. This overlap leads to a DC shift, and the answer is **G**.

Part f. As seen in previous parts of this problem, shifting a signal $f_i(t)$ or part of such a signal by an integer number of seconds does not affect the output, since $g(t)$ is periodic in t with $T = 1$. We can simplify this part of the problem by shifing the second triangular piece of $f_5(t)$ to the left by 1 second and then summing with the first triangular piece. The result is a rectangular pulse and the result of convolving with $g(t)$ is shown below.



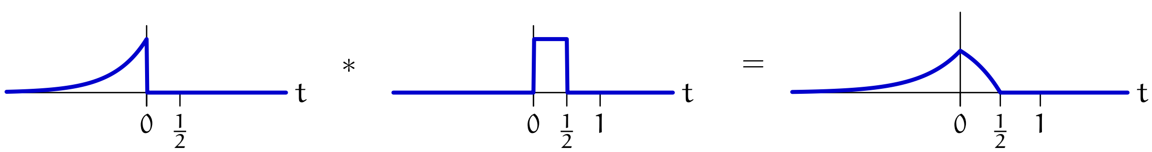
Summing shifted versions of this reponse yields **A**.

Part g. Start by convolving $f_6(t)$ with one rectangle pulse from $g(t)$. below.



Summing shifted versions of this reponse yields **F**, where the DC shift results because the duration of $f_6(t)$ exceeds 1 second.

Part h. Start by convolving $f_7(t)$ with one rectangle pulse from $g(t)$. below.



Summing shifted versions of this reponse yields **H**, where the DC shift results because the duration of $f_7(t)$ exceeds 1 second.

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