6.300 Problem Set 8

Answers

Problem 1: Frequencies

Part A

Consider the following CT signal:

$$x(t) = 6\cos(42\pi t) + 4\cos(18\pi t - 0.5\pi)$$

Now imagine that this signal is sampled with a sampling rate of $f_s = 60\text{Hz}$ to obtain a discrete-time signal, which is periodic.

- To start, we want to find the DFT coefficients associated with this signal when analyzed with N equal to the fundamental period of $x[n]$. Under that analysis, for which values of k will $X[k] = 0$? Explain your reasoning.

> Our original function has a fundamental period of $T = 1/3$ seconds.
>
> When we sample with a sample rate of 60Hz, our DT signal is periodic in $N = 20$ samples (which corresponds to $\frac{1}{3}$ seconds). Our sampled signal, then, looks like $x[n] = 6\cos(42\pi\frac{n}{60}) + 4\cos(18\pi\frac{n}{60} - 0.5\pi)$.
>
> Our goal with any Fourier representation is to express this as a sum of terms of the form $x[n] = \sum X[k]e^{j\frac{2\pi k}{20}n}$
>
> There are many ways we could write this, but the most direct is to represent each $\cos(\theta)$ as $e^{j\theta} + e^{-j\theta}$. And in order to match the frequencies of those waves, we would have nonzero $X[k]$ at $k \in \{3, 7, -3, -7\}$, which correspond to $\Omega \in \{\frac{3\pi}{10}, \frac{7\pi}{10}, \frac{-3\pi}{10}, \frac{-7\pi}{10}\}$.
>
> The last step is to remember that $X[k] = X[k+N]$. As such, the k values in the range $[0, 20)$ that correspond to nonzero $X[k]$ are 3, 7, 13 $(-7+20)$, and 17 $(-3+20)$.

- If you instead analyze with $N = 80$, which values of k in the range $[0, 80)$ are associated with non-zero $X[k]$? Explain your reasoning.

> We can think about this largely the same way as the last part, but now, with $N = 80$, our Fourier representation looks like $x[n] = \sum X[k]e^{j\frac{2\pi k}{80}n}$
>
> So in order to have $\Omega \in \{\frac{3\pi}{10}, \frac{7\pi}{10}, \frac{-3\pi}{10}, \frac{-7\pi}{10}\}$, we now need
>
> $k \in \{12, 28, -12, -28\}$. And since these coefficients are periodic in 80, the k values in the range $[0, 80)$ that are associated with nonzero $X[k]$ are 12, 28, 68 $(80-12)$, and 52 $(80-28)$.

- If you instead analyze with $N = 42$, which values of k in the range $[0, 42)$ are associated with non-zero $X[k]$? Explain your reasoning.

> Note here that, with $N = 42$, our signal is no longer periodic in the analysis window! This leads to nonzero $X[k]$ for all k.

Part B

Consider an arbitrary CT signal $x_c(\cdot)$, which is "band-limited" so that it does not contain any frequency content for any ω such that $|\omega| \geq (2\pi \times 1000)$, i.e., $X(\omega) = 0$ for all $|\omega| \geq (2\pi \times 1000)$. This signal is then sampled with a sampling rate f_s to produce a new DT signal $x[\cdot]$, where $x[n] = x_c(n/f_s)$.

Then, length-N portions of $x[\cdot]$ are analyzed using the DFT. For reasons associated with computational efficiency, we will assume N is a power of 2. Both f_s and N can be chosen at will, subject to the constraint that aliasing must be avoided and $N = 2^v$ for some integer v.

Determine the *minimum* values of N and f_s so that the frequency spacing between DFT coefficients is less than or equal to 2Hz and aliasing is avoided completely.

We can separate this into two separate concerns:

- we need to pick a sampling rate such that no aliasing occurs, and
- we need to pick N such that the f values associated with adjacent k values are at least 2Hz apart.

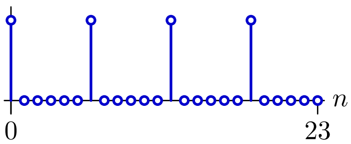
We can start by thinking about f_s. Since we know that there is no frequency content about $\omega = 2\pi \times 1000$ (rad / second), we know that there is no frequency content about $f = 1000$Hz. Since the maximum frequency which will be preserved by sampling is $f_{\text{max}} = \frac{f_s}{2}$, we need $f_s \geq 2000$Hz to guarantee that aliasing does not occur.

We know that the spacing between adjacent k values is $\frac{f_s}{N}$. So we need $\frac{f_s}{N} \leq 2$Hz, where N is a power of 2. We can achieve this with $N = 1024$ samples.

Problem 2: Matching Magnitudes

Part A

Each of the following plots shows the first 24 samples of a discrete-time signal. Find the plot on the following page that corresponds to the 24-point Discrete Fourier Transform (DFT) for each of these signals. Enter the letter of the plot (A-N) in the box provided.

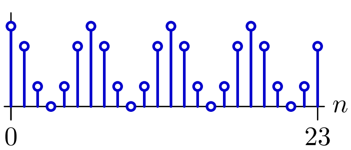


Matching DFT magnitudes (A-N): B

$$x_1[n] = \delta[n] + \delta[n-6] + \delta[n-12] + \delta[n-18]$$

$$X_1[k] = \frac{1}{24}\sum_{n=0}^{23} x[n]e^{-j2\pi kn/24} = 1 + e^{-j2\pi k/4} + e^{-j4\pi k/4} + e^{-j6\pi k/4} = 1 + (-j)^k + (-1)^k + j^k$$

$$= \begin{cases} 1/6 & \text{if } k = 0, 4, 8, 12, 16, 20 \\ 0 & \text{otherwise} \end{cases}$$

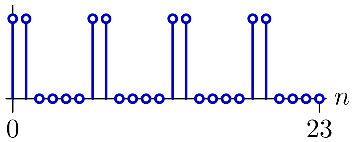


Matching DFT magnitudes (A-N): I

$$x_2[n] = 1 + \cos(2\pi n/6) = 1 + \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6}$$

So:

$$X_2[k] = \delta[k] + \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k+4]$$
$$= \delta[k] + \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-20]$$



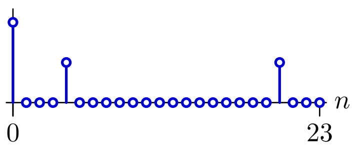
Matching DFT magnitudes (A-N): F

$$x_3[n] = x_1[n] + x_1[n-1]$$

so

$$X_3[k] = \left(1 + e^{-j2\pi k/24}\right) X_1[k] = 2e^{-j\pi k/24}\cos(\pi k/24) X_1[k]$$

Similar to plot B except components get more attenuated as we approach $k = 12$.

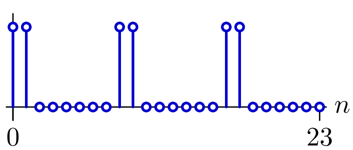


Matching DFT magnitudes (A-N): J

$$x_4[n] = \delta[n] + \frac{1}{2}\delta[n-4] + \frac{1}{2}\delta[n-20]$$

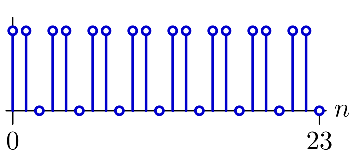
so

$$X_4[k] = \frac{1}{24} + \frac{1}{48}e^{-j2\pi k/6} + \frac{1}{48}e^{j2\pi k/6} = \frac{1}{24}\left(1 + \cos(2\pi k/6)\right)$$



Matching DFT magnitudes (A-N): E

x_5 is similar to x_3 except the period is 8 instead of 6. Therefore X_5 has non-zero components at $k = 0, 3, 6, ...$ and components near $k = 12$ are still attenuated in a cosine-like shape.

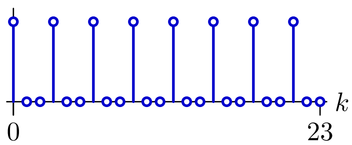


Matching DFT magnitudes (A-N): H

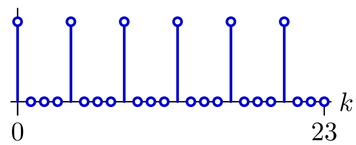
x_6 is similar to x_3 except the period is 3 instead of 6. Therefore X_6 has non-zero components at $k = 0, 8, 16$ and components near $k = 12$ are still attenuated in the same way.

Each of the following plots shows the magnitude of a DFT computed with an analysis window $N = 24$. The vertical scale for each plot is different: it has been normalized so that the peak value in each plot is 1.

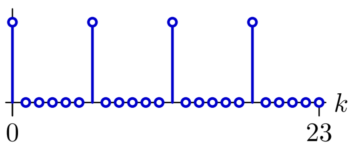
A



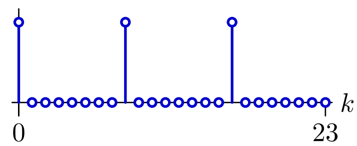
B



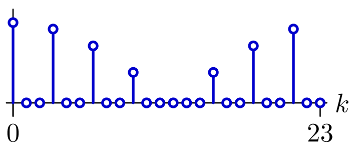
C



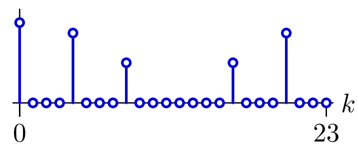
D



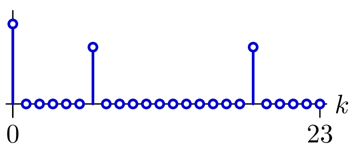
E



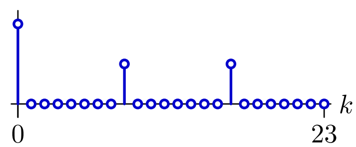
F



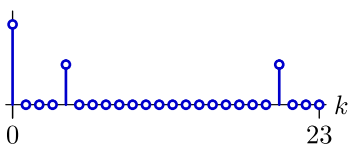
G



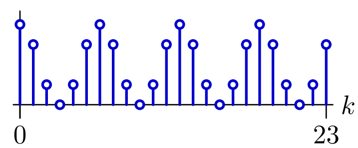
H



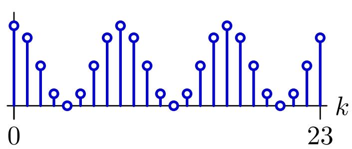
I



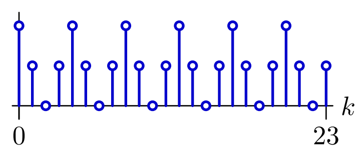
J



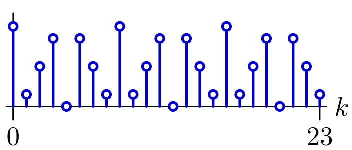
K



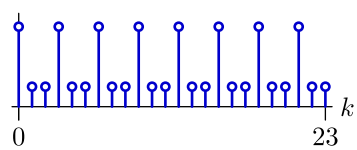
L



M



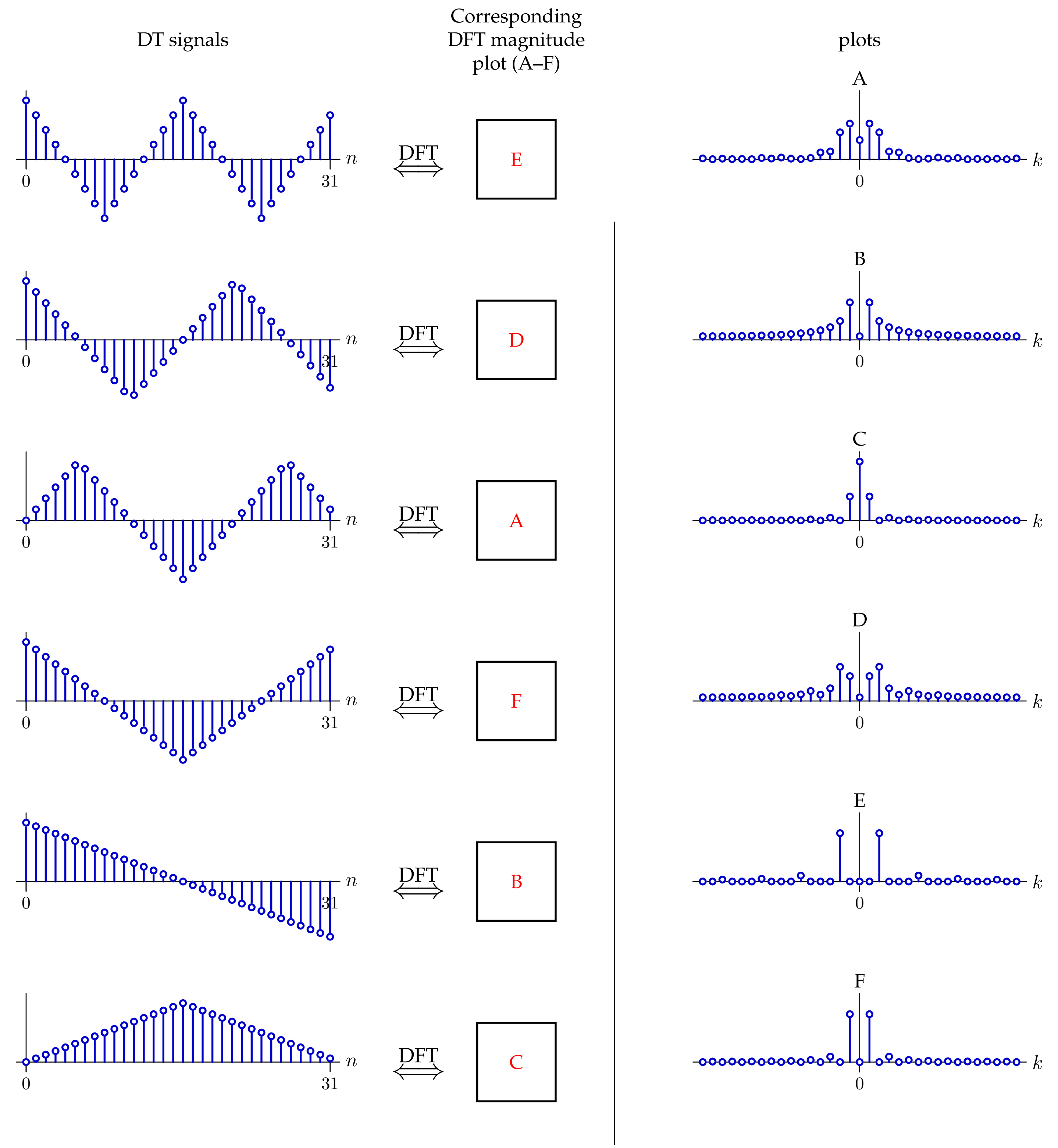
N



Part B

The left column below shows six discrete-time signals for $0 \le n \le 31$. The right column shows plots of the magnitudes of six DFTs computed for $N = 32$.

For each discrete-time signal in the left column below, find the matching DFT magnitude (one of plots A–F) and enter its letter in the box provided.



(reasoning for answers from previous page)

The top signal shows two full cycles of a triangle wave. Therefore the fundamental frequency of the triangle wave falls at $k = 2$. There could also be harmonics of $k = 2$ (i.e., at $k = 4, 6, 8, ...$).
$\rightarrow$ plot E

The next two signals show 1.5 full cycles of a triangle wave. Therefore the magnitude will peak between $k = 1$ and $k = 2$. If the first of these is periodically extended, it will have a big discontinuity between periods. The second of these has a much smaller discontinuity. Also, the DC value of the second is much larger than the first.
$\rightarrow$ the second signal corresponds to plot D
$\rightarrow$ the third signal corresponds to plot A

The fourth signal shows 1 full cycle of a triangle wave. Therefore $k = 1$.
$\rightarrow$ plot F

When periodically extended, the fifth signal will be a sawtooth with $k = 1$. There will also be a large discontinuity at the period boundaries, so that will generate contributions at nearby k's.
$\rightarrow$ plot B

When periodically extended, the last signal will make a triangle wave at $k = 1$. Notice however that there is a large DC component.
$\rightarrow$ plot C

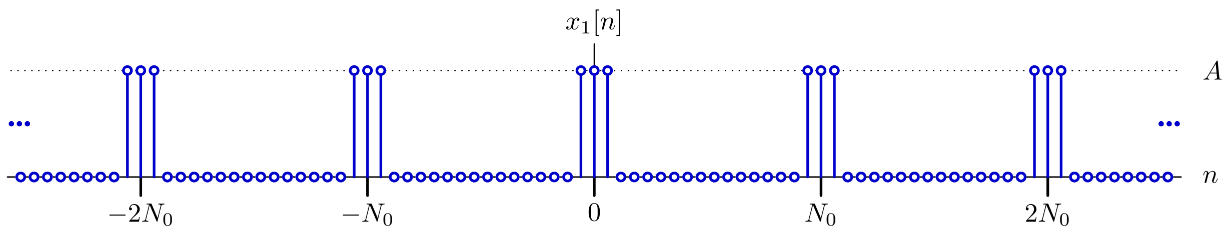
Problem 3: Interpreting Fourier Relationships

Part A

Let $x_0[n] = \delta[n+1] + \delta[n] + \delta[n-1]$, and let $x_1[\cdot]$ be a scaled and periodically-extended version of $x_0[\cdot]$, with repetitions every N_0 samples:

$$x_1[n] = \sum_{m=-\infty}^{\infty} Ax_0[n - mN_0] = \sum_{m=-\infty}^{\infty} A\delta[n - mN_0 + 1] + A\delta[n - mN_0] + A\delta[n - mN_0 - 1]$$

As an example of the general shape of this function, here is an example with $N_0 = 17$ (though you should not assume that $N_0 = 17$ throughout the problem):



Also let $x_2[n] = B\cos(\Omega_0 n)$ for some value Ω_0, and let $x_3[n] = x_1[n] + x_2[n]$.

Each of the plots on the following page shows the (purely real) DTFT of some function. Which of the graphs (1-6) corresponds to $X_3(\Omega)$, the DTFT of $x_3[n] = x_1[n] + x_2[n]$? And what are the values of A, N_0, B, and Ω_0? Enter a single number in each box below:

Graph number: 5

$A =$ $\frac{8}{3\pi}$

$N_0 =$ 8

$B =$ $\frac{2}{\pi}$

$\Omega_0 =$ $\frac{\pi}{3}$

(see rationale on following page)

Let's not jump straight into math here. Rather, we should actually be able to eliminate a whole bunch of possibilities here before doing any math, which should make the rest of the process a little bit easier for us.
Firstly, let's look at properties that the signal $x_3[n]$ has:

- It is periodic because we're adding together two periodic signals.
- It is symmetric because we're adding together two symmetric signals.

From just those two properies of $x_3[n]$, we can tell a lot about $X_3(\Omega)$:

- Because it could be represented by a series, its DTFT should only be nonzero at certain frequencies: integer multiples of $\frac{2\pi}{N_0}$. Thus, we can eliminate all of the continuous graphs.
- Because it is a real-valued, symmetric function of time, its DTFT should be real and symmetric.

From those two facts, we've reduced our possibilities down to either graph 5 or 6 without doing any complicated math! We can also use this to solve for some of the values.

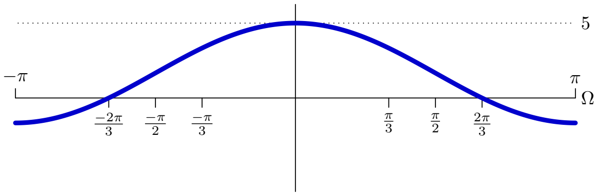
We should have deltas at $\frac{2\pi k}{N_0}$ from x_1, and we should have a pair of deltas at $\pm\Omega_0$ from x_2. The deltas seem to repeat regularly at $\frac{\pi}{4}$, which tells us that $N_0 = 8$; and the extra pair of sticks is at $\pm\frac{\pi}{3}$, so $\Omega_0 = \frac{\pi}{3}$.

Furthermore, since the deltas at $\pm\frac{\pi}{3}$ have heights of 2, we can find B. $B\cos\left(\frac{\pi}{3n}\right) = \frac{1}{2\pi}\left(2e^{j\frac{\pi}{3}} + 2e^{-j\frac{\pi}{3}}\right)$, so $B = \frac{2}{\pi}$.

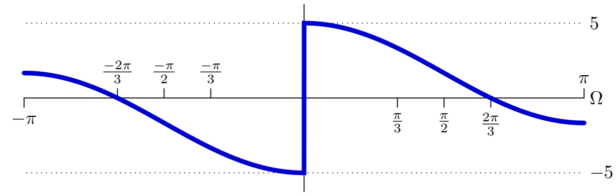
If we do a little math, we can also definitively say which of the graphs is appropriate. The DTFT of x_1 should be given by $X_{1,\text{DTFT}}(\Omega) = \sum_k 2\pi X_{1,\text{DTFS}}[k]\delta\left(\Omega - \frac{2\pi k}{8}\right)$. And we can find the DTFS coefficients of x_1 easily enough: the sticks at $n = \pm 1$ give us a $\frac{2A}{8}\cos\left(\frac{2\pi k}{N}\right)$, and the stick at $n = 0$ gives us a DC offset of $\frac{A}{8}$. From this, we can see that the deltas should definitely not all have the same heights, so we must be interested in graph 5.

And since we have an exact expression for the DTFT of x_1, we can solve for A as well. The height of the delta at $\Omega = 0$ is 2; and the value of the DTFS coefficient at $k = 0$ is $\frac{3A}{8}$. So we must have $2\pi\left(\frac{3A}{8}\right) = 2$, implying that $A = \frac{8}{3\pi}$

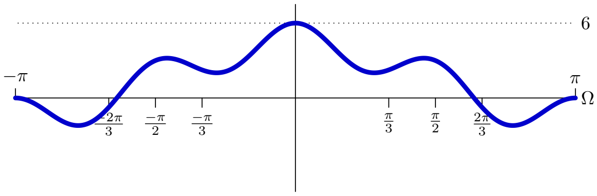
Graph 1



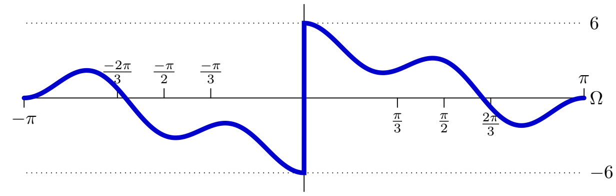
Graph 2



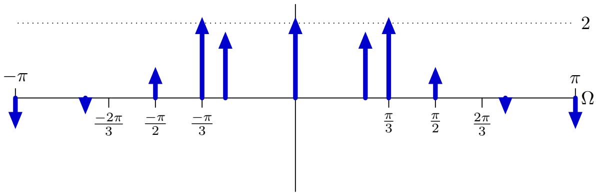
Graph 3



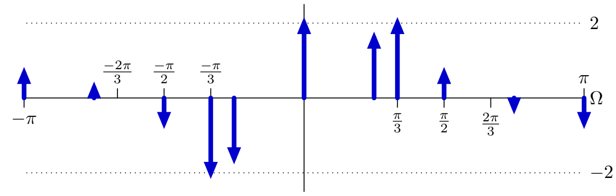
Graph 4



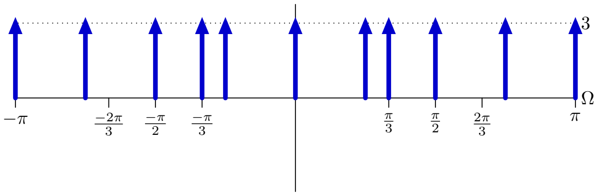
Graph 5



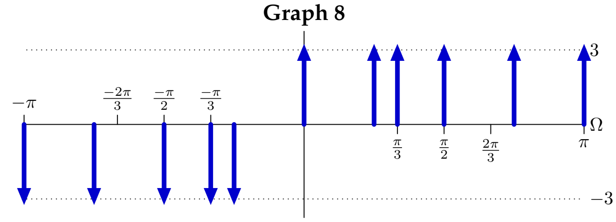
Graph 6



Graph 7



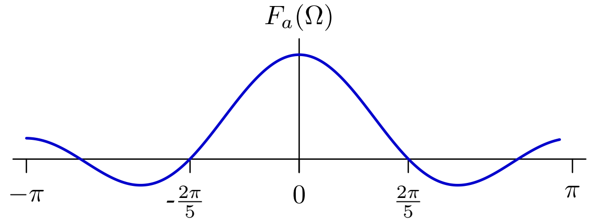
Graph 8



Part B

Let $F_a(\Omega)$ represent the Discrete-Time Fourier Transform of a discrete-time signal $f_a[n]$, which is a pulse of unknown width:

$$f_a[n] = \begin{cases} 1 & \text{if } -W \le n \le W \\ 0 & \text{otherwise} \end{cases} \quad \overset{\text{DTFT}}{\Longleftrightarrow}$$



We can find an exact representation of $F_a(\Omega)$, which may be useful here: $F_a(\Omega) = \frac{\sin(\Omega(W+1/2))}{\sin(\frac{\Omega}{2})}$

We can use this to solve for W: when $\Omega = \frac{2\pi}{5}$, the argument to the sin function in the numerator is π (our first zero crossing), so $\frac{2\pi}{5}(W+0.5) = \pi$. Solving, we find $W = 2$, i.e., we have a pulse with a width of 5.

and let $F_b[k]$ represent a sampled version of $F_a(\Omega)$:

$$F_b[k] = F_a\left(\frac{2\pi k}{9}\right)$$

If we were to compute the DFT of f_a directly with $N = 9$, we would have $F_a[k] = \frac{1}{9}F_a\left(\frac{2\pi k}{9}\right)$. So $F_b[k] = 9F_a[k]$.

where k is an integer.

Use an inverse Discrete Fourier Transform with length $N = 9$ to find $f_b[n]$:

$$f_b[n] \quad \overset{\text{DFT}(N=9)}{\Longleftrightarrow} \quad F_b[k]$$

Enter your answers in the boxes below.

Taking the inverse transform gets us a 5-wide rectangular pulse with a height of 9 (since $F_b[k] = 9F_a[k]$), but our new signal is also periodic in $N = 9$, so the values from $n = -1$ and $n = -2$ in our original signal wrap back around, giving us:

$f_b[0] =$ 9 $\quad f_b[1] =$ 9 $\quad f_b[2] =$ 9

$f_b[3] =$ 0 $\quad f_b[4] =$ 0 $\quad f_b[5] =$ 0

$f_b[6] =$ 0 $\quad f_b[7] =$ 9 $\quad f_b[8] =$ 9

Part C

Consider a signal $x[n]$, which is nonzero only for $n = 0, 1, 2, 3$.

You know the values of the signal's **DTFT** at two points only. In particular, you know the values $X(\frac{\pi}{2})$ and $X(\frac{3\pi}{2})$. You also know the value of $\sum_m x[m]$ and $\sum_m e^{j\pi m} x[m]$.

Explain (in detail) how you could determine $x[n]$ from this information.

$x_3[n]$ is non-zero for $n = 0, 1, 2, 3$. The DTFT of $x_3[n]$ is:

$$X_3(\Omega) = \sum_{n=-\infty}^{\infty} x_3[n] e^{-j\Omega n}$$
$$= \sum_{n=0}^{3} x_3[n] e^{-j\Omega n}$$

Let $X_3[K]$ be the DFT of $x_3[n]$ with analysis window $N = 4$. Then,

$$X_3[0] = \frac{1}{4}\sum_{n=0}^{3} x_3[n] e^{-j\frac{2\pi}{4}n\cdot 0} = \frac{1}{4}\sum_{n=0}^{3} x_3[n]$$
$$X_3[1] = \frac{1}{4}\sum_{n=0}^{3} x_3[n] e^{-j\frac{2\pi}{4}n} = \frac{1}{4} X_3\left(\frac{\pi}{2}\right)$$
$$X_3[2] = \frac{1}{4}\sum_{n=0}^{3} x_3[n] e^{-j\frac{2\pi}{4}n\cdot 2} = \frac{1}{4}\sum_{n=0}^{3} x_3[n] e^{-j\pi n} = \frac{1}{4}\sum_{n=0}^{3} x_3[n] e^{j\pi n}$$
$$X_3[1] = \frac{1}{4}\sum_{n=0}^{3} x_3[n] e^{-j\frac{2\pi}{4}n\cdot 3} = \frac{1}{4} X_3\left(\frac{3\pi}{2}\right)$$

The right hand side of each of the equations are given as known in the problem. We can determine $x_3[n]$ by applying the DFT synthesis equation using these four values.