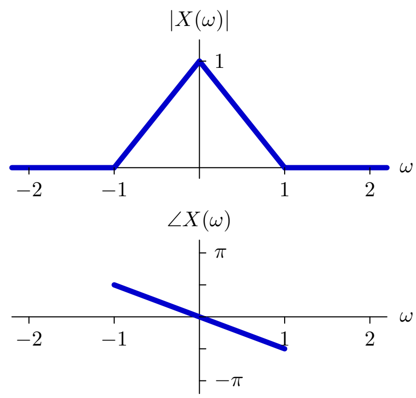
6.300 Problem Set 6
Answers

Problem 1: Fourier Match

The magnitude and angle of the Fourier transform of a signal $x(t)$ are given in the following plots.



Seven magnitude plots (M1-M7) and seven angle plots (A1-A7) are shown on the following page. Determine which of these plots is associated with each of the derived signals below:

- Which plot shows the magnitude of the Fourier transform of $\frac{dx(t)}{dt}$?

If a signal is differentiated in time, its Fourier transform is multiplied by $j\omega$. Thus the magnitude is multipled by $|\omega|$. The result is symmetric about $\omega = 0$. The shape is parabolic over the interval $(0, 1)$, reaching a peak value of $\frac{1}{4}$ at $\omega = \pm\frac{1}{2}$. Therefore the answer is **M5**.

- Which plot shows the angle of the Fourier transform of $\frac{dx(t)}{dt}$?

If a signal is differentiated in time, its Fourier transform is multiplied by $j\omega$. Thus the angle is increased by $\frac{\pi}{2}$ (since $j = e^{j\frac{\pi}{2}}$) if $\omega > 0$ and decreased by $\frac{\pi}{2}$ if $\omega < 0$. The result is an antisymmetric function of ω (as the angle must be if the time function is real), tapering from $\frac{\pi}{2}$ at $\omega = 0$ to 0 at $\omega = 1$. Therefore the answer is **A4**.

- Which plot shows the magnitude of the Fourier transform of $(x * x)(t)$?

If a signal is convolved with itself, the magnitude of the Fourier transform is squared. The result is 1 at $\omega = 0$ and 0 for $|\omega| > 1$. The square of a number that is between 0 and 1 is less than the number. For example, the result at $\omega = \frac{1}{2}$ is $\frac{1}{4}$. Therefore the answer is **M3**.

- Which plot shows the angle of the Fourier transform of $(x * x)(t)$?

If a signal is convolved with itself, the Fourier transform is squared. Thus the angle is doubled. Therefore the answer is **A2**.

- Which plot shows the magnitude of the Fourier transform of $x\left(t-\frac{\pi}{2}\right)$?

Delaying in time alters the angle of the Fourier transform but not the magnitude. Therefore the answer is **M1**.

- Which plot shows the angle of the Fourier transform of $x\left(t-\frac{\pi}{2}\right)$?

Delaying a signal by $\frac{\pi}{2}$ seconds multiples the transform by $e^{-j\frac{\pi}{2}\omega}$. This decreases the angle by $\frac{\pi}{2}\omega$ for $\omega > 0$ and increases the angle by $\frac{\pi}{2}|\omega|$ for $\omega < 0$. Therefore the answer is **A2**.

- Which plot shows the magnitude of the Fourier transform of $x(2t)$?

Compressing in time stretches proportionally in frequency. Therefore the answer is **M4**.

- Which plot shows the angle of the Fourier transform of $x(2t)$?

Compressing in time stretches proportionally in frequency. Therefore the answer is **A3**.

- Which plot shows the magnitude of the Fourier transform of $x^2(t)$?

If a signal is squared, its Fourier transform is convolved with itself. Convolution of a triangle of width 2 with itself produces a result of width 4. Between $\omega=-2$ and $\omega=-1$, the magnitude grows with the square of ω. Between -1 and 0, it grows more slowly than the square. And the result is symmetric in ω. Therefore the answer is **M6**.

- Which plot shows the angle of the Fourier transform of $x^2(t)$?

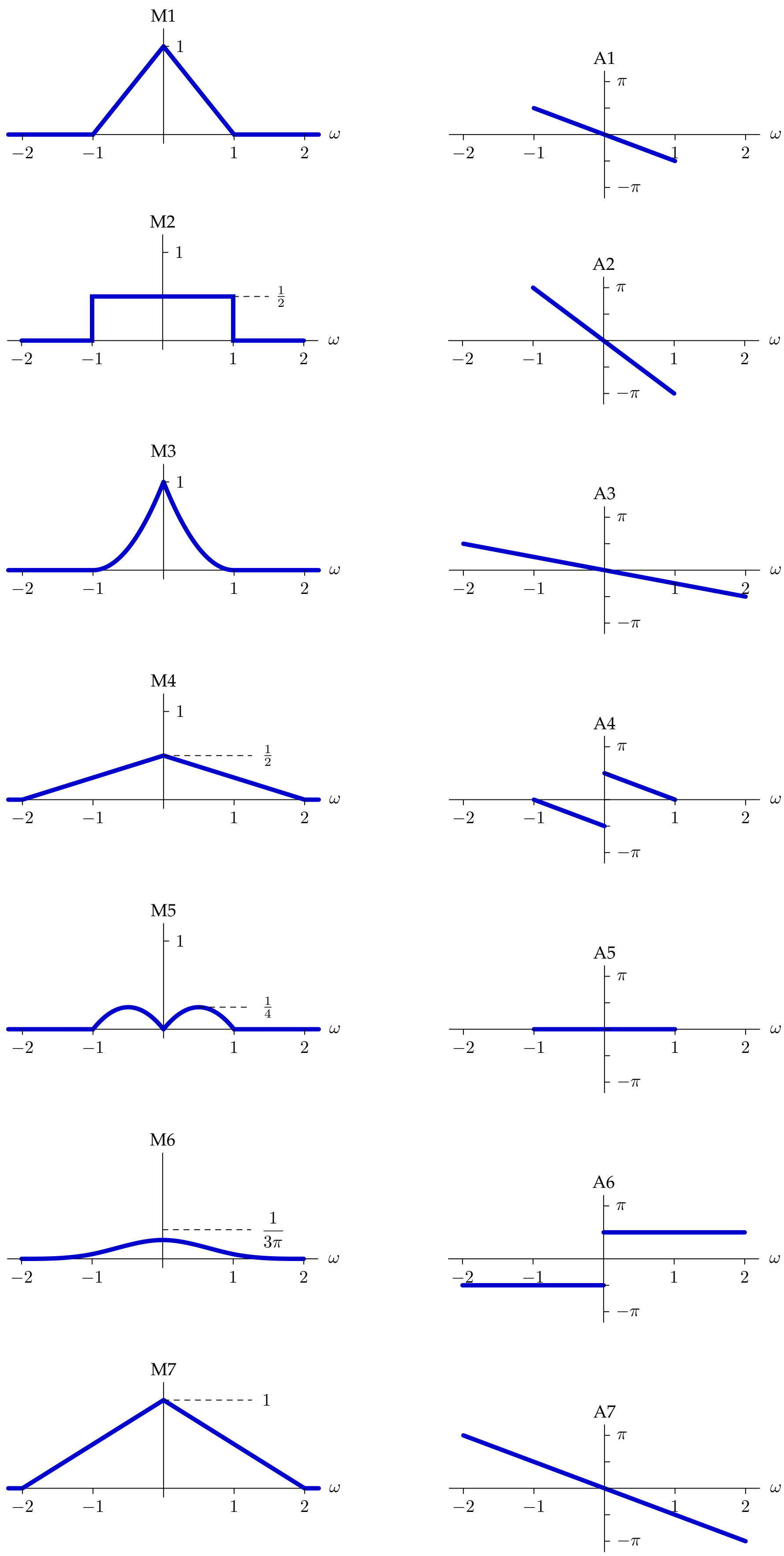
Squaring a signal corresponds to convolving its Fourier transform with itself. Let $X(\omega)$ represent the original transform. Then:

$$(X*X)(\omega)=\frac{1}{2\pi}\int\Big|X(\lambda)\Big|\times e^{j\angle X(\lambda)}\times\Big|X(\omega-\lambda)\Big|\times e^{j\angle X(\omega-\lambda)}\mathrm{d}\lambda$$

We can see from the original graph that $\angle X(\omega)=-\frac{\pi}{2}\omega$, which we can substitute in to the above equation:

$$\begin{aligned}(X*X)(\omega)&=\frac{1}{2\pi}\int\Big|X(\lambda)\Big|\times e^{j\angle X(\lambda)}\times\Big|X(\omega-\lambda)\Big|\times e^{j\angle X(\omega-\lambda)}\mathrm{d}\lambda\\
&=\frac{1}{2\pi}\int\Big|X(\lambda)\Big|\times e^{-j\frac{\pi}{2}\lambda}\times\Big|X(\omega-\lambda)\Big|\times e^{-j\frac{\pi}{2}(\omega-\lambda)}\mathrm{d}\lambda\\
&=\frac{1}{2\pi}\int\Big|X(\lambda)\Big|\times\Big|X(\omega-\lambda)\Big|\times e^{-j\frac{\pi}{2}\lambda}\times e^{-j\frac{\pi}{2}(\omega-\lambda)}\mathrm{d}\lambda\\
&=\frac{1}{2\pi}\int\Big|X(\lambda)\Big|\times\Big|X(\omega-\lambda)\Big|\times e^{-j\frac{\pi}{2}\omega}\mathrm{d}\lambda\\
&=\frac{1}{2\pi}e^{-j\frac{\pi}{2}\omega}\int\Big|X(\lambda)\Big|\times\Big|X(\omega-\lambda)\Big|\mathrm{d}\lambda\end{aligned}$$

This form shows us that we find the magnitude of this new signal by convolving the magnitudes of the original signal, and that the angle of the new signal is $\angle X_5(\omega)=-\frac{\pi}{2}\omega$.
This must be true everywhere that $|X_5(\omega)|$ is nonzero, which is the range from -2 to 2 (for other ω values, the angle is undefined). Thus, the answer must be **A7**.


M1
A1
M2
A2
M3
A3
M4
A4
M5
A5
M6
A6
M7
A7
ω
π
$-\pi$
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{3\pi}$
−2
−1
1
2

Problem 2: DT Convolution

For these problems, keep in mind both the superposition view of convolution as well as the DT convolution formula:

$$(x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \ldots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \ldots$$

Part A

Let y represent the DT signal that results when f is convolved with g, i.e, $y[n] = (f * g)[n]$.

Determine closed-form expressions for the following variants of y:

function	$f[n]$	$g[n]$
$y_a[n]$	$u[n]$	$u[n]$
$y_b[n]$	$u[n]$	$\left(\frac{1}{2}\right)^n u[n]$
$y_c[n]$	$\left(\frac{1}{2}\right)^n u[n]$	$\left(\frac{1}{3}\right)^n u[n]$
$y_d[n]$	$\left(\frac{1}{2}\right)^n u[n]$	$\left(\frac{1}{2}\right)^n u[n]$

In all cases, $u[n]$ represents the DT unit step function: $u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

For y_a:

$$y_a[n] = \sum_{m=-\infty}^{\infty} u[m]u[n-m]$$

Since $u[m] = 0$ if $m < 0$ and $u[n-m] = 0$ if $n - m < 0$, we can limit the bounds of the sum:

$$y_a[n] = \sum_{m=0}^{n} 1 = \begin{cases} n+1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$y_a[n] = (n+1)u[n]$$

For y_b:

$$y_b[n] = \sum_{m=-\infty}^{\infty} u[m] \left(\frac{1}{2}\right)^{n-m} u[n-m]$$

$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^{n-m} = \sum_{m=n}^{0} \left(\frac{1}{2}\right)^{m}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} u[n]$$

$$= \left(2 - \left(\frac{1}{2}\right)^{n}\right) u[n]$$

Notice that this is equivalent to the running sum of $\left(\frac{1}{2}\right)^n u[n]$.

For y_c:

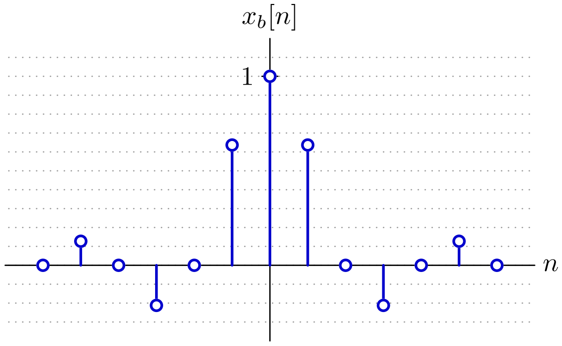
$$y_c[n] = \sum_{m=-\infty}^{\infty} \left(\frac{1}{3}\right)^m u[m] \left(\frac{1}{2}\right)^{n-m} u[n-m]$$
$$= \left(\frac{1}{2}\right)^n \sum_{m=0}^{n} \left(\frac{2}{3}\right)^m$$
$$= \left(\frac{1}{2}\right)^n \frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}$$
$$= \left(3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right) u[n]$$

For y_d:

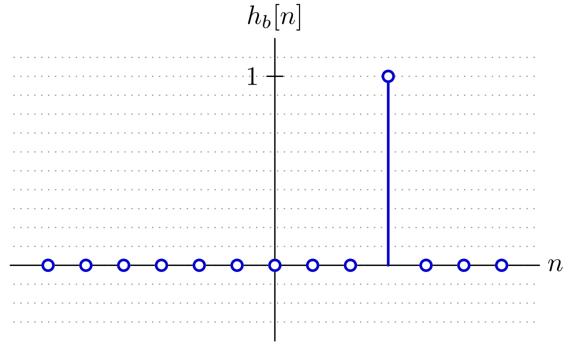
$$y_d[n] = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m] \left(\frac{1}{2}\right)^{n-m} u[n-m]$$
$$= \left(\frac{1}{2}\right)^n \sum_{m=0}^{n} 1$$
$$= (n+1)\left(\frac{1}{2}\right)^n$$
$$= y_d[n] = (n+1)\left(\frac{1}{2}\right)^n u[n]$$

Part B

Determine a closed-form expression for $y_b[n] = (x_b * h_b)[n]$ with x_b and h_b as shown below:



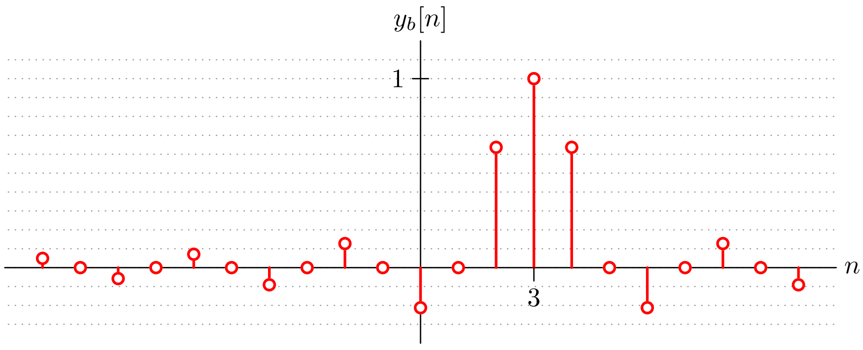
$x_b[n] = \frac{\sin(n\pi/2)}{n\pi/2}$



$h_b[n] = \delta[n-3]$

Sketch a graph of your result, labeling key points.

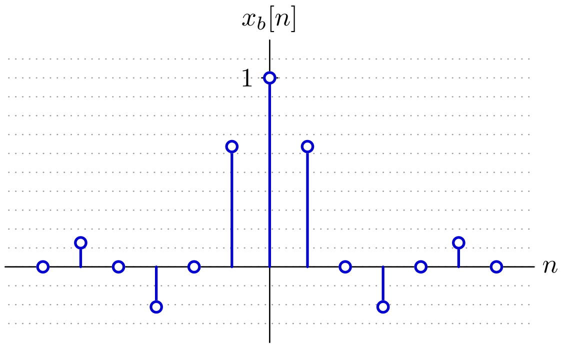
By superposition, this places a copy of x_b (scaled by 1) centered at $n = 3$, so the result of the convolution is $x_b[n-3]$:



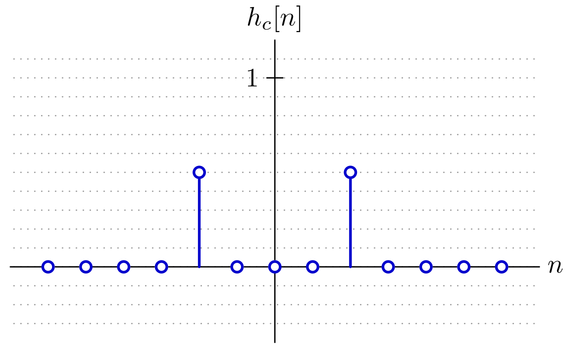
$y_b[n] = x_b[n-3]$

Part C

Consider the same signal x_b from the previous part, now convolved with a different signal h_c:

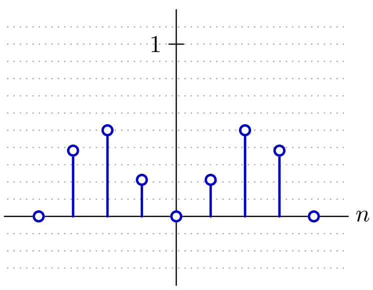


$x_b[n] = \frac{\sin(n\pi/2)}{n\pi/2}$

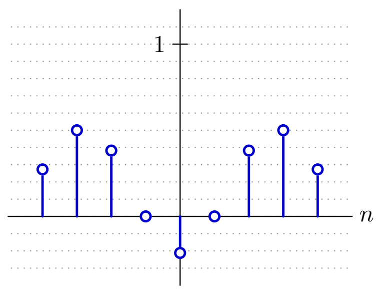


$h_c[n] = \frac{\delta[n-2]+\delta[n+2]}{2}$

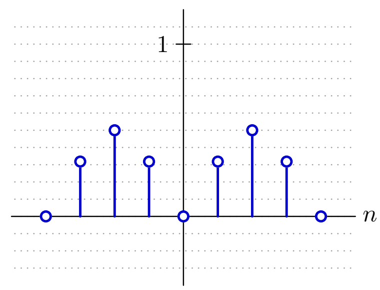
Which of the following represents the result of convolving x_b with h_c? Explain your reasoning.



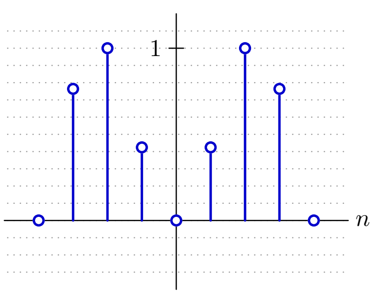
Graph A



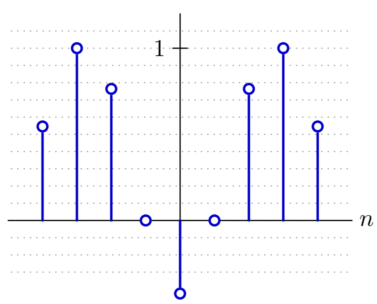
Graph B



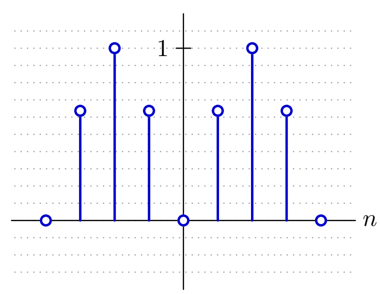
Graph C



Graph D



Graph E

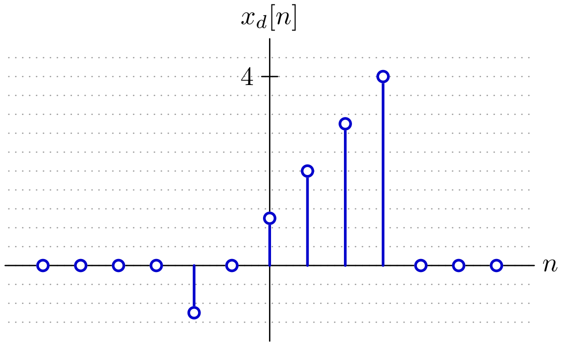


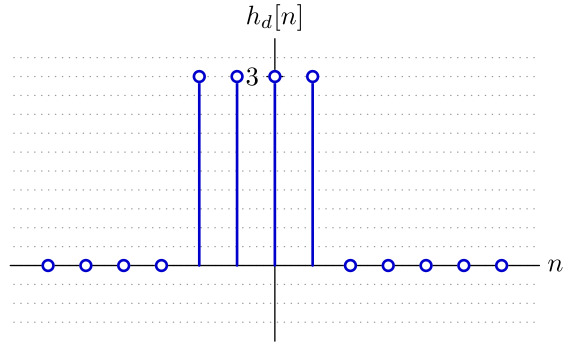
Graph F

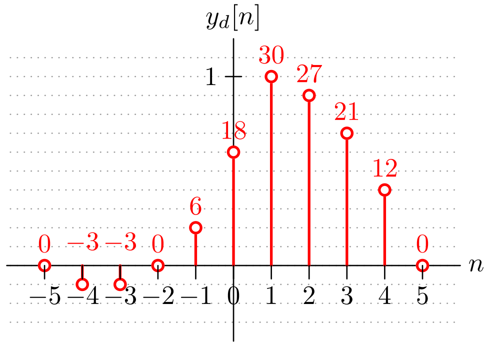
Graph A

Part D

Compute $y_d[n] = (x_d * h_d)[n]$ and plot it, including and labeling all nonzero values. The signals x_d and h_d are zero outside the region shown. You do not need to determine a closed-form expression for $y_d[n]$.







Problem 3: CT Convolution

For these problems, keep in mind the CT convolution formula:

$$(x * h)(t) = \int_{-\infty}^{\infty} x(u)h(t-u)\mathrm{d}u$$

Part A

Let y represent the CT signal that results when f is convolved with g, i.e, $y[n] = (f * g)(t)$.

Determine closed-form expressions for the following variants of y:

function	$f(t)$	$g(t)$
$y_a(t)$	$u(t)$	$u(t)$
$y_b(t)$	$u(t)$	$e^{-t}u(t)$
$y_c(t)$	$e^{-t}u(t)$	$e^{-2t}u(t)$
$y_d(t)$	$e^{-t}u(t)$	$e^{-t}u(t)$

In all cases, $u(t)$ represents the DT unit step function: $u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

For y_a:

$$y_a(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \int_0^t d\tau = tu(t)$$

For y_b:

$$y_b(t) = \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = e^{-t}\int_0^t e^{\tau}d\tau = e^{-t}(e^t - 1)u(t) = (1 - e^{-t})u(t)$$

For y_c:

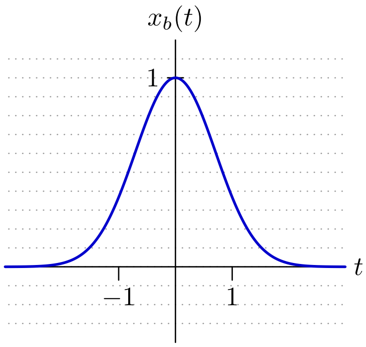
$$y_c(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau = e^{-2t}\int_0^t e^{\tau}d\tau = e^{-2t}(e^t - 1)u(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

For y_d:

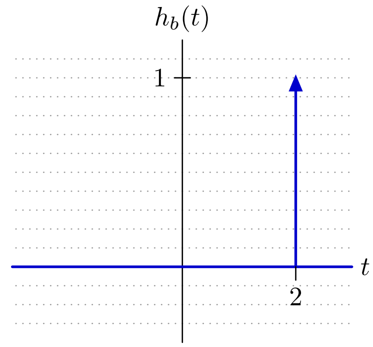
$$y_d(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = e^{-t}\int_0^t d\tau = e^{-t}tu(t) = te^{-t}u(t)$$

Part B

Determine a closed-form expression for $y_b(t) = (x_b * h_b)(t)$ with x_b and h_b as shown below:



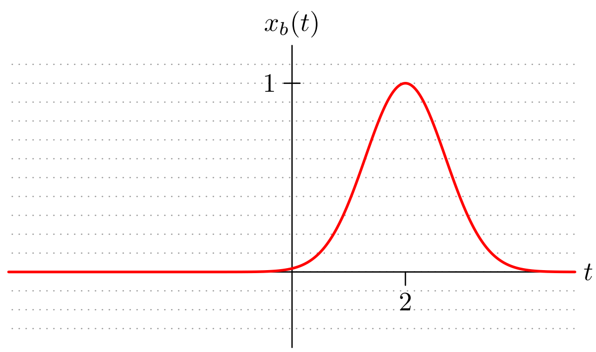
$x_b(t) = e^{-t^2}$



$h_b(t) = \delta(t-2)$

Sketch a graph of your result, labeling key points.

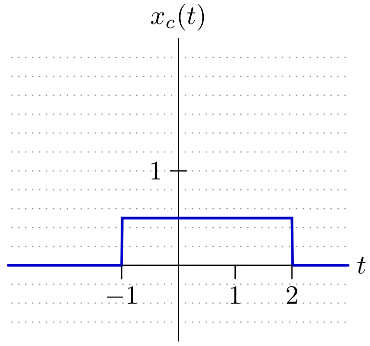
This places a copy of x_b (scaled by 1) centered at $t = 2$, so the result of the convolution is $x_b(t-2)$:

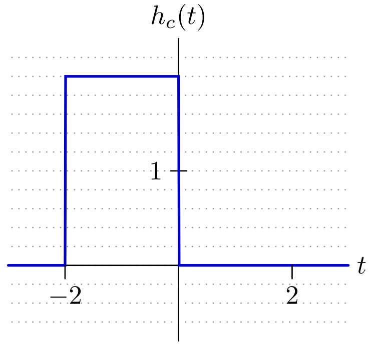


$y_b(t) = x_b(t-2)$

Part C

Consider the signals x_c and h_c shown below:

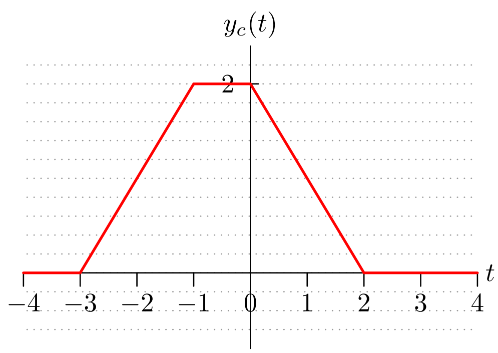




$$x_c(t) = \begin{cases} 1/2 & \text{if } -1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$h_c(t) = \begin{cases} 2 & \text{if } -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

Sketch and fully label a graph of $y_c(t) = (x_c * h_c)(t)$.



Part D

Consider a polynomial term $p(t)$ that is clamped to zero before $t = 0$, so that $p(t) = \alpha t^k u(t)$, where k is a non-negative integer. Determine a closed-form expression for $y_d(t) = (p * u)(t)$.

$$y_d(t) = \left(\frac{\alpha}{k+1}\right)\left(t^{k+1}\right)u(t)$$