6.300 Problem Set 2
Answers

Problem sets and computational labs are both released on Thursdays at 4:00 p.m.

Both are due the following Wednesday at 10:00 p.m.

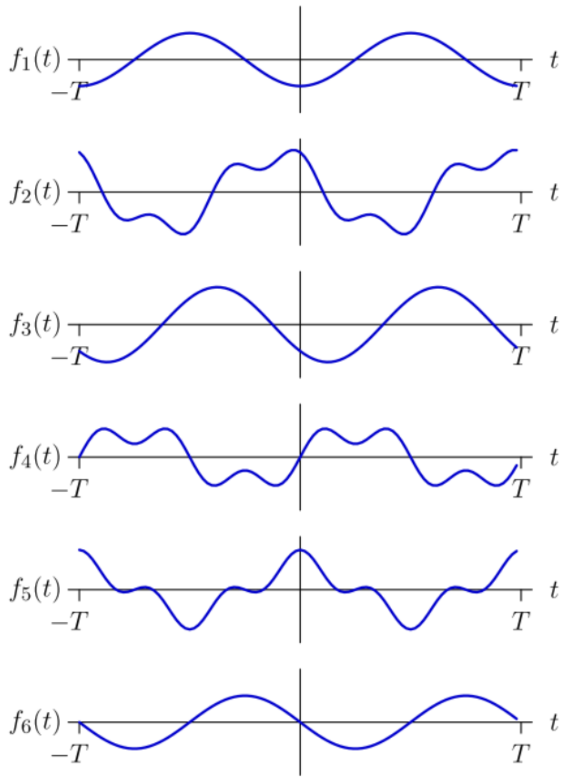
To receive credit for a lab check-in, the check-in must be completed by the following Monday at 9:00 p.m. Lab check-ins may be completed during common hours. A schedule of common hours is posted at `sigproc.mit.edu`.

Problem 1: Graphical Fourier Series

Part A

Each of plots $f_1(\cdot)$ through $f_6(\cdot)$ shows a periodic function with period $t = T$ that can be represented by a Fourier series of the following form:

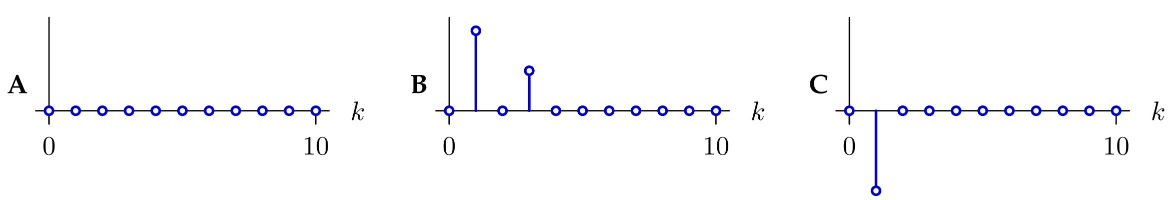
$$f(t) = c_0 + \sum_{k=1}^{10} c_k \cos(2\pi kt/T) + \sum_{k=1}^{10} d_k \sin(2\pi kt/T).$$



Each of the functions $f_i(\cdot)$ can be represented by an expansion of the following form:

$$f_i(t) = \sum_{k=0}^{\infty} c_k \cos\left(\frac{2\pi k}{T}t\right) + \sum_{k=0}^{\infty} d_k \sin\left(\frac{2\pi k}{T}t\right)$$

where c_k and d_k are represented by either **A**, **B**, or **C** (shown below), where values are assumed to be 0 for all values of k not pictured:



For each $f_i(\cdot)$, indicate which of **A**, **B**, or **C** corresponds to its c_k and d_k coefficients. Justify your answers.

$f_1(t)$ is a negative cosine at the fundamental frequency. Therefore $c_1 < 0$ and $d_1 = 0$. The c_k coefficients are those in panel C. The d_k coefficients are those in panel A.

$f_2(t)$ has both fundamental and higher order terms. The peak of the fundamental component occurs slightly before $t = 0$, as would happen if $c_1 = 1$ and $d_1 = -1$. Therefore $c_1 < 0$ and $d_1 = 0$. The c_k coefficients are those in panel B. The d_k coefficients are those in panel C.

$f_3(t)$ is a sinusoid at the fundamental frequency that has neither sine nor cosine phase. Therefore both c_1 and d_1 must be non-zero and all other coefficients must be zero. The c_k coefficients are those in panel C. The d_k coefficients are those in panel C.

$f_4(t)$ has both fundamental and higher order terms. Since it is anti-symmetric about $t = 0$, $c_k = 0$ for all k. The c_k coefficients are those in panel A. The d_k coefficients are those in panel B.

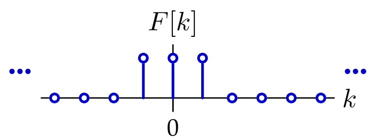
$f_5(t)$ has both fundamental and higher order terms. Since it is symmetric about $t = 0$, $d_k = 0$ for all k. The c_k coefficients are those in panel B. The d_k coefficients are those in panel A.

$f_6(t)$ is a negative sine wave at the fundamental frequency. Therefore $d_1 < 0$ and $c_1 = 0$. The c_k coefficients are those in panel A. The d_k coefficients are those in panel C.

Part B

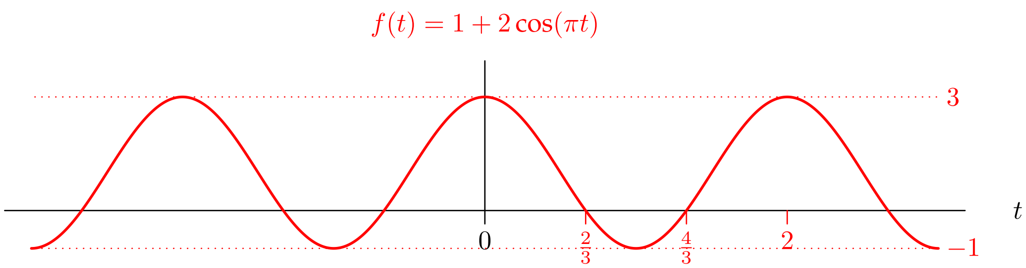
Consider a function $f(\cdot)$ that is periodic in $T = 2$ seconds. Its CTFS coefficients $F[\cdot]$ are shown below:

$$F[k] = \begin{cases} 1 & \text{if } -1 \le k \le 1 \\ 0 & \text{otherwise} \end{cases}$$

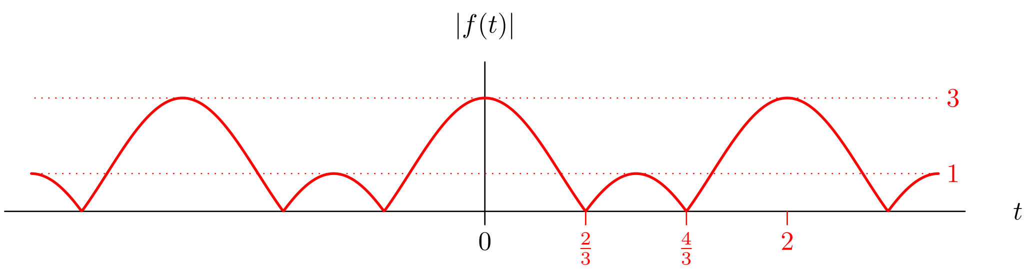


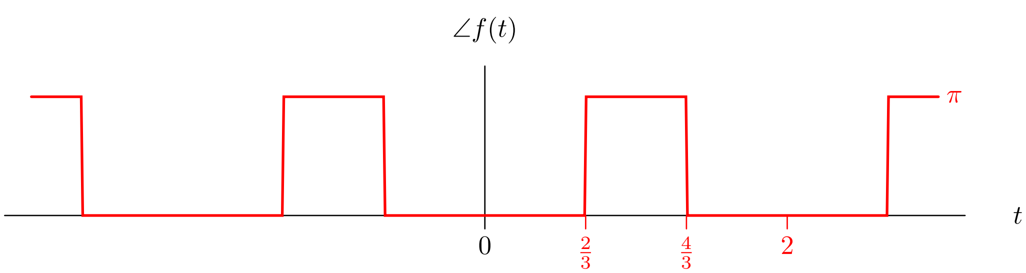
such that $f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi k}{T}t}$

$f(t)$ is purely real. Sketch a plot of $f(t)$ versus t:



Now, thinking of $f(t)$ as a number in the complex plane, sketch the magnitude and angle of $f(t)$ as functions of t:

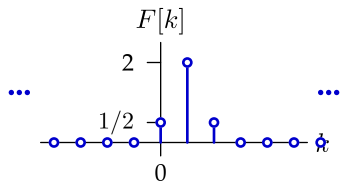




Part C

Consider a function $f(\cdot)$ that is periodic in $T = 3$ seconds. Its CTFS coefficients $F[\cdot]$ are shown below:

$$F[k] = \begin{cases} 1/2 & \text{if } k \in \{0, 2\} \\ 2 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$



such that $f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi k}{T}t}$

$f(t)$ can be expressed in the following form: $f(t) = \left(e^{j\omega_1 t}\right) \times (\cos(\omega_2 t) + c)$

What are the necessary values of ω_1, ω_2, and c? Show your work.

Applying the synthesis formula directly, we have:

$$f(t) = \frac{1}{2} + 2e^{j\frac{2\pi}{3}t} + \frac{1}{2}e^{j\frac{4\pi}{3}t}\mathrm{d}t$$

We would like to coax this into the following form:

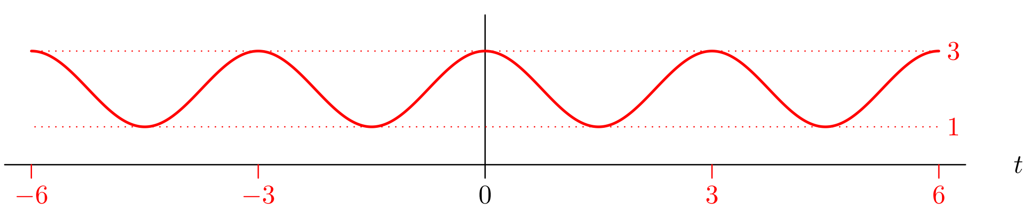
$$f(t) = \left(e^{j\omega_1 t}\right) \times (\cos(\omega_2 t) + c) = \frac{e^{j(\omega_1+\omega_2)t}}{2} + \frac{e^{j(\omega_1-\omega_2 t)}}{2} + ce^{j\omega_1 t}$$

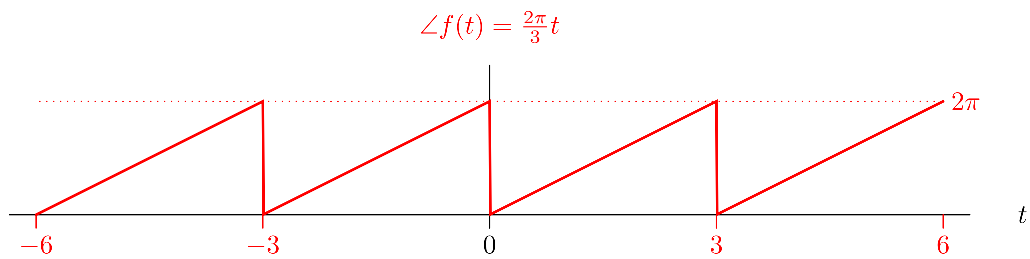
which we can do with $\omega_1 = \omega_2 = \frac{2\pi}{3}$ and $c = 2$.

This tells us that we have $f(t) = e^{j\frac{2\pi}{3}t}\left(\cos\left(\frac{2\pi}{3}t\right) + 2\right)$

Now, thinking of $f(t)$ as a number in the complex plane, sketch the magnitude and angle of $f(t)$ as functions of t:

$|f(t)| = 2 + \cos\left(\frac{2\pi}{3}t\right)$





One nice method here is to think first about a related function $f_a(t) = 2 + \cos\left(\frac{2\pi}{3}t\right)$ first. This function is always positive and real, so its magnitude is just the function itself, and its angle is zero.

Then we can think about the effect of multiplying by $e^{j\frac{2\pi}{3}t}$. If we previously had $f_a(t) = r_t e^{j\phi_t}$, then: $f(t) = e^{j\frac{2\pi}{3}t} f_a(t) = r_t e^{j(\phi_t + \frac{2\pi}{3}t)}$

So we have $|f(t)| = |f_a(t)| = 2 + \cos\left(\frac{2\pi}{3}t\right)$, and $\angle f(t) = \frac{2\pi}{3}t + \angle f_a(t) = \frac{2\pi}{3}t$.

Here, we've sketched the angle in the range $[0, 2\pi)$, but that's not strictly necessary since the angle wraps around at 2π.

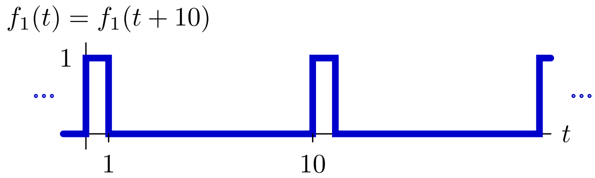
Notice that, unlike the function we looked at in a similar problem last week, $f(t)$ is not purely real-valued! What feature(s) of the coefficients are consequences of this fact?

We've seen before that real-valued signals always have Fourier series coefficients that are conjugate symmetric, i.e., if $x(\cdot)$ is real-valued, then $X[k] = X^*[-k]$. Here, we don't see that symmetry, which is the main thing to notice to let us know that $f(t)$ in this problem is complex-valued.

Problem 2: Fourier Series

Part A

Let $f_1(t)$ represent the following function, which is periodic in t with period $T = 10$:



Find the coefficients $F_1[k]$ for a Fourier series expansion of $f_1(t)$ in complex-exponential form:

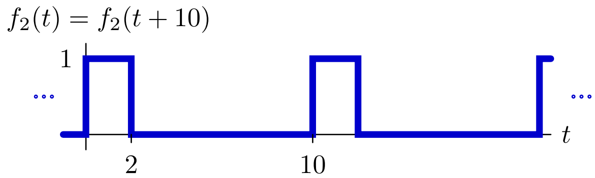
$$f_1(t) = \sum_{k=-\infty}^{\infty} F_1[k] e^{j\frac{2\pi k}{T}t}$$

Directly apply the Fourier analysis formula. The expression is ill-formed for $k = 0$, but the $F_1[0]$ term can be determined by computing the average value of $f_1(t)$ over one period. There are multiple equivalent expressions for the Fourier series coefficients. Here's one.

$$F_1[k] = \begin{cases} 1/10 & k = 0 \\ \frac{1}{j2\pi k}(1 - e^{-j\pi k/5}) & k \neq 0 \end{cases}$$

Part B

Let f_2 represent the following function that is periodic in t with period $T = 10$:



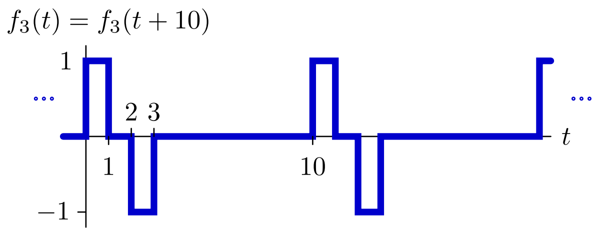
Find the Fourier series coefficients associated with this function in complex-exponential form.

Directly apply the Fourier analysis formula. The expression is ill-formed for $k = 0$, but the $F_2[0]$ term can be determined by computing the average value of $f_2(t)$ over one period. There are multiple equivalent expressions for the Fourier series coefficients. Here's one.

$$F_2[k] = \begin{cases} 1/5 & k = 0 \\ \frac{1}{j2\pi k}(1 - e^{-j2\pi k/5}) & k \neq 0 \end{cases}$$

Part C

Let f_3 represent the following function that is periodic in t with period $T = 10$:



Find the Fourier series coefficients associated with this function in complex-exponential form.

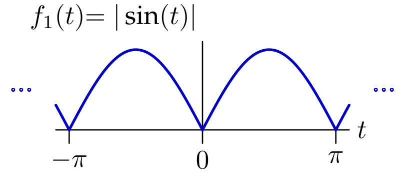
$$f_3(t) = f_1(t) - f_1(t-2)$$

$$F_3[k] = F_1[k] - e^{-j\frac{2\pi}{T}(2k)}F_1[k] = (1 - e^{-j2\pi k/5})F_1[k]$$

$$F_3[k] = \begin{cases} 0 & k = 0 \\ (1 - e^{-j2\pi k/5})\frac{1}{j2\pi k}(1 - e^{-j\pi k/5}) & k \neq 0 \end{cases}$$

Part D

Find the complex-exponential Fourier series coefficients $F_1[k]$ for the following function:



Using direct application of the analysis formula:

$$\begin{aligned}
F_1[k] &= \frac{1}{T}\int_T f_2(t)e^{-j\frac{2\pi k}{T}t}\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^\pi \sin(t)e^{-j2kt}\mathrm{d}t \\
&= \frac{1}{2\pi j}\int_0^\pi \left(e^{jt} - e^{-jt}\right)e^{-j2kt}\mathrm{d}t \\
&= \frac{1}{2\pi j}\int_0^\pi e^{jt(1-2k)}\mathrm{d}t + \frac{1}{2\pi j}\int_0^\pi e^{-jt(1+2k)}\mathrm{d}t \\
&= \left(\frac{1}{2\pi j}\frac{1}{j(1-2k)}\,e^{jt(1-2k)}\Big|_{t=0}^{\pi}\right) - \left(\frac{1}{2\pi j}\frac{1}{-j(1+2k)}\,e^{-jt(1+2k)}\Big|_{t=0}^{\pi}\right) \\
&= \left(\frac{-1}{2\pi - 4\pi k}\,e^{jt(1-2k)}\Big|_{t=0}^{\pi}\right) + \left(\frac{-1}{2\pi + 4\pi k}\,e^{-jt(1+2k)}\Big|_{t=0}^{\pi}\right) \\
&= \frac{-1}{2\pi - 4\pi k}\left(\underbrace{e^{j(\pi-2\pi k)}}_{-1} - \underbrace{e^{j0}}_{1}\right) + \frac{-1}{2\pi + 4\pi k}\left(\underbrace{e^{-j(\pi+2\pi k)}}_{-1} - \underbrace{e^{j0}}_{1}\right) \\
&= \frac{2}{2\pi - 4\pi k} + \frac{2}{2\pi - 4\pi k} \\
&= \frac{1}{\pi}\left(\frac{1}{1-2k} + \frac{1}{1+2k}\right) \\
&= \frac{1}{\pi}\left(\frac{1+2k+1-2k}{1-4k^2}\right) \\
&= \frac{2}{\pi}\left(\frac{1}{1-4k^2}\right)
\end{aligned}$$

Part E

Find the complex-exponential Fourier series coefficients $F_2[k]$ for the following function:

$$f_2(t) = \cos\left(\frac{2t\pi}{3}\right)\sin\left(\frac{2t\pi}{9}\right)$$

This form is already *really* close to the form we want. We're looking to express this as a sum of complex exponentials, so let's rearrange things a little bit to start, to see if we can put it in that form (rather than using the analysis formula, which might be harder):

$$\begin{aligned} f_2(t) &= \cos\left(\frac{2t\pi}{3}\right)\sin\left(\frac{2t\pi}{9}\right) \\ &= \left(\frac{e^{j\frac{6\pi}{9}t}+e^{-j\frac{6\pi}{9}t}}{2}\right)\left(\frac{e^{j\frac{2\pi}{9}t}-e^{-j\frac{2\pi}{9}t}}{2j}\right) \\ &= \frac{e^{j\frac{8\pi}{9}t}}{4j}+\frac{e^{-j\frac{4\pi}{9}t}}{4j}-\frac{e^{j\frac{4\pi}{9}t}}{4j}-\frac{e^{-j\frac{8\pi}{9}t}}{4j} \end{aligned}$$

After these steps, this function is in precisely the form we're looking for in the synthesis equation:

$$f_2(t) = \sum_{k=-\infty}^{\infty} F_2[k]e^{j\frac{2\pi k}{T}t}$$

Here, we have $T = 9$ seconds, and so to make these match, we need:

$$F_2[k] = \begin{cases} \frac{1}{4j} & \text{if } k \in \{-2, 4\} \\ \frac{-1}{4j} & \text{if } k \in \{2, -4\} \\ 0 & \text{otherwise} \end{cases}$$

Problem 3: Fourier Series — Trigonometric Form vs. Complex Form

In this problem, we compare two methods for expanding a function $f(\omega_0 t)$ as a series of the following form:

$$f(\omega_0 t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)$$

Part A

Use trigonometric identities and the rules of ordinary algebra to determine the values of the non-zero coefficients c_k and d_k needed to expand the function $f_1(\omega_0 t) = \cos^5(\omega_0 t)$. Show your work.

Using trig identities:

$$\begin{aligned}
f_1(\omega_0 t) = \cos^5(\omega_0 t) &= \cos^2(\omega_0 t)\cos^2(\omega_0 t)\cos(\omega_0 t) \\
&= \left(\frac{1+\cos(2\omega_0 t)}{2}\right) \times \left(\frac{1+\cos(2\omega_0 t)}{2}\right) \times \cos(\omega_0 t) \\
&= \left(\frac{1+2\cos(2\omega_0 t)+\cos^2(2\omega_0 t)}{4}\right) \times \cos(\omega_0 t) \\
&= \left(\frac{1+2\cos(2\omega_0 t)+\frac{1+\cos(4\omega_0 t)}{2}}{4}\right) \times \cos(\omega_0 t) \\
&= \left(\frac{3}{8}+\frac{1}{2}\cos(2\omega_0 t)+\frac{1}{8}\cos(4\omega_0 t)\right)\cos(\omega_0 t) \\
&= \frac{3}{8}\cos(\omega_0 t)+\frac{1}{2}\cos(2\omega_0 t)\cos(\omega_0 t)+\frac{1}{8}\cos(4\omega_0 t)\cos(\omega_0 t) \\
&= \frac{3}{8}\cos(\omega_0 t)+\frac{1}{4}\cos(\omega_0 t)+\frac{1}{4}\cos(3\omega_0 t)+\frac{1}{16}\cos(3\omega_0 t)+\frac{1}{16}\cos(5\omega_0 t) \\
&= \frac{5}{8}\cos(\omega_0 t)+\frac{5}{16}\cos(3\omega_0 t)+\frac{1}{16}\cos(5\omega_0 t)
\end{aligned}$$

The coefficient $c_1 = 5/8$, $c_3 = 5/16$, and $c_5 = 1/16$.The other coefficients are zero.

Part B

An alternative to trigonometric identities is to use complex exponentials. Determine the non-zero coefficients c_k and d_k as in the previous part – but this time use Euler's formula and complex numbers, but no trigonometric identifies.

We can use Euler's formula to rewrite the target function as:

$$f_1(\omega_0 t) = \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)^5$$

Then we can expand the fifth power by repeated multiplication (or by using the binomial equation) to get:

$$f_1(\omega_0 t) = \frac{1}{32}\left(e^{j5\omega_0 t} + 5e^{j3\omega_0 t} + 10e^{j\omega_0 t} + 10e^{-j\omega_0 t} + 5e^{-3j\omega_0 t} + e^{-5j\omega_0 t}\right)$$

Finally, we can pair complex exponentials with their negative partners and use Euler's formula to obtain

$$f_1(\omega_0 t) = \frac{5}{8}\cos(\omega_0 t) + \frac{5}{16}\cos(3\omega_0 t) + \frac{1}{16}\cos(5\omega_0 t)$$

As before, the coefficient $c_1 = 5/8$, $c_3 = 5/16$, and $c_5 = 1/16$. The others are zero.

Part C

Use trigonometric identities plus the rules of ordinary algebra to determine the values of the non-zero coeeficients c_k and d_k needed to expand the function $f_2(\omega_0 t) = \sin^5(\omega_0 t)$. Show your work.

$$\begin{aligned}
f_2(\omega_0 t) &= \sin^5(\omega_0 t) = \sin^2(\omega_0 t)\sin^2(\omega_0 t)\sin(\omega_0 t) \\
&= \left(\frac{1-\cos(2\omega_0 t)}{2}\right) \times \left(\frac{1-\cos(2\omega_0 t)}{2}\right) \times \sin(\omega_0 t) \\
&= \left(\frac{1 - 2\cos(2\omega_0 t) + \cos^2(2\omega_0 t)}{4}\right) \times \sin(\omega_0 t) \\
&= \left(\frac{1 - 2\cos(2\omega_0 t) + \frac{1+\cos(4\omega_0 t)}{2}}{4}\right) \times \sin(\omega_0 t) \\
&= \left(\frac{3}{8} - \frac{1}{2}\cos(2\omega_0 t) + \frac{1}{8}\cos(4\omega_0 t)\right)\sin(\omega_0 t) \\
&= \frac{3}{8}\sin(\omega_0 t) - \frac{1}{2}\cos(2\omega_0 t)\sin(\omega_0 t) + \frac{1}{8}\cos(4\omega_0 t)\sin(\omega_0 t) \\
&= \frac{3}{8}\sin(\omega_0 t) - \frac{1}{4}\sin(3\omega_0 t) + \frac{1}{4}\sin(\omega_0 t) + \frac{1}{16}\sin(5\omega_0 t) - \frac{1}{16}\sin(3\omega_0 t) \\
&= \frac{5}{8}\sin(\omega_0 t) - \frac{5}{16}\sin(3\omega_0 t) + \frac{1}{16}\sin(5\omega_0 t)
\end{aligned}$$

The coefficient $d_1 = 5/8$, $d_3 = -5/16$, and $d_5 = 1/16$. The other coefficients are zero.

Part D

Determine the non-zero coefficients c_k and d_k as in the previous part – but this time use Euler's formula and complex numbers, but no trigonometric identities.

Use Euler's formula to rewrite the target function as:

$$f_2(\omega_0 t) = \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)^5$$

Then expand the fifth power by repeated multiplication (or by using the binomial formula) to obtain:

$$f_2(\omega_0 t) = \frac{1}{32j}\left(e^{j5\omega_0 t} - 5e^{j3\omega_0 t} + 10e^{j\omega_0 t} - 10e^{-j\omega_0 t} + 5e^{-3j\omega_0 t} - e^{-5j\omega_0 t}\right)$$

Then pair the complex exponentials with their negative partners and use Euler's formula to obtain:

$$f_2(\omega_0 t) = \frac{5}{8}\sin(\omega_0 t) - \frac{5}{16}\sin(3\omega_0 t) + \frac{1}{16}\sin(5\omega_0 t)$$

As before, the coefficient $d_1 = 5/8$, $d_3 = -5/16$, and $d_5 = 1/16$. The others are zero.

Part E

List the mathematical relations that you used in each of the previous parts. Briefly describe the pros and cons of using trigonometric identities versus Euler's formula.

The solution to part a used the product of cosines rule repeatedly. The solution to part c used the product of sines and the product of sines with cosines. Thus these solutions are similar, but used different trig identities.

The solution to parts b and d used Euler's formula to convert the original trig functions tocomplex exponentials. The resulting expression was expanded with the binomial theorem (Pascal's triangle). And that result was converted back to trig form using Euler's formula.

The point of this problem is that a single equation – namely Euler's formula – substitutes for any number of trigonometric identities. Rather than remembering and learning to use the many trig identities that exist, we can remember and learn to use a single equation (Euler's formula) if we use complex numbers.