6.300 Signal Processing

Week 10, Lecture A:
Quiz Review-Properties of Fourier Transforms

Lecture slides are available on CATSOOP:

https://sigproc.mit.edu/fall24

Quiz 2: Thursday November 7, 2-4pm 50-340 (2:05-3:45pm exam; 3:45-3:55 scan and upload)

- Closed book except for two page of notes (8.5'' x 11'' both sides)
- No electronic devices (No headphones, cell phones, calculators, …) but you will need your phone at the end! Also please make sure you scan the first page with your name!
- Coverage up to Week #8 (DFT) today's lecture and recitation also useful
- practice quiz as a study aid, no HW#9

Topics we have covered till now

- Fourier Series (CTFS, DTFS)
- Sampling and Aliasing
- Fourier Transforms (CTFT, DTFT)
- Systems:
  - LTI systems & Difference(differential) equation description
  - Impulse response & Convolution
  - Frequency Response & Filtering
- DFT (& circular convolution)

Properties of Fourier Transforms

Continuous-Time Fourier Transform

Property	$y(t)$	$Y(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Time delay	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Scaling time	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$
Time derivative	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency derivative	$tx(t)$	$j\frac{d}{d\omega} X(\omega)$

Discrete-Time Fourier Transform

Property	$y[n]$	$Y(\Omega)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\Omega) + bX_2(\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time delay	$x[n - n_0]$	$e^{-j\Omega n_0} X(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Frequency derivative	$nx[n]$	$j\frac{d}{d\Omega} X(\Omega)$

FT Properties: Conjugation

If: $y(t) = x^*(t)$

then: $Y(\omega) = X^*(-\omega)$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \cdot e^{j(-\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} (x\,(t) \cdot e^{-j(-\omega)t} dt)^*$$

$$= X^*(-\omega)$$

If: $y[n] = x^*[n]$

then: $Y(\Omega) = X^*\,(-\Omega)$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*[n] \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*[n] \cdot e^{j(-\Omega)n}$$

$$= \sum_{n=-\infty}^{\infty} (x[n] \cdot e^{-j(-\Omega)n})^*$$

$$= X^*(-\Omega)$$

FT Properties: Time Derivative/Difference

If: $$y(t) = \frac{d}{dt}x(t)$$

then: $$Y(\omega) = j\omega \cdot X(\omega)$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$$

differentiate both sides w.r.t t

$$y(t) = \int_{-\infty}^{\infty} X(\omega) \cdot (j\omega) \cdot e^{j\omega t}\, d\omega$$

$$y(t) = \int_{-\infty}^{\infty} (j\omega \cdot X(\omega)) \cdot e^{j\omega t}\, d\omega$$

since

$$y(t) = \int_{-\infty}^{\infty} Y(\omega) \cdot e^{j\omega t}\, d\omega \Rightarrow Y(\omega) = j\omega \cdot X(\omega)$$

If: $$y[n] = x[n] - x[n-1]$$

then: $$Y(\Omega) = (1 - e^{-j\Omega}) \cdot X(\Omega)$$

This can be derived based on the time delay property of DTFT.

FT Properties: Frequency Derivative

If: $y(t) = tx(t)$

then: $Y(\omega) = j\frac{d}{d\omega}X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

differentiate both sides w.r.t ω

$$\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot (-jt) \cdot e^{-j\omega t}\, dt$$

$$j\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} (tx(t)) \cdot e^{-j\omega t}\, dt$$

since

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t}\, dt \Rightarrow Y(\omega) = j\frac{d}{d\omega}X(\omega)$$

If: $y[n] = nx[n]$

then: $Y(\Omega) = j\frac{d}{d\Omega}X(\Omega)$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

differentiate both sides w.r.t Ω

$$\frac{d}{d\Omega}X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot (-jn) \cdot e^{-j\Omega n}$$

$$j\frac{d}{d\Omega}X(\Omega) = \sum_{n=-\infty}^{\infty} nx[n] \cdot e^{-j\Omega n}$$

since

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j\Omega n} \Rightarrow Y(\Omega) = j\frac{d}{d\Omega}X(\Omega)$$

FT Properties: Scaling Time

If: $y(t) = x(At)$, A>0

then: $Y(\omega) = \frac{1}{A} X\left(\frac{\omega}{A}\right)$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(At) \cdot e^{-j\omega t}\, dt$$

$$let\ u = At, then\ t = \frac{u}{A}, du = Adt$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(u) \cdot e^{-j\omega \frac{u}{A}} (\frac{1}{A} du)$$

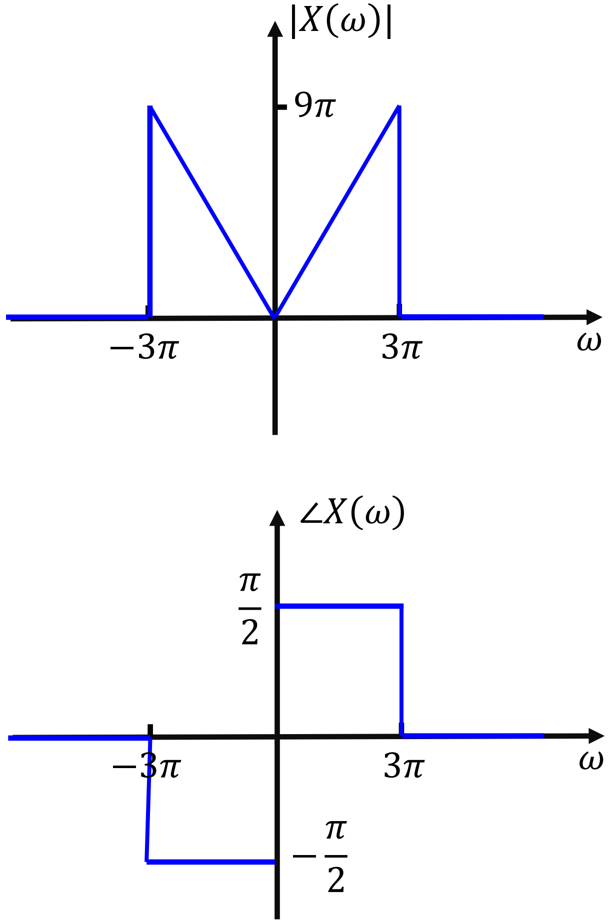
$$= \frac{1}{A} \int_{-\infty}^{\infty} x(u) \cdot e^{-j\left(\frac{\omega}{A}\right)u}\, du = \frac{1}{A} X\left(\frac{\omega}{A}\right)$$

Exercise I

- The magnitude and phase of signal $x(t)'s$ Fourier Transform $X(\omega)$ is shown on the right, find signal $x(t)$.

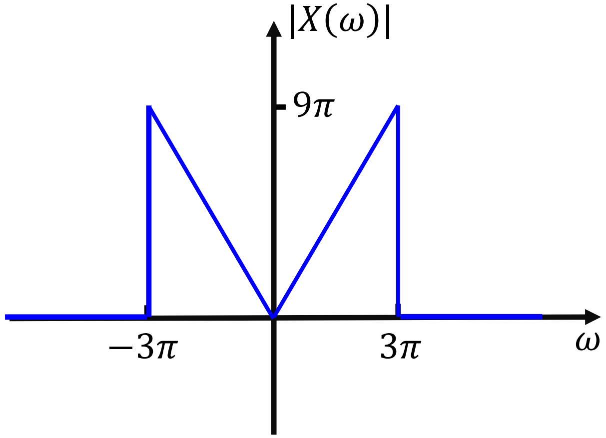
Let us find $X(\omega)$ first:

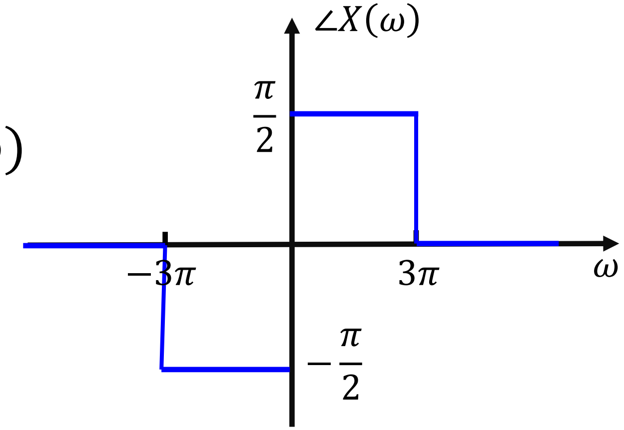
participation question



Exercise I

- The magnitude and phase of signal $x(t)'s$ Fourier Transform $X(\omega)$ is shown on the right, find signal $x(t)$.





From the magnitude and phase plot we can find that

$$X(\omega) = \begin{cases} j3\omega, & -3\pi < \omega < 3\pi \\ 0, & Otherwise \end{cases}$$

Let $Y(\omega) = \begin{cases} 3, & -3\pi < \omega < 3\pi \\ 0, & Otherwise \end{cases}$, $y(t) \leftrightarrow Y(\omega)$, $\frac{d}{dt}y(t) \leftrightarrow j\omega Y(\omega) = X(\omega)$

$$y(t) = \frac{1}{2\pi}\int_{-3\pi}^{3\pi} 3e^{j\omega t}\, d\omega = \frac{3}{2\pi}\frac{e^{j\omega t}\Big|_{-3\pi}^{3\pi}}{jt} = \frac{3}{\pi t}\sin(3\pi t)$$

$$x(t) = \frac{d}{dt}y(t) = \frac{9}{t}\cos(3\pi t) - \frac{3}{\pi t^2}\sin(3\pi t)$$

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Discrete Fourier Transform (DFT)

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n} \qquad X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi k}{N}n}$$

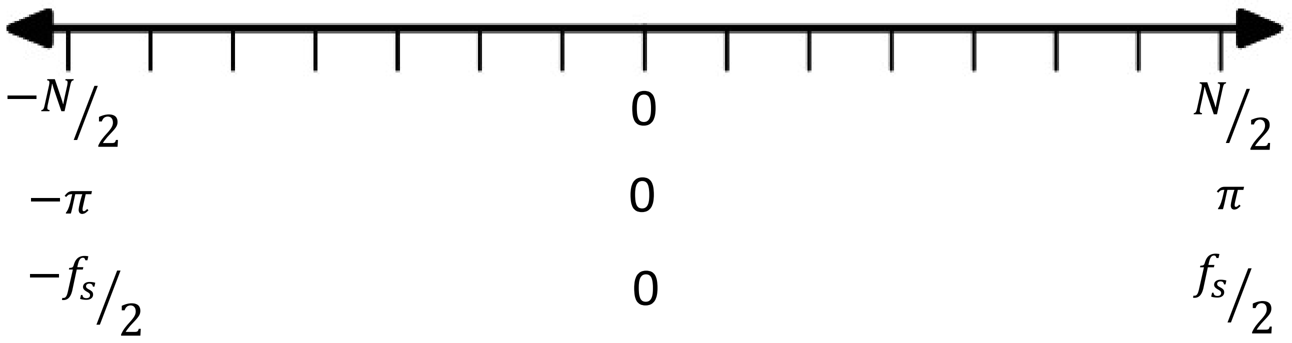
DFT (especially efficient FFT algorithm) significantly facilitate computation.

Connection with DTFS and DTFT:

- with DTFS: DFT is the DTFS of periodically extended version, $x_p[n]$
- with DTFT: $X[k] = \frac{1}{N} X_w(\frac{2\pi k}{N})$ $\quad X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n]\cdot e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]w[n]\cdot e^{-j\Omega n}$

Frequency Resolution in DFT: fs/N in Hz

$$\Omega = \frac{2\pi k}{N} \qquad \Omega = \frac{2\pi f}{f_s}$$



k: integer (frequency)

Ω: rad/sample

f: cycles/second (Hz)

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Systems

Three different representations for Linear, Time-Invariant (LTI) systems:

- Difference/Differential Equation: e.g. $y[n] - \alpha y[n-1] + \beta y[n-2] = x[n], \quad 0 < \alpha, \beta < 1$

$$y(t) + \alpha \frac{dy(t)}{dt} = x(t), \alpha > 0$$

- Convolution:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \qquad y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

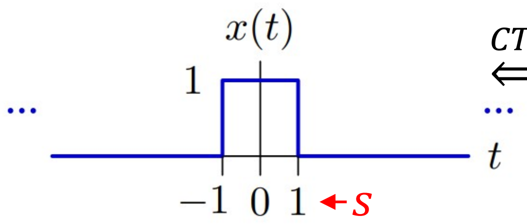
- Filter:

$$e^{j\Omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(\Omega)\, e^{j\Omega n} \qquad Y(\Omega) = H(\Omega) X(\Omega)$$

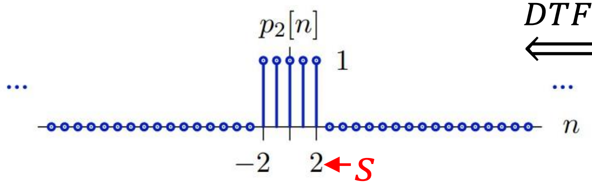
We can carry out filtering either in time domain or frequency domain.
Freq. domain multiplication correspond to time domain (circular) convolution:

$$(x * h)[n] \overset{DTFT}{\Longleftrightarrow} H(\Omega)X(\Omega) \qquad \frac{1}{N}(x \circledast h)[n] \overset{DFT}{\Longleftrightarrow} H[k]X[k]$$

Useful Signals and Their Fourier Transforms



$\overset{CTFT}{\Longleftrightarrow}$

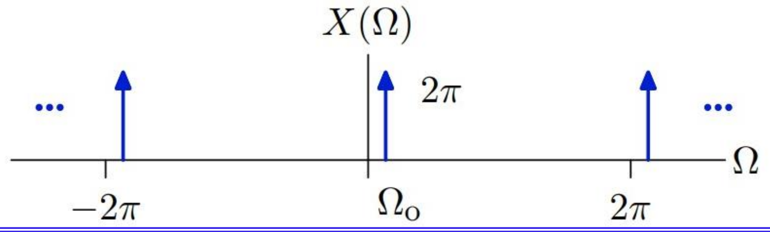


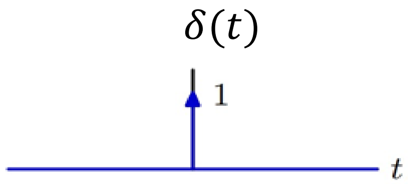
$\overset{DTFT}{\Longleftrightarrow}$

$x(t) = e^{j\omega_0 t} \overset{CTFT}{\Longleftrightarrow}$

$x[n] = e^{j\Omega_0 n} \overset{DTFT}{\Longleftrightarrow}$

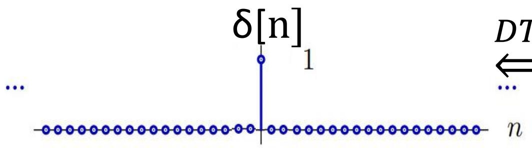
$e^{j\Omega_0 n} \overset{DTFT}{\Longleftrightarrow}$





$X(\omega)$?

$1 \overset{CTFT}{\Longleftrightarrow}$



$\overset{DTFT}{\Longleftrightarrow}$

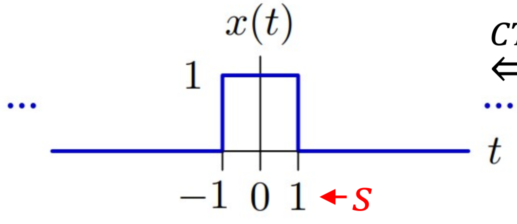
X(Ω)=?

$1 \overset{DTFT}{\Longleftrightarrow}$

Duality

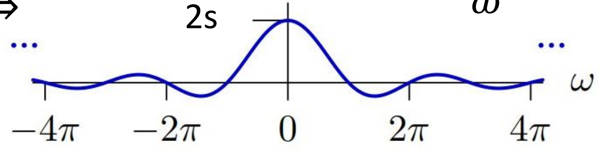
If $x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$ then $X(t) \overset{CTFT}{\Longleftrightarrow} 2\pi x(-\omega)$

Useful Signals and Their Fourier Transforms

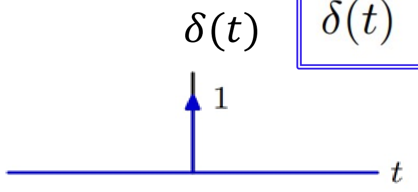


CTFT $\Longleftrightarrow$

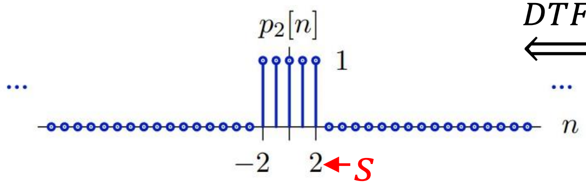
$$X(\omega) = 2\frac{\sin(\omega S)}{\omega}$$



$$\delta(t) \overset{\text{CTFT}}{\Longleftrightarrow} 1$$

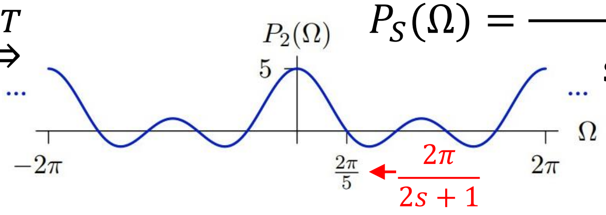


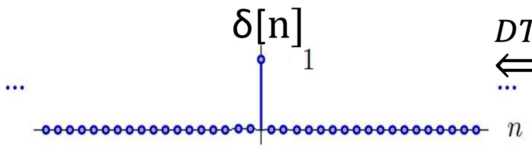
$$1 \overset{\text{CTFT}}{\Longleftrightarrow} 2\pi\delta(\omega)$$



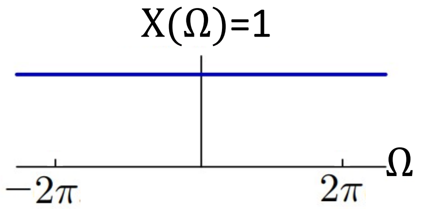
DTFT $\Longleftrightarrow$

$$P_S(\Omega) = \frac{\sin\left(\Omega\left(S+\frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}$$





DTFT $\Longleftrightarrow$

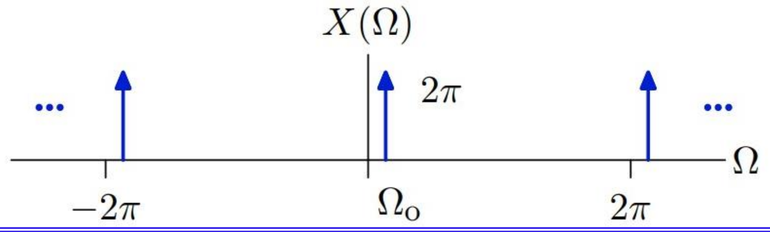


$$1 \overset{\text{DTFT}}{\Longleftrightarrow} \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi m)$$

$$x(t) = e^{j\omega_0 t} \overset{CTFT}{\Longleftrightarrow} X(\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x[n] = e^{j\Omega_0 n} \overset{DTFT}{\Longleftrightarrow} X(\Omega) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 + 2\pi m)$$

$e^{j\Omega_0 n} \overset{\text{DTFT}}{\Longleftrightarrow}$



Duality

If $x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$ then $X(t) \overset{CTFT}{\Longleftrightarrow} 2\pi x(-\omega)$

Exercise II

- A LTI system has its frequency response $H(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j2\Omega}}$, find the system's unit sample response.

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- A LTI system has its frequency response $H(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j2\Omega}}$, find the system's unit sample response.

If we use $h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(\Omega)e^{j\Omega n}\, d\Omega$, the integration will be complicated.

Since we know $H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$, if we consider $H(\Omega)$ as a result of the geometric series (with $0 < \alpha < 1$)

$H(\Omega) = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$, we can see $\alpha = \frac{1}{2}e^{-j2\Omega}$, therefore

$$H(\Omega) = \sum_{n=0}^{\infty} \alpha^n = \sum_{n=0}^{\infty} \frac{1}{2} e^{-j\Omega 2n}$$

$$h[n] = \begin{cases} (\frac{1}{2})^{n/2}, & n = 0,2,4,6,8,\dots,\infty \\ 0, & otherwise \end{cases}$$