6.300 Problem Set 5

Answers

Problem 1: Series and Transforms

Part A

Find the Fourier series coefficients of the signal $x_1(\cdot)$, analyzed with T chosen to be the fundamental period of $x_1(\cdot)$.

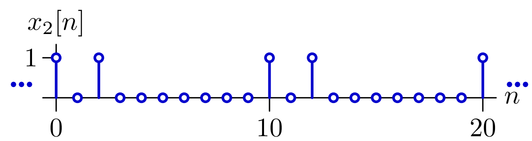
$$x_1(t) = 2\cos\left(\frac{\pi}{2}t\right) + 4\cos\left(\frac{\pi}{3}t\right)$$

Determine a closed-form (no integrals or infinite sums) expression for $X_1[k]$.

$$X_1[k] = \delta[k+3] + 2\delta[k+2] + 2\delta[k-2] + \delta[k-3]$$

Part B

Find the Fourier series coefficients of the signal $x_2[\cdot]$, shown below, which is periodic in $N = 10$.



Determine a closed-form expression for $X_2[k]$.

$$X_2[k] = \frac{1}{10}(1 + e^{-j(2\pi/5)k})$$

Part C

Find the Fourier transform of the signal $x_3(\cdot)$ as defined below:

$$x_3(t) = \begin{cases} 1 & \text{if } -1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine a closed-form expression for $X_3(\omega)$.

$$X_3(\omega) = \frac{2\sin(\frac{3}{2}\omega)}{\omega} e^{-j\omega/2}$$

Part D

Find the Fourier transform of the signal $x_4[\cdot]$, defined below:

$$x_4[n] = \delta[n+3] + \delta[n+1] - \delta[n-1] + \delta[n-3]$$

Determine a simple closed-form expression for $X_4(\Omega)$.

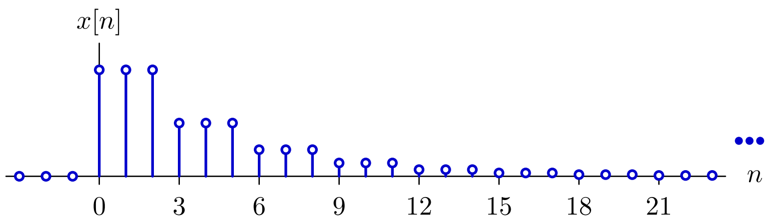
$$X_4(\Omega) = 2\cos(3\Omega) + 2j\sin(\Omega)$$

Part E

Let $x[n]$ represent the following discrete-time signal

$$x[n] = \begin{cases} 0 & \text{if } n < 0 \\ a^0 & \text{if } n \in \{0,1,2\} \\ a^1 & \text{if } n \in \{3,4,5\} \\ a^2 & \text{if } n \in \{6,7,8\} \\ \ldots \end{cases}$$

where a is a real number between 0 and 1, as shown in the plot below:



Determine a closed form expression for $X(\Omega)$, which is the discrete-time Fourier transform of $x[n]$.

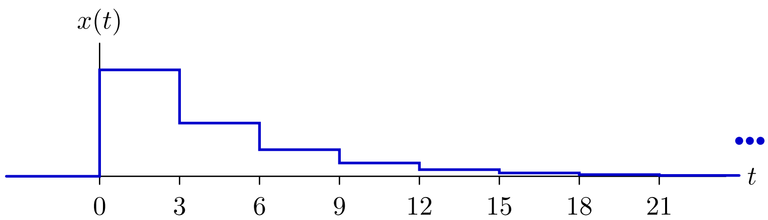
$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= \sum_{m=0}^{\infty} a^m \left(e^{-j\Omega 3m} + e^{-j\Omega(3m+1)} + e^{-j\Omega(3m+2)}\right) \\ &= \sum_{m=0}^{\infty} a^m e^{-j\Omega 3m}\left(1 + e^{-j\Omega} + e^{-j2\Omega}\right) \\ &= \frac{1 + e^{-j\Omega} + e^{-j2\Omega}}{1 - ae^{-j3\Omega}} \end{aligned}$$

Part F

Let $x(t)$ represent the following continuous-time signal

$$x(t) = \begin{cases} 0 & \text{if } < 0 \\ a^0 & \text{if } 0 \le t < 3 \\ a^1 & \text{if } 3 \le t < 6 \\ a^2 & \text{if } 6 \le t < 9 \\ \dots \end{cases}$$

where a is a real number between 0 and 1, as shown in the plot below.



Determine a closed-form expression for $X(\omega)$, which is the continuous-time Fourier transform of $x(t)$.

Start with a simpler signal consisting of only the first "box":

$$x_1(t) = \begin{cases} 1 & \text{if } 0 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of $x_1(t)$ can be found a number of different ways, but we end up with the following, or one of its equivalent forms:

$$X_1(\omega) = \frac{1 - e^{-j3\omega}}{j\omega}$$

With this in mind, we can define $x(t) = \sum_{m=0}^{\infty} a^m x_1(t - 3m)$

By linearity and the time-shift property, then:

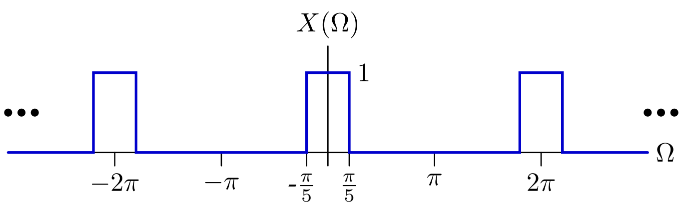
$$\begin{aligned} X(\omega) &= \sum_{m=0}^{\infty} a^m e^{-j3m\omega} X_1(\omega) \\ &= X_1(\omega) \sum_{m=0}^{\infty} \left(ae^{-j3\omega}\right)^m \\ &= X_1(\omega) \left(\frac{1}{1 - ae^{-j3\omega}}\right) \\ &= \left(\frac{1 - e^{-j3\omega}}{j\omega}\right) \left(\frac{1}{1 - ae^{-j3\omega}}\right) \end{aligned}$$

Problem 2: Slowing Down

Let $x[n]$ represent a discrete time signal whose DTFT is given by

$$X(\Omega) = \begin{cases} 1 & \text{if } |\Omega| < \frac{\pi}{5} \\ 0 & \text{if } \frac{\pi}{5} < |\Omega| < \pi \end{cases}$$

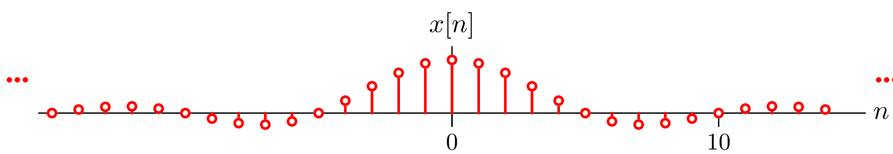
and is periodic in Ω with period 2π as shown below.



Part A

Determine an expression for $x[n]$. Sketch a plot of $x[n]$ and label the important features of your plot.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi/5}^{\pi/5} e^{j\Omega n}\mathrm{d}\Omega = \frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\right]_{-\pi/5}^{\pi/5} = \frac{\sin(\pi n/5)}{\pi n}$$



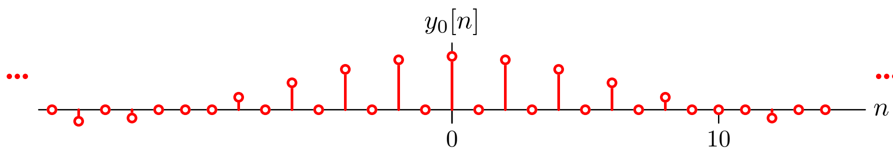
The function has the form of $\sin(n)/n$. Its value at $n = 0$ is $\frac{1}{5}$ which is what we expect since the area under $X(\Omega)$ for $-\pi < \Omega < \pi$ divided by 2π is $1/5$. The function $x[n] = 0$ at $n = \pm 5, \pm 10, \pm 15, \ldots$.

Part B

A new signal $y_0[n]$ is derived by stretching $x[n]$ as follows:

$$y_0[n] = \begin{cases} x\left[\frac{n}{2}\right] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Sketch a plot of $y_0[n]$ and label its key features.



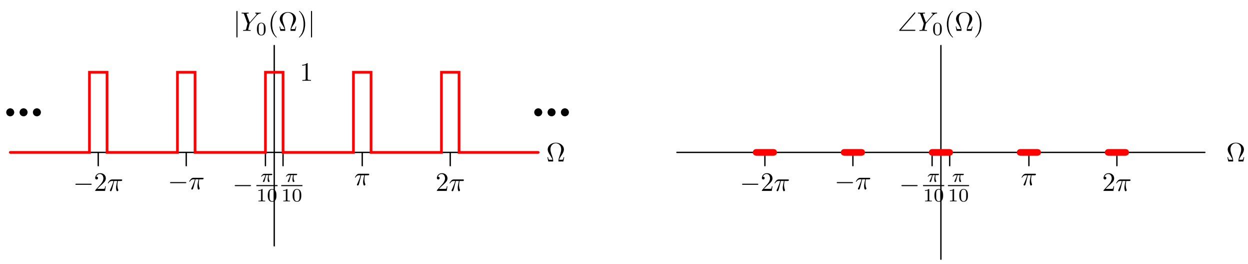
Part C

Determine an expression for $Y_0(\Omega)$ in terms of $X(\Omega)$. Sketch the magnitude and angle of $Y_0(\Omega)$ and label all important parameters of your plots.

$$Y_0(\Omega) = \sum_{n=-\infty}^{\infty} y_0[n]e^{-j\Omega n} = \sum_{n \text{ even}} x\left[n/2\right]e^{-j\Omega n}$$

If we let $n = 2m$, then:

$$Y_0(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega 2m} = X(2\Omega)$$



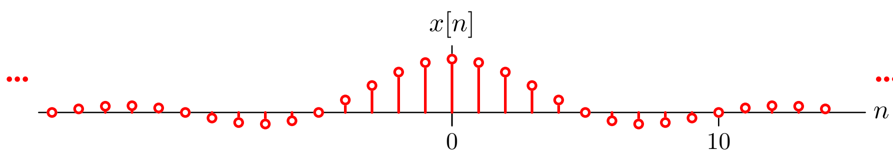
The passband of $Y_0(\Omega)$ is only half as wide as that of $X(\Omega)$. Also, unlike $X(\Omega)$, there are copies of the passband of $Y_0(\Omega)$ at Ω equal to odd multiples of π.

Part D

The $y_0[n]$ signal alternates between non-zero and zero values. To reduce the effect of the zero values, we define

$$y_1[n] = \frac{1}{2}y_0[n-1] + y_0[n] + \frac{1}{2}y_0[n+1]$$

Sketch a plot of $y_1[n]$ and label the important features of your plot. Briefly describe the relation between $y_0[n]$ and $y_1[n]$.



The zero-values that were inserted into $y_0[n]$ have been replaced with the average of the samples on either side of them.

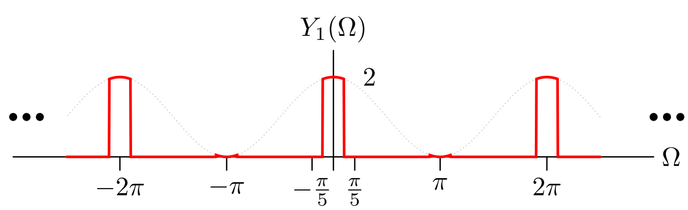
Part E

Determine an expression for $Y_1(\Omega)$ (the Fourier transform of $y_1[n]$) in terms of $Y_0(\Omega)$.

Make a plot of $Y_1(\Omega)$ and briefly describe the relationship between $Y_0(\Omega)$ and $Y_1(\Omega)$

$$Y_1(\Omega) = \sum_{n=-\infty}^{\infty} y_1[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}y_0[n-1] + y_0[n] + \frac{1}{2}y_0[n+1]\right)e^{-j\Omega n}$$

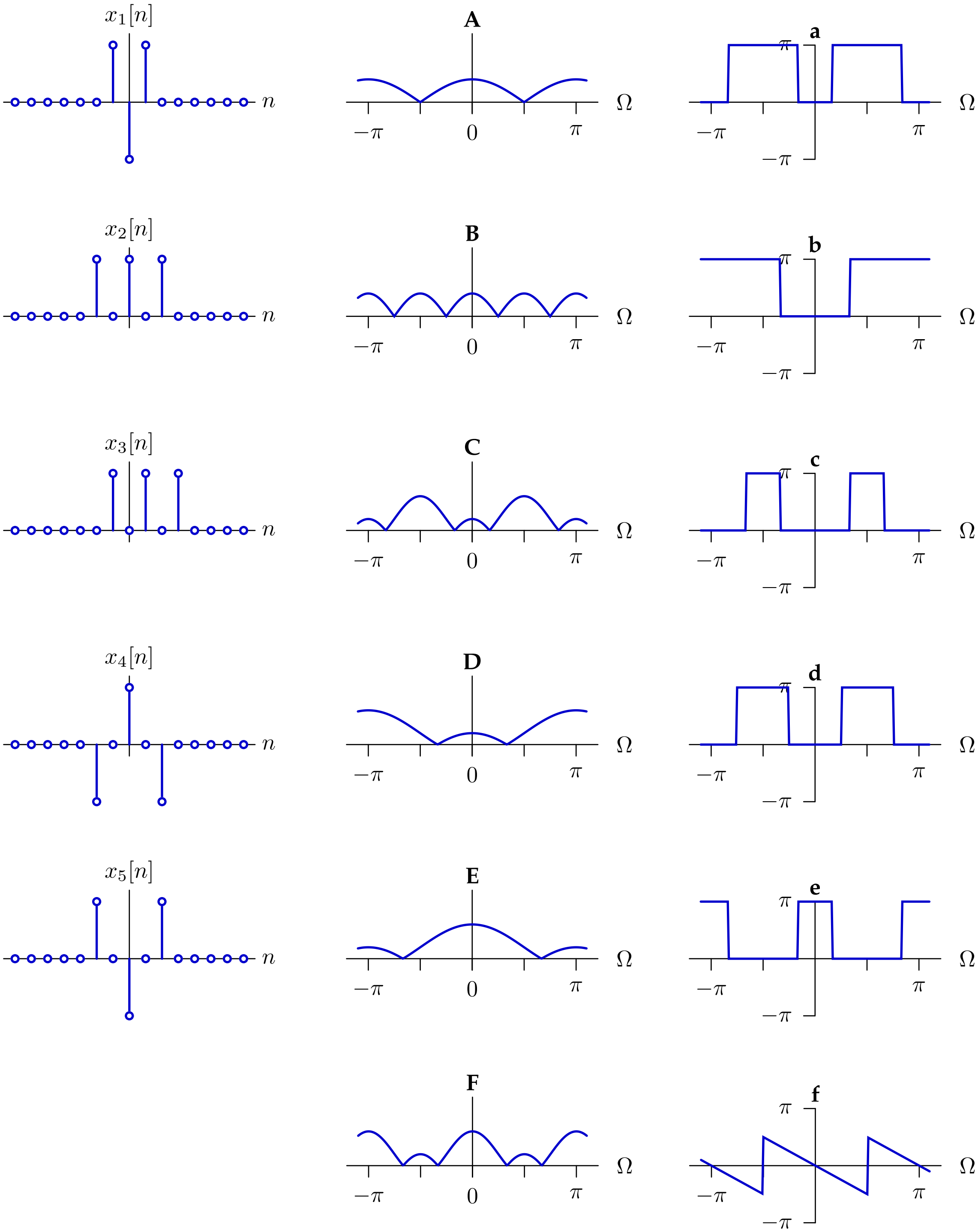
$$= \frac{1}{2}e^{-j\Omega}Y_0(\Omega) + Y_0(\Omega) + \frac{1}{2}e^{j\Omega}Y_0(\Omega) = (1+\cos(\Omega))\,Y_0(\Omega)$$



The overall amplitude of $Y_1(\Omega)$ is twice that of $Y_0(\Omega)$. This results because the values of $y_0[n]$ are zero for odd values of n, while those for $y_1[n]$ are not. Components of $Y_1(\Omega)$ near $\Omega = \pi$ are greatly reduced in magnitude relative to those in $Y_0(\Omega)$ because of multiplying by $1 + \cos(\Omega)$.

Problem 3: Transforms

The diagrams below show five DT signals (x_1 through x_5), six DTFT magnitude plots (labeled **A** through **F**), and six DTFT angle plots (labeled **a** through **f**).



For each signal in the left column, identify its magnitude (A-F or none) and angle (a-f or none).

(answers on following page)

Part 1.
$X_1(\Omega) = 2\cos(\Omega) - 1$

magnitude: D
angle: b

Part 2.
$X_2(\Omega) = 2\cos(\Omega) + 1$

magnitude: F
angle: c

Part 3.
$x_3[n] = x_2[n-1]$, so $X_3(\Omega) = X_2(\Omega)e^{-j\Omega}$

Thus we have $|X_3(\Omega)| = |X_2(\Omega)|$. Then we also have $\angle X_3(\Omega) = \angle X_2(\Omega) - \Omega$. So we are looking for a phase that is linearly decreasing as Ω increases. Graph f *almost* looks right (the phase is linear in Ω), but it doesn't have jumps by $\pm\pi$ at the same points that graph c does, so the correct answer is `none`.

magnitude: F
angle: `none`

Part 4.
$X_4(\Omega) = 1 - 2\cos(2\Omega)$

magnitude: C
angle: e

Part 5.
$x_5[n] = -x_4[n]$, so $X_5(\Omega) = -X_4(\Omega)$ Thus, its magnitude should be the same as $|X_4(\Omega)|$, and its phase graph should have the same rough shape as X_4's, but with 0 and π swapped.

magnitude: C
angle: a