6.300 Problem Set 3

Answers

Problem 1: Sampling Sinusoids

1.1 From Continuous Time to Discrete Time

Consider two periodic signals:

- $f_a(t)$ is periodic with fundamental period $T_a = 2/3$ seconds.
- $f_b(t)$ is periodic with fundamental period $T_b = 2/7$ seconds.

Each of these signals contains a component at its fundamental frequency as well as harmonics 2, 3, 4, and 5 — but no other frequencies. Each of these continuous-time (CT) signals is then sampled at a rate of 10 samples per second to produce related discrete-time (DT) signals:

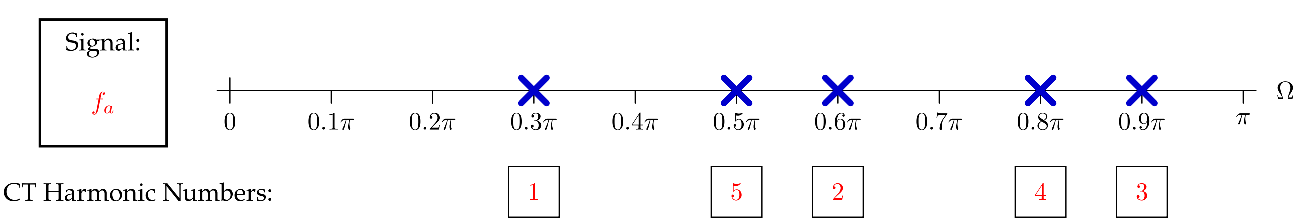
- $f_a[n] = f_a(n/10)$
- $f_b[n] = f_b(n/10)$

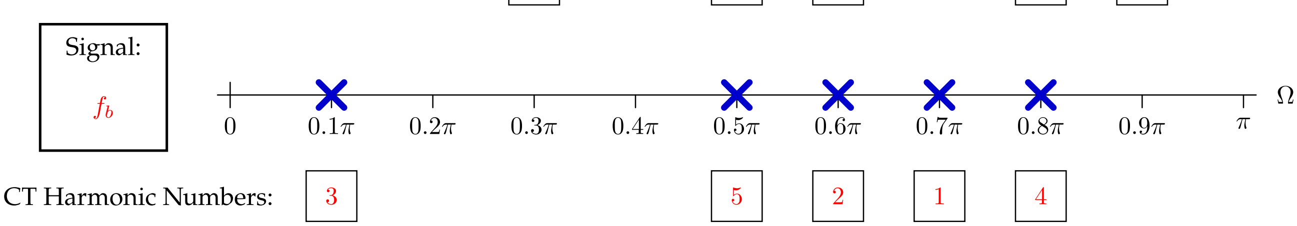
Each of these DT signals contains exactly five discrete-time sinusoidal components with frequencies in the range $0 \le \Omega \le \pi$.

The plots below each represent one of these DT signals. Each plot has five $\times$'s representing the DT frequencies present in the associated DT signal, each of which is associated with one of the harmonics in the CT signal.

In the box next to each plot, write the name of the corresponding signal: f_a or f_b. Then, for each DT frequency, write the number of the associated CT harmonic: 1, 2, 3, 4, or 5.

$f_a(t)$ has components at $\omega = 3\pi, 6\pi, 9\pi, 12\pi$, and 15π.
$f_a[n]$, therefore, has components at $\Omega = .3\pi, .6\pi, .9\pi, 1.2\pi$, and 1.5π.
The values where $0 < \Omega < \pi$ are already in the baseband, so $k = 1, 2, 3$ show up at $\Omega = .3\pi, .6\pi$, and $.9\pi$, respectively.
The values where $\pi < \Omega \le 2\pi$ alias down to $2\pi - \Omega$, so 1.2π ($k = 4$) aliases to $.8\pi$ and 1.5π ($k = 5$) aliases to $.5\pi$.
$f_b(t)$ has components at $\omega = 7\pi, 14\pi, 21\pi, 28\pi$, and 35π.
So $f_b[n]$ has components at $\omega = .7\pi, 1.4\pi, 2.1\pi, 2.8\pi$, and 3.5π.
The value $.7\pi$ is in the range $0 < \Omega < \pi$ and it shows up precisely at that spot. The $\Omega = 1.4\pi$ component ($k = 2$) aliases to $2\pi - 1.4\pi = .6\pi$. The $\Omega = 2.1\pi$ and $\Omega = 2.8\pi$ components ($k = 3$ and 4, respectively) alias to $\Omega - 2\pi$, so $.1\pi$ and $.8\pi$, respectively. And, finally, the $\Omega = 3.5\pi$ component ($k = 5$) aliases to $4\pi - \Omega = .5\pi$.





Let $f(t)$ represent the following continuous-time signal:

$$f(t) = 4\cos(300\pi t) + 2\sin(400\pi t) + \cos(600\pi t).$$

1.2 Fundamental Period I

Let $f_a[n]$ represent a discrete-time signal that is obtained by sampling $f(t)$ with sampling frequency $F_a = 100$ Hz, so that

$$f_a[n] = f(n/F_a).$$

Determine the fundamental period of $f_a[n]$, if one exists. Briefly explain.

$$f_a[n] = f(n/100) = 4\cos(3\pi n) + 2\sin(4\pi n) + \cos(6\pi n) = 4(-1)^n + 0 + 1.$$

This function is periodic in n with periods $N = 2, 4, 6, \cdots$. Therefore the fundamental period is $N = 2$.

1.3 Fundamental Period II

Let $f_b[n]$ represent a discrete-time signal that is obtained by sampling $f(t)$ with sampling frequency $F_b = 200$ Hz, so that

$$f_b[n] = f(n/F_b).$$

Determine the fundamental period of $f_b[n]$, if one exists. Briefly explain.

$$f_b[n] = f(n/200) = 4\cos(3\pi n/2) + 2\sin(2\pi n) + \cos(3\pi n) = 4\cos(3\pi n/2) + 0 + (-1)^n.$$

The fundamental period of this function is $N = 4$.

1.4 Fundamental Period III

Let $f_c[n]$ represent a discrete-time signal that is obtained by sampling $f(t)$ with sampling frequency $F_c = 300$ Hz, so that

$$f_c[n] = f(n/F_c).$$

Determine the fundamental period of $f_c[n]$, if one exists. Briefly explain.

$$f_c[n] = f(n/300) = 4\cos(\pi n) + 2\sin(4\pi n/3) + \cos(2\pi n) = 4(-1)^n + 2\sin(4\pi n/3) + 1.$$

The fundamental period of this function is $N = 6$.

1.5 Sampling Frequency

Determine a sampling frequency F_d for which

$$f_d[n] = f(n/F_d).$$

is not periodic, if such a frequency exists. Briefly explain.

If F_d is an irrational number, then $300\pi/F_d$, $400\pi/F_d$, and $600\pi/F_d$ will not be integer multiples of 2π. Therefore F_d can be any irrational number, e.g., $F_d = \pi$.

Problem 2: Stroboscopy

In lecture and recitation, we saw examples of aliasing primarily as bad things — unintended side effects of sampling that we wanted to avoid. However, aliasing can be used to our benefit as well. One place where aliasing shows its usefulness is in intentionally aliasing something fast to slow down its apparent speed.

Illuminating a moving object with a flashing light can change the apparent speed of the object's motion. This is called the stroboscopic principle, and it is widely used to slow the apparent speed of fast motions — such as the rotation speed of a motor — so as to make them easier for a human to observe. Some neat illustrations of this idea are available at:

- https://www.youtube.com/watch?v=D1LnySRbO5U
- https://www.youtube.com/watch?v=0txlQTmx1LE

For this problem, we will consider an object moving in 1 dimension with a periodic motion — with period T — such that $x(t) = x(t+T)$ represents the object's position at time t.

Assume that we observe $x(\cdot)$ using a flashing light that flashes every Δ seconds. This is equivalent to sampling $x(\cdot)$ to derive a discrete-time signal $y[\cdot]$, given by $y[n] = x(n\Delta)$.

Throughout the problem, as is the case with many fast-moving signals, we'll assume the motion is so fast that our sampling rate is limited to be $\Delta > T$. That is, we cannot sample faster than once per period of the object's motion.

2.1 Stationary Image

Subject to the constraint above, can we set the sampling rate so that the object appears to stand still — so that $y[n]$ is constant for all n? If so, what value(s) of Δ will make this happen? If not, why not?

Yes, this is possible. If we let Δ be an integer multiple of T, then every time we sample, the underlying function $x(t)$ will have gone through exactly an integer number of full periods, ending up back at the same spot, resulting in $x[n]$ being constant for all n.

2.2 A Tenth of a Period

Now assume that we wish to find ten samples of $x(t)$ that are uniformly spaced from $t = 0$ to T, i.e., $y[n] = x\left(\frac{nT}{10}\right)$. Find a value of Δ such that $\Delta > T$ and $y[n] = x\left(\frac{nT}{10}\right)$. Explain your result and your process.

By the strobe equation, we have $y[n] = x(n\Delta)$.

We would like $y[n] = x(nT/10)$.

We could choose $\Delta = T/10$, but that violates the requirement that $\Delta > T$. However, $x(t)$ is periodic in T, so $y[n] = x(nT/10) = x(kT + nT/10)$ where k is an integer. If we let $k = n$ then $y[n] = x(nT/10) = x(nT + nT/10) = x(1.1nT)$

So we can achieve the desired result by setting $\Delta = 1.1T$ or $2.1T$ or $3.1T$ or $\ldots$

2.3 Reverse Mode

Is it possible to choose Δ so that the apparent motion seems to run backward? If so, find Δ such that $y[n] = x\left(-\frac{nT}{10}\right)$. If not, prove why this is not possible.

For values $\Delta = 1.9T, 2.9T, 3.9T, \ldots$ the motion will appear to run backward.

2.4 Periodicity

For what values of Δ is $y[n]$ periodic?

If $y[n]$ is periodic in N, then $y[n] = y[n+N]$.

Since $y[n] = x(n\Delta)$, we have $y[n] = x(n\Delta) = y[n+N] = x((n+N)\Delta)$, and $x(n\Delta) = x(n\Delta + N\Delta)$.

Or, equivalently, $x(t) = x(t + N\Delta)$.

Since x is periodic in T, $N\Delta = kT$ where k is an integer. It follows that Δ must be a rational multiple of T: $\Delta = \frac{k}{N}T$

2.5 Discrete Frequency

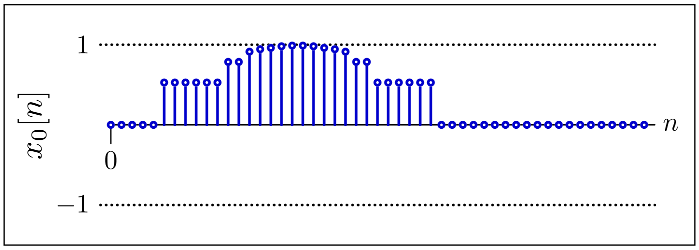
Assume that $x(t)$ is a cosine with radian frequency 2π, $x(t) = \cos(2\pi t)$. Find all the values of Δ such that $y[n]$ can be written in the form $y[n] = \cos(\Omega n)$ where Ω is in the range $0 < \Omega < \frac{\pi}{2}$.

Sampling $x(t)$ gives us values like $\cos(2\pi\Delta n)$. The values of Δ in the range $(0, 1/4)$ all work, as do $(-1/4, 0)$. But those are not the only values that work. In general, any values of Δ in the range $(q - 1/4, q)$ OR $(q, q + 1/4)$ also work, where q is an integer.

The constrant that $\Delta > T$ limits us to values of Δ that are greater than 1 second.

Problem 3: Dome, Sweet Dome

Ben Bitdiddle created a signal $x_0[n]$ representing the MIT dome, but he only saved the DTFS coefficients $X_0[k]$ (and not the original signal). However, he knew that one period of the original signal (which is periodic in $N = 51$) looked like this:



Ben tried several different methods of recovering the original image based on $X_0[k]$, by applying the DTFS synthesis equation to the following sets of coefficients.

For each set of Fourier coefficients described below (X_A through X_I), determine the corresponding signal from the 24 options shown on the next page (x_1 through x_{24}).

Assume that all 24 of those signals are purely real and are periodic in $N = 51$. If the required signal would be complex-valued, record your answer as "must be complex." Otherwise, write the name of the signal from the following page. Justify your answer for each part.

a) $X_A[k] = \text{Re}\left(X_0[k]\right)$

$$X_A[k] = \text{Re}\left(X_0[k]\right) = \frac{1}{2}X_0[k] + \frac{1}{2}X_0^*[k] \quad \text{(property of complex numbers)}$$

Now find the effect of conjugating $X[k]$.

$$X[k] = \frac{1}{N}\sum x[n]e^{-\frac{j2\pi kn}{N}} \quad \text{(Fourier analysis equation)}$$
$$X^*[k] = \frac{1}{N}\sum x^*[n]e^{\frac{j2\pi kn}{N}} \quad \text{(conjugate both sides)}$$
$$X^*[k] = \frac{1}{N}\sum x^*[-n]e^{-\frac{j2\pi kn}{N}} \quad (n \to -n)$$
$$x^*[-n] \overset{\text{DTFS}}{\Longrightarrow} X^*[k] \quad \text{(Fourier analysis equation)}$$

Then, since $x_0[n]$ is real-valued:

$$x_A[n] = \frac{1}{2}x_0[n] + \frac{1}{2}x_0^*[-n] = \frac{1}{2}x_0[n] + \frac{1}{2}x_0[-n]$$

The flipped signal $x_0[-n]$ looks a lot like $x_0[n]$ (since that function is symmetric about $n = 18.5$) but it is shifted by 15 samples. Thus when $x_0[n]$ is added to $x_0[-n]$, part of the dome from $x_0[n]$ overlaps part of the dome from $x_0[-n]$. The result looks like $x_{16}[n]$.

We can think about symmetry properties as a way to check this answer. The sum of $x_0[n]$ and $x_0[-n]$ (which is a flipped version about $n = 0$) will be a symmetric function of n. Since $x_0[n]$ is also periodic in $n = 51$, the result of adding $x_0[n]$ to $x_0[-n]$ is also symmetric about $n = 25.5$. There are only four signals with this symmetry: x_9, x_{11}, x_{16}, and x_{22}. (Notice that x_{14} is not quite right since there are only four leading values of zero.) However, the signal is clearly not zero, eliminating x_{11}. Also x_9 is upside down and x_{22} is upside-down plus a constant.

Thus the answer must be x_{16}.

b) $X_B[k] = \text{Im}\left(X_0[k]\right)$

$$X_B[k] = \text{Im}\left(X_0[k]\right) = \frac{1}{2j}X_0[k] - \frac{1}{2j}X_0^*[k] \quad \text{(property of complex numbers)}$$

Then:

$$x_B[n] = \frac{1}{2j}x_0[n] - \frac{1}{2j}x_0^*[-n] = \frac{1}{2j}x_0[n] - \frac{1}{2j}x_0[-n]$$

Since $x_0[n]$ is real-valued, $x_B[n]$ must be complex-valued (and, thus, can't be represented by any of the pictures).

c) $X_C[k] = j\text{Im}\left(X_0[k]\right)$

$$X_C[k] = j\text{Im}\left(X_0[k]\right) = \frac{1}{2}X_0[k] - \frac{1}{2}X_0^*[k] \quad \text{(property of complex numbers)}$$

Since $x_0[n]$ is real-valued:

$$x_C[n] = \frac{1}{2}x_0[n] - \frac{1}{2}x_0^*[-n] = \frac{1}{2}x_0[n] - \frac{1}{2}x_0[-n]$$

When $x_0[-n]$ is subtracted from $x_0[n]$, the result is an antisymmetric function of n. Since $x_0[n]$ is also periodic in $N = 51$, the result is also antisymmetric about $n = 25.5$. The result looks like $x_8[n]$.

x_{11} has the right symmetry properties, but our answer is clearly not zero. Also x_{13} clearly has the wrong shape. x_{21} is the negative of the right answer, i.e.,$x[-n] - x[n]$.

So the answer must be x_8.

d) $X_D[k] = \begin{cases} 0 & \text{if } k = 0 \\ X_0[k] & \text{otherwise} \end{cases}$

By setting $k = 0$ in the analysis equation,

$$X_0[k] = \frac{1}{N}\sum x_0[n]e^{\frac{-j2\pi kn}{N}}$$

we can see $X_0[0]$ is the average value of $x_0[n]$. Let $\bar{x}$ represent the average value of $x_0[n]$. Then by linearity:

$$x_0[n] - \bar{x} \overset{\text{DTFS}}{\Longrightarrow} X_0[k] - X_0[0]$$

Setting $X_0[0]$ to zero is thus equivalent to subtracting the average value of $x_0[n]$ from $x[n]$ for all n.

Two signals $x_6[n]$ and $x_{19}[n]$ are simple vertical shifts of $x_0[n]$. Since $x_{19}[n]$ is shifted in the wrong direction, the answer must be $x_6[n]$.

e) $X_E[k] = \begin{cases} 0 & \text{if } k = 25 \\ X_0[k] & \text{otherwise} \end{cases}$

Setting the twenty-fifth component of the Fourier series to zero is equivalent to subtracting a complex exponential with frequency of $\frac{2\pi 25}{51}$ from $x_0[n]$.

So our new signal would be $x_E[n] = x_0[n] - X_0[25]e^{j2\pi(25/51)n}$. Unless $X_0[25] = 0$, this extra term will be complex-valued.

f) $X_F[k] = X_0[k] + 1/51$

By linearity, adding a constant to $X_0[k]$ adds a signal $y[n]$ to $x_0[n]$ where $y[n]$ is the signal whose Fourier series $Y[k]$ is $1/51$ for all k:

$$y[n] = \sum \frac{1}{51} e^{\frac{j\omega kn}{51}}$$

By orthogonality, $y[n]$ must be $\delta[n]$ since the above sum goes to zero except at $n = 0$.

Thus the solution is $x_{20}[n]$.

g) $X_G[k] = e^{j\pi} X_0[k]$

The multiplier $e^{j\pi}$ is equal to -1. Therefore the new signal is flipped about the horizontal axis. The solution must be $x_{23}[n]$.

h) $X_H[k] = \begin{cases} X_0[0] & \text{if } k = 0 \\ e^{j\pi} X_0[k] & \text{otherwise} \end{cases}$

The multiplier here is the same as in the last part. However, the DC term is still that of the original signal (which is positive). The resulting effect is that $x_H[n] = 2X_0[0] - x_0[n]$ (i.e., it is reflected about the horizontal axis, and then shifted to account for the change in DC value).

The solution is $x_{15}[n]$.

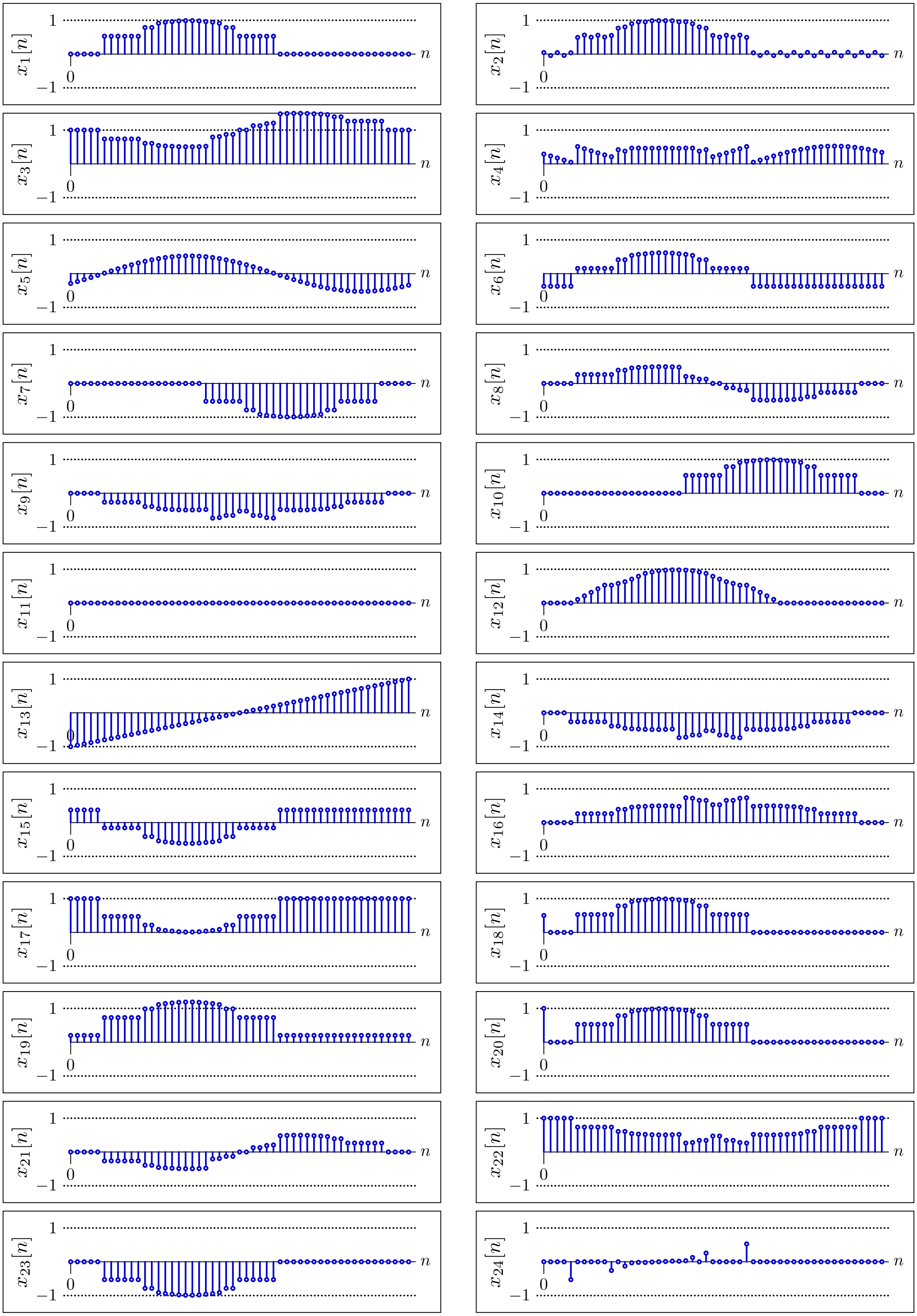
i) $X_I[k] = |X_0[k]| e^{j(-\angle X_0[k])}$

Negating the angle of a complex number while holding the magnitude constant has the same effect as taking the complex conjugate of the original number. This follows from thinking about the definition of magnitude and angle of a complex number a:

$$a = |a| e^{j\angle a}$$
$$a^* = |a| e^{-j\angle a}$$

Thus $X_I[k] = X_0^*[k]$.

Conjugating the Fourier series has the effect of conjugating the time function and then flipping it about $n = 0$. Since $x_0[n]$ is real-valued, the result is just a time flip, and the answer is x_{10}.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