6.300 Signal Processing

Week 8, Lecture A:
Discrete Fourier Transform (I)

- Relations to Discrete-Time Fourier Series (DTFS)
- Relations to Discrete-Time Fourier Transform (DTFT)
- Effect of windowing

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

Why do we need another Fourier Representation?

Fourier Series represent signals as sums of sinusoids. They provide insights that are not obvious from time representations, but FS are only limited to periodic signals.

Fourier Transform has no periodicity constraint:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

If we want to compute this using computers (not analytical solution by hand), two things stand in the way:

- infinite sum
- continuous function of frequency

Solutions:

- only consider a finite number of samples in time, and
- only consider a finite number of frequencies.

Toward DFT:

Start with the DTFT analysis equation:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

- only consider N samples
- take N uniformly spaced frequencies from the range $0 \leq \Omega < 2\pi$

The DFT:

$$X[k] = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n}$$

DFT: Definition

$$X[k] = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n}$$

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

DFT (**D**iscrete **F**ourier **T**ransform) is discrete in both domains.
Computationally feasible (opens doors to analyzing complicated signals).

Most modern signal processing is based on the DFT, and we'll use the DFT almost exclusively moving forward in 6.300.

The FFT (**F**ast **F**ourier **T**ransform) is an algorithm for computing the DFT efficiently.

DFT: Comparison to Other Fourier Representations

	Analysis	Synthesis
DFT	$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi k}{N}n}$	$x[n] = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}$
DTFS	$X[k] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\frac{2\pi k}{N}n}$	$x[n] = \sum_{k=<N>} X[k]e^{j\frac{2\pi k}{N}n}$
DTFT	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$

DTFS: x[·] is periodic in N
DTFT: x[·] is arbitrary
DFT: only a portion of an arbitrary x[·] is considered, N is only the analysis window

Relation Between DFT and DTFS

If a signal is periodic in the DFT analysis window N, then the DFT coefficients are equal to the DTFS coefficients.

Consider $x_1[n] = \cos\left(\frac{2\pi}{64}n\right)$, when analyzed with N = 64, the DFT coefficients are:

$$X_1[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n] \cdot e^{-j\frac{2\pi k}{N}n} \quad = \frac{1}{64}\sum_{n=0}^{63} \frac{1}{2}\left(e^{j\frac{2\pi}{64}n} + e^{-j\frac{2\pi}{64}n}\right) \cdot e^{-j\frac{2\pi k}{64}n}$$

$$= \frac{1}{2} \cdot \frac{1}{64}\sum_{n=0}^{63} e^{-j\frac{2\pi}{64}(k-1)n} \quad + \frac{1}{2} \cdot \frac{1}{64}\sum_{n=0}^{63} e^{-j\frac{2\pi}{64}(k+1)n} = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1]$$

DTFS

$$X[k] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\frac{2\pi k}{N}n}$$

The DFT coefficients are the same as the Fourier series coefficients.

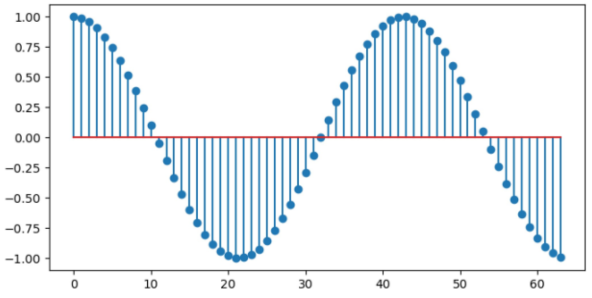
Relation Between DFT and DTFS

If a signal is not periodic in the DFT analysis window N, then there are no DTFS coefficients to compare.

Consider $x_2[n] = \cos\left(\frac{3\pi}{64}n\right)$, when analyzed with N = 64, the DFT coefficients are:

$$X_2[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n}$$

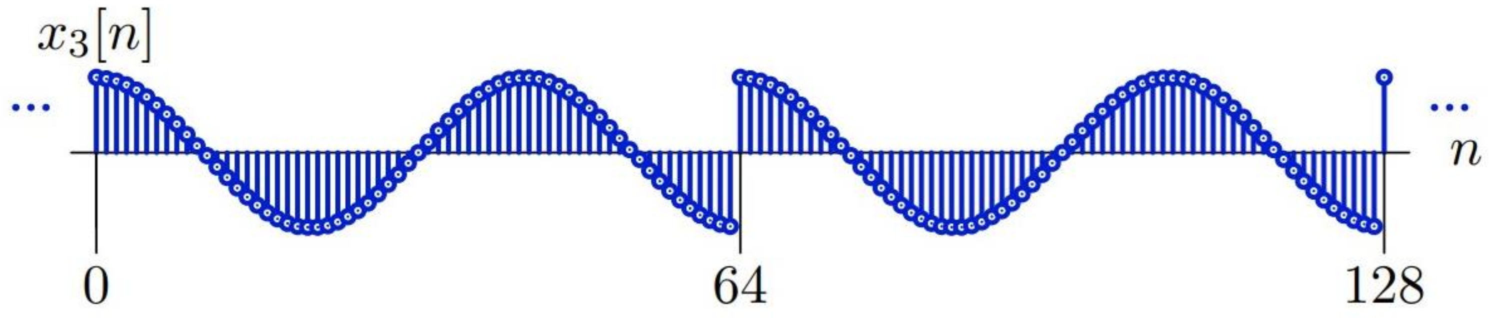
The reason is that $x_2[n]$ is not periodic with period N = 64.



Even though $x_2[n]$ contains a single frequency Ω = 3π/64, there are large coefficients at many different frequencies k.

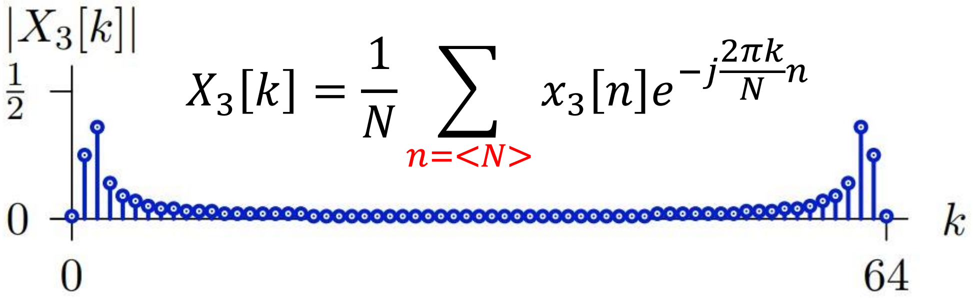
Relation Between DFT and DTFS

Although $x_2[n] = \cos\left(\frac{3\pi}{64}n\right)$ is not periodic in N=64, we can define a signal $x_3[n]$ that is equal to $x_2[n]$ for $0 \le n < 64$ and that is periodic with N=64.

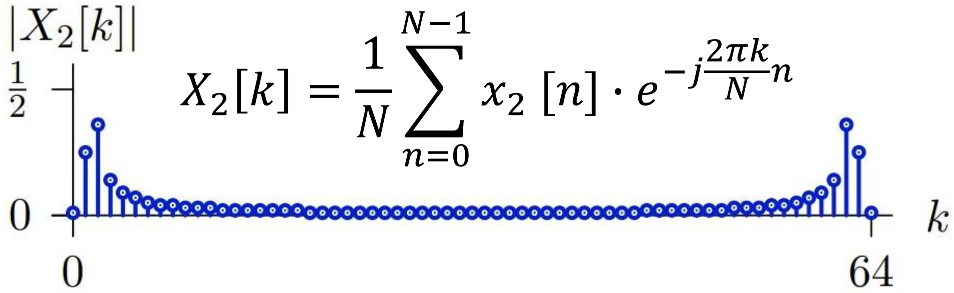


The DTFS coefficients of $x_3[n]$ equal the DFT coefficients of $x_2[n]$. The large number of non-zero coefficients are necessary to produce the step discontinuity at n = 64.

DTFS



$$X_3[k] = \frac{1}{N}\sum_{n=<N>} x_3[n]e^{-j\frac{2\pi k}{N}n}$$

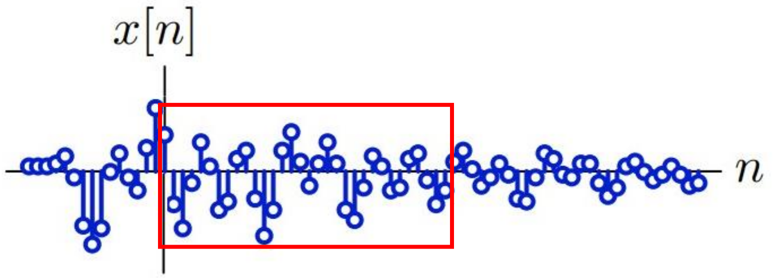


$$X_2[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_2\,[n]\cdot e^{-j\frac{2\pi k}{N}n}$$

DFT

Basically DFT of a signal with analysis window N is equivalent to take those N samples and generate periodic extensions and do DTFS.

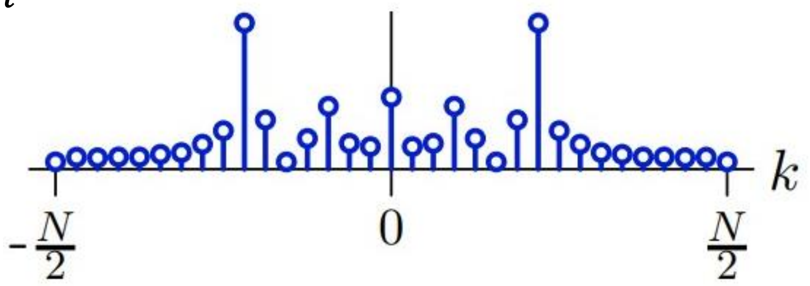
DFT: Relation to DTFT



$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\frac{2\pi k}{N}n}$$

DFT

$$X[k] = \frac{1}{N} X_w\left(\frac{2\pi k}{N}\right)$$

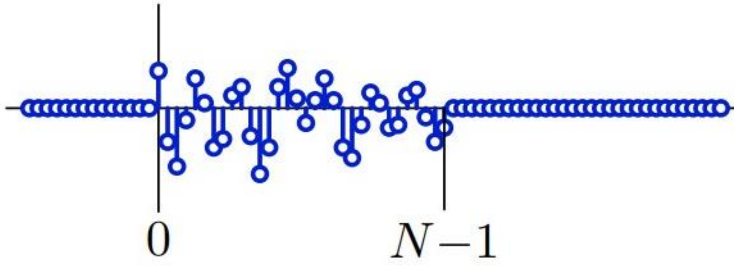


window: $w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$

sample: $\Omega \to \frac{2\pi k}{N}$

scale: $1/N$

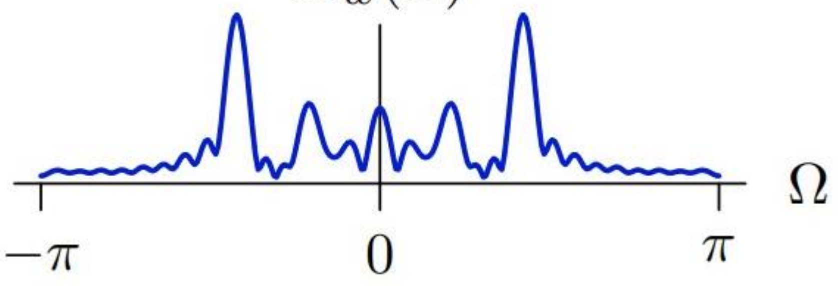
$x_w[n] = x[n]w[n]$



DTFT

$$X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\Omega n}$$

$X_w(\Omega)$



Note: here we only have ⟹ for DFT because:

$x[n] \overset{DFT}{\Longrightarrow} X[k] \overset{iDFT}{\Longrightarrow} x_p[n]$, with $x_p[n]$ being the periodically extended version of $x[n]$ with $0 \le n \le N-1$

Relation Between DFT and DTFT

The DFT can also be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $\frac{1}{N}$.

Let $x_w[n] = x[n] \times w[n]$ represent a **windowed** version of $x[n]$ where

$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Then the Fourier transform of $x_w[n]$ is

$$X_w(\Omega) = \sum_{n=-\infty}^{\infty} x_w[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$$

Scale both sides of this equation by $\frac{1}{N}$. Then **sample** the resulting function of Ω at $\Omega = \frac{2\pi k}{N}$ to obtain an expression for the DFT of $x[n]$:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} X_w\left(\frac{2\pi k}{N}\right)$$

Two Ways to Think About DFT

We can think about the DFT in two different ways:

1. The DFT is equal to the DTFS of the periodic extension of the first N samples of a signal.

2. The DFT is equal to (scaled) samples of the DTFT of a "windowed" version of the original signal.

These views are equivalent, but they highlight different phenomena.

Effect of Windowing on Fourier Representations

Example: characterize the effect of windowing on **complex exponential** signals, which are the basis functions for Fourier analysis.

Step 1: Find $X(\Omega)$, the DTFT of a complex exponential signal:
$x[n] = e^{j\Omega_0 n}$

Step 2: Find $X_w(\Omega)$, the DTFT of a windowed version of $x[n]$:
$x_w[n] = x[n]w[n]$

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

Check yourself

Step 1:

Let $x[n] = e^{j\Omega_o n}$

Participation question for Lecture

Find $X(\Omega)$, which is the DTFT of $x[n]$.

Check yourself

$x[n] = e^{j\Omega_0 n}$ Find $X(\Omega)$.

Compute the DTFT:

We need to find $X(\Omega)$ such that : $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega) \cdot e^{j\Omega n}\, d\Omega = e^{j\Omega_0 n}$

which is the same as: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n}\, d\Omega = e^{j\Omega_0 n}$

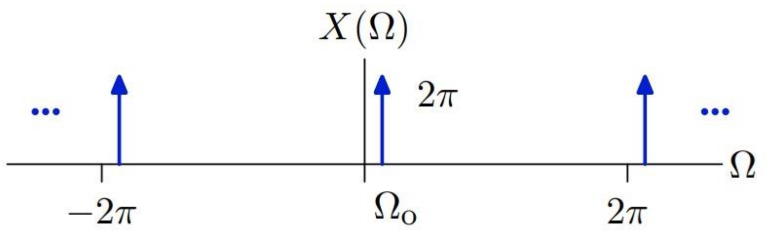
If we take a look at: $\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega - \Omega_0) \cdot e^{j\Omega n}\, d\Omega$

Based on the sifting properties of δ function, we will have: $\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega - \Omega_0) \cdot e^{j\Omega n}\, d\Omega = \frac{1}{2\pi} e^{j\Omega_0 n}$

which means $X(\Omega)$ should be$2\pi\delta(\Omega - \Omega_0)$, but since $X(\Omega)$ is also periodic with 2π, we have:

$$X(\Omega) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 + 2\pi m)$$

$e^{j\Omega_0 n} \overset{\text{DTFT}}{\Longleftrightarrow}$



Effect of Windowing on Fourier Representations

Example: characterize the effect of windowing on **complex exponential** signals, which are the basis functions for Fourier analysis.

Step 1: Find $X(\Omega)$, the DTFT of a complex exponential signal:
$x[n] = e^{j\Omega_0 n}$

Step 2: Find $X_w(\Omega)$, the DTFT of a windowed version of $x[n]$:
$x_w[n] = x[n]w[n]$

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

Effect of Windowing on Fourier Representations

Apply a rectangular window $w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & otherwise \end{cases}$

to the complex exponential signal $x[n] = e^{j\Omega_0 n}$

such that $x_w[n] = x[n] \cdot w[n] = e^{j\Omega_0 n} w[n]$

What the Fourier Transform $X_w(\Omega)$ look like?

$$X_w(\Omega) = \sum_{n=-\infty}^{\infty} x_w[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} w[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} w[n] e^{-j(\Omega - \Omega_0) n}$$

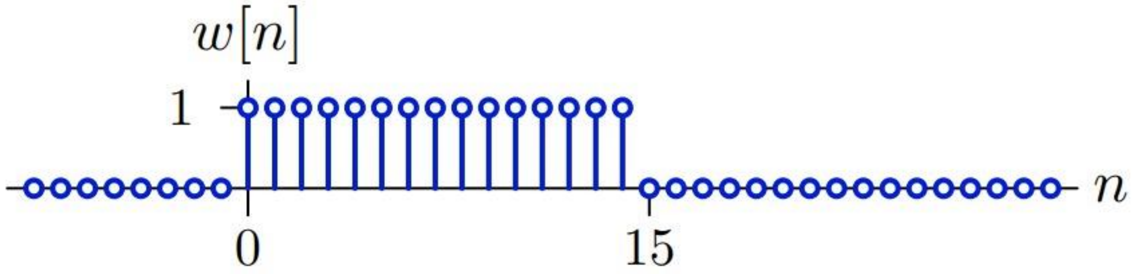
$$= W(\Omega - \Omega_0)$$

$$e^{j\Omega_0 n} w[n] \overset{DTFT}{\Longleftrightarrow} W(\Omega - \Omega_0)$$

→ Need to know $W(\Omega)$.

Effect of Windowing on Fourier Representations

Let $w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & otherwise \end{cases}$ As shown below for N=15



What is the DTFT of $w[n]$?

$$W(\Omega) = \sum_{n=-\infty}^{\infty} w[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n}$$

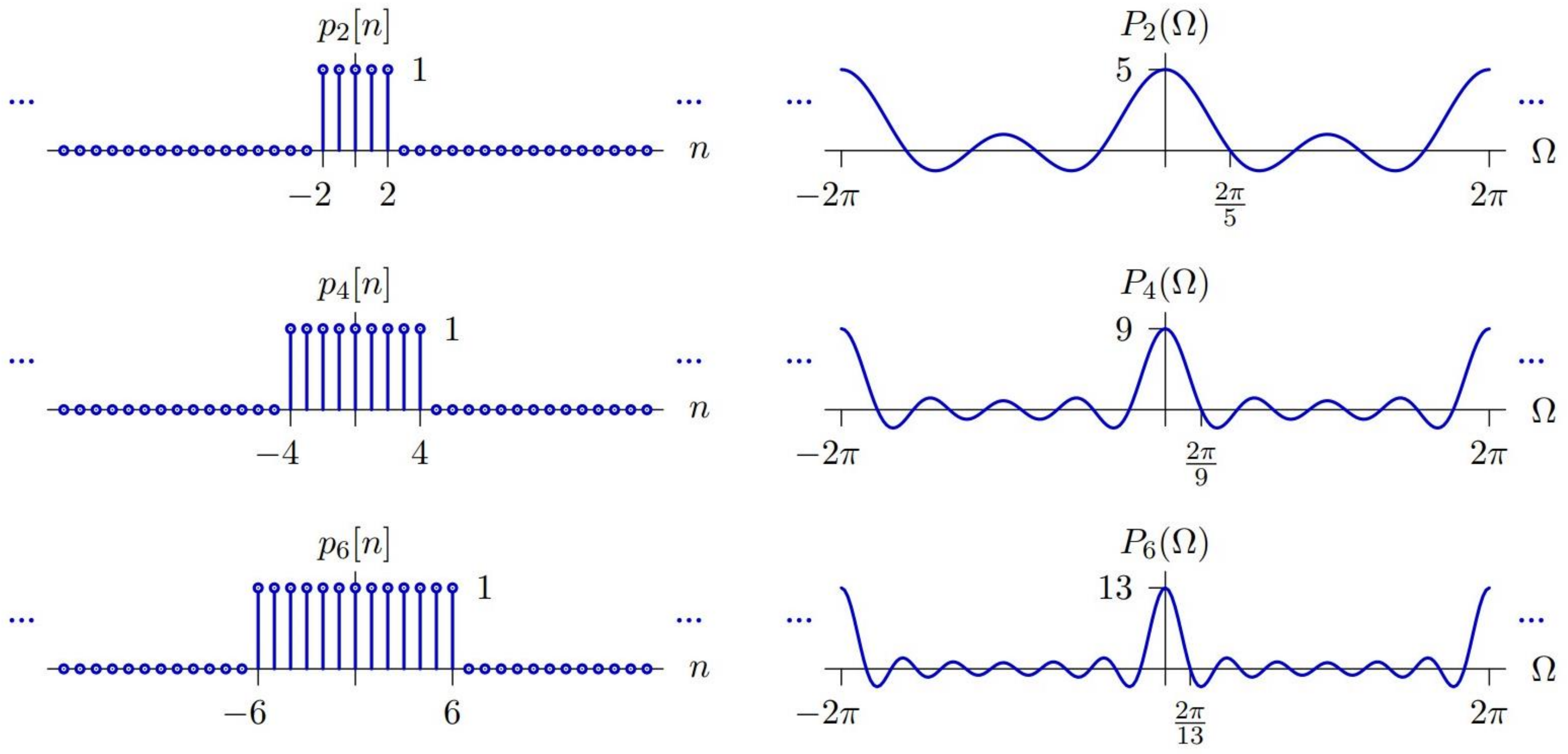
If Ω=0, $W(\Omega = 0) = N$

If $\Omega \neq 0$:

$$W(\Omega) = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = \frac{e^{-j\Omega\frac{N}{2}}(e^{j\Omega\frac{N}{2}} - e^{-j\Omega\frac{N}{2}})}{e^{-j\Omega\frac{1}{2}}(e^{j\Omega\frac{1}{2}} - e^{-j\Omega\frac{1}{2}})} = \frac{\sin(\Omega\frac{N}{2})}{\sin(\frac{\Omega}{2})} e^{-j\Omega\frac{N-1}{2}}$$

From Lecture 04B:

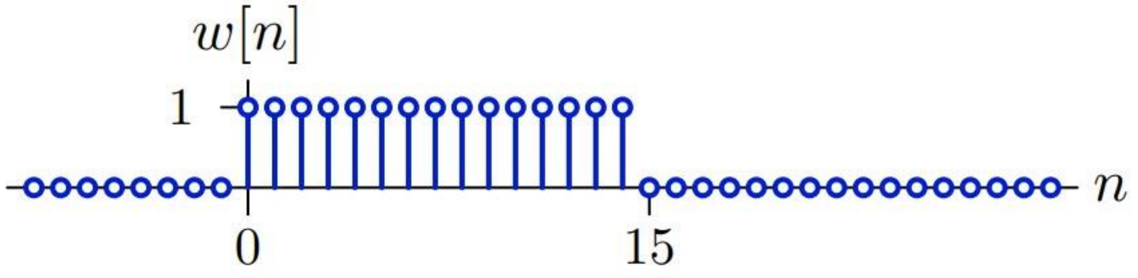
Slide #17



$$P_S(\Omega) = \frac{\sin(\Omega(S + \frac{1}{2}))}{\sin(\frac{\Omega}{2})}$$

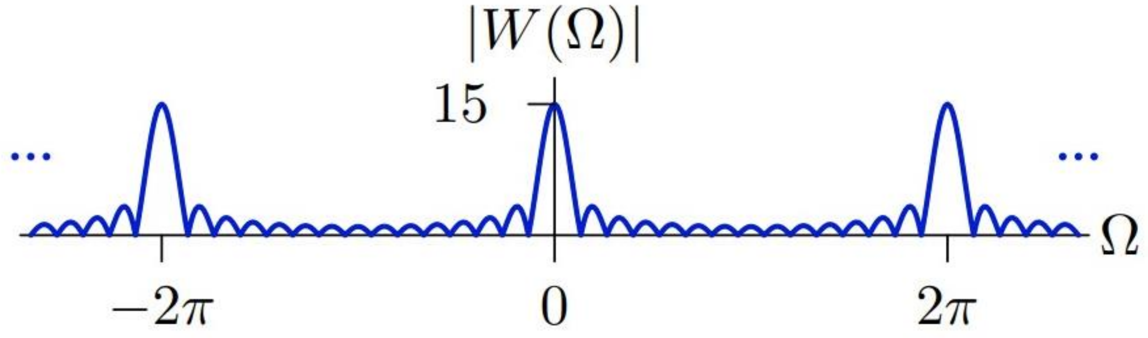
Effect of Windowing on Fourier Representations

Let $w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & otherwise \end{cases}$ As shown below for N=15



What is the DTFT of $w[n]$?

$$W(\Omega) = \sum_{n=-\infty}^{\infty} w[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n} = \begin{cases} N & \Omega = 0 \\ \dfrac{\sin(\Omega \frac{N}{2})}{\sin(\frac{\Omega}{2})} e^{-j\Omega\frac{N-1}{2}}, & \Omega \neq 0 \end{cases}$$

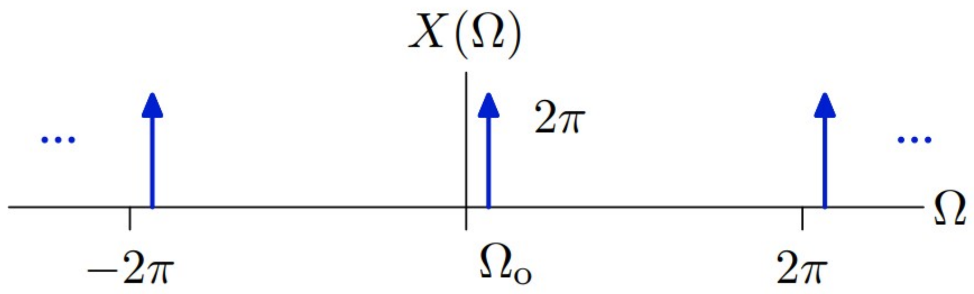


Effect of Windowing on Fourier Representations

The effect of windowing can be seen from this example of complex exponential signal:

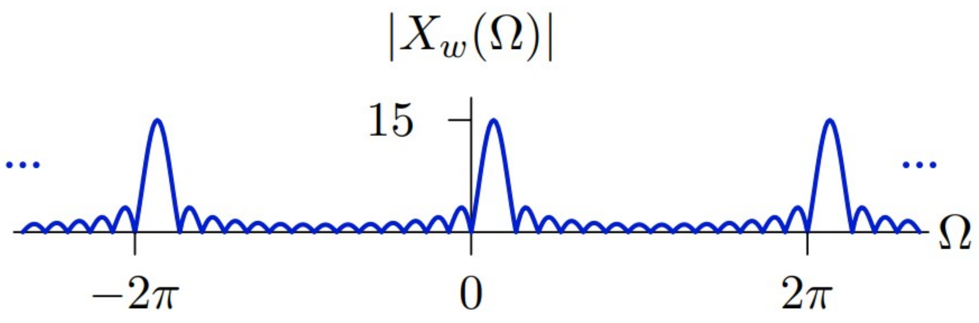
$$x_w[n] = e^{j\Omega_0 n} w[n] \overset{DTFT}{\Longleftrightarrow} X_w(\Omega) = W(\Omega - \Omega_0)$$

$$x[n] = e^{j\Omega_o n} \overset{\text{DTFT}}{\Longleftrightarrow}$$



The frequency content of X(Ω) is at discrete frequencies $\Omega = \Omega_o + 2\pi m$

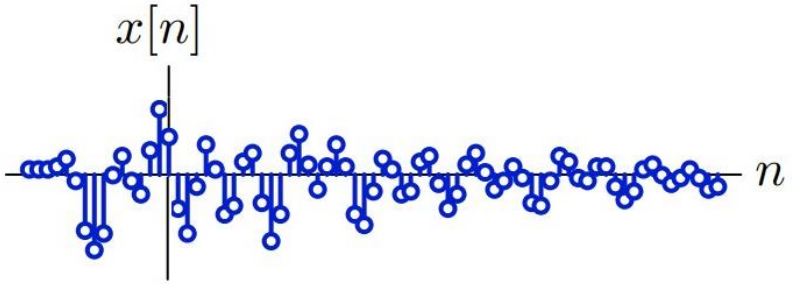
$$x_w[n] = x[n]w[n] \overset{\text{DTFT}}{\Longleftrightarrow}$$



The frequency content of $X_w(\Omega)$ is most dense at these same frequencies, but is spread out over almost all other frequencies as well.

Effect of windowing: spectrum smear

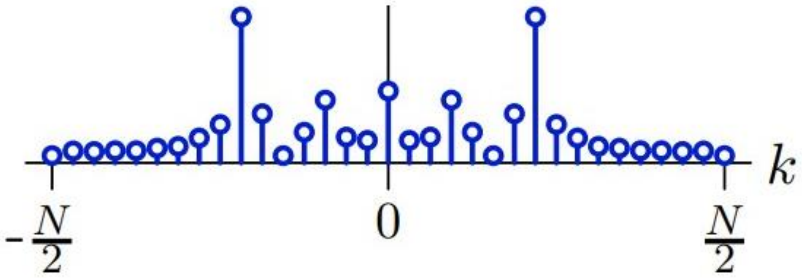
DFT: Relation to DTFT



$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-j\frac{2\pi k}{N}n}$$

DFT

$$X[k] = \frac{1}{N} X_w\left(\frac{2\pi k}{N}\right)$$

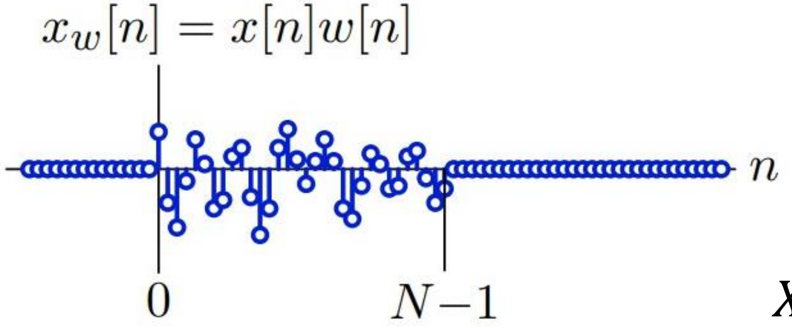


window

$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & otherwise \end{cases}$$

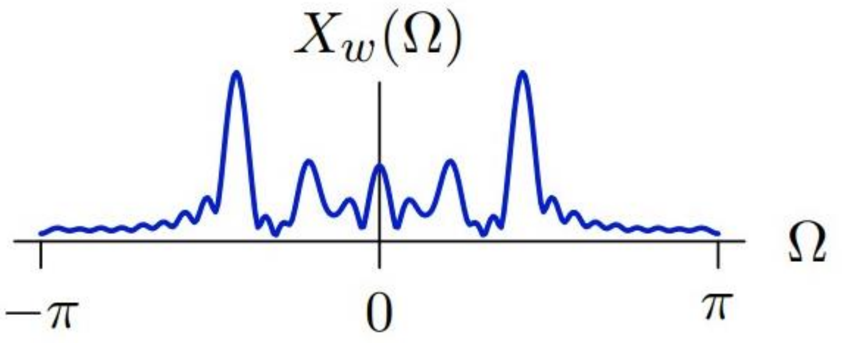
sample: $\Omega \rightarrow \frac{2\pi k}{N}$

scale: $1/N$



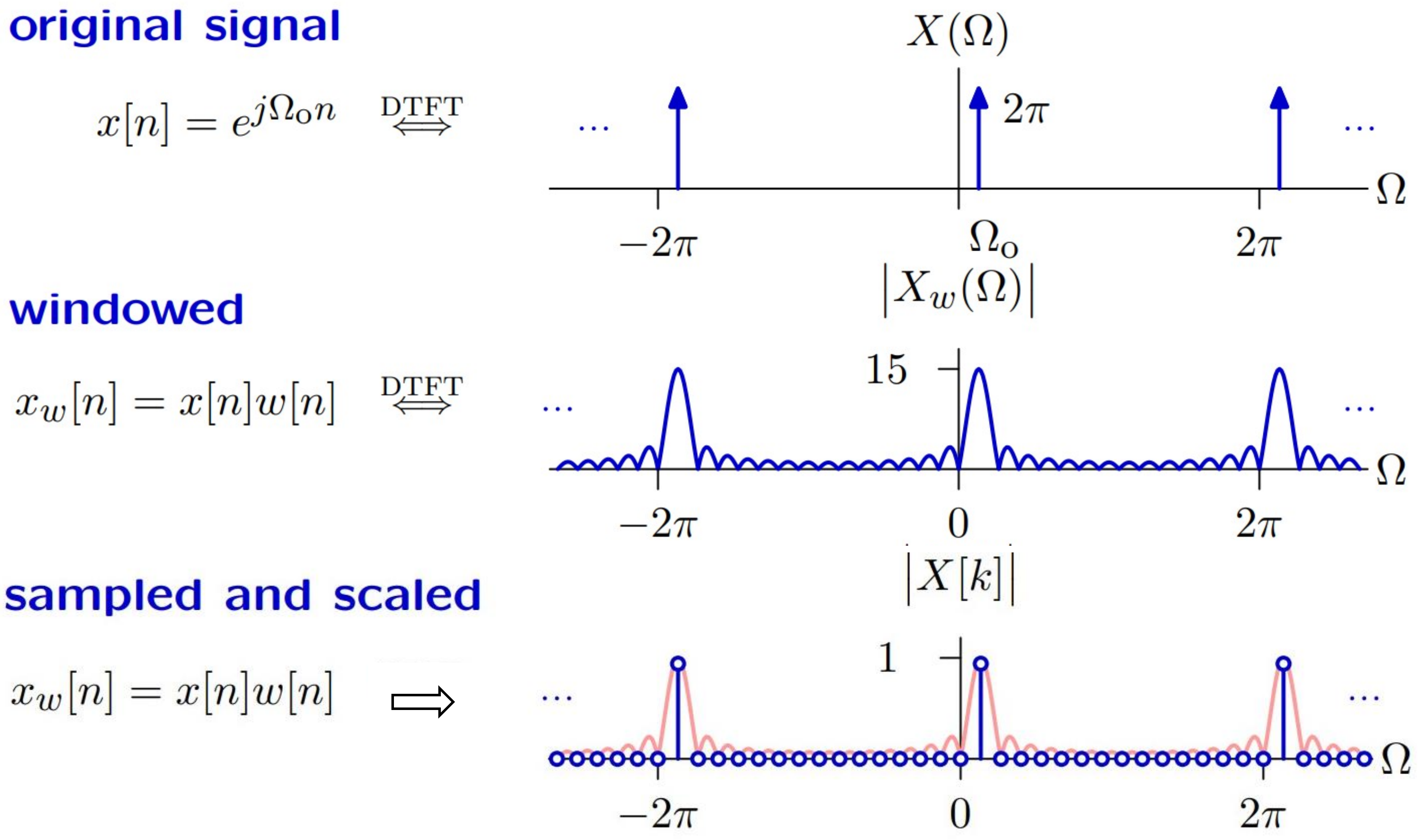
DTFT

$$X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n]\cdot e^{-j\Omega n}$$



Effect of Windowing on Fourier Representations

Considering the example of $x[n] = e^{j\Omega_0 n}$ and $x_w[n] = e^{j\Omega_0 n} w[n]$ with $\Omega_0 = \frac{2\pi}{15}$:



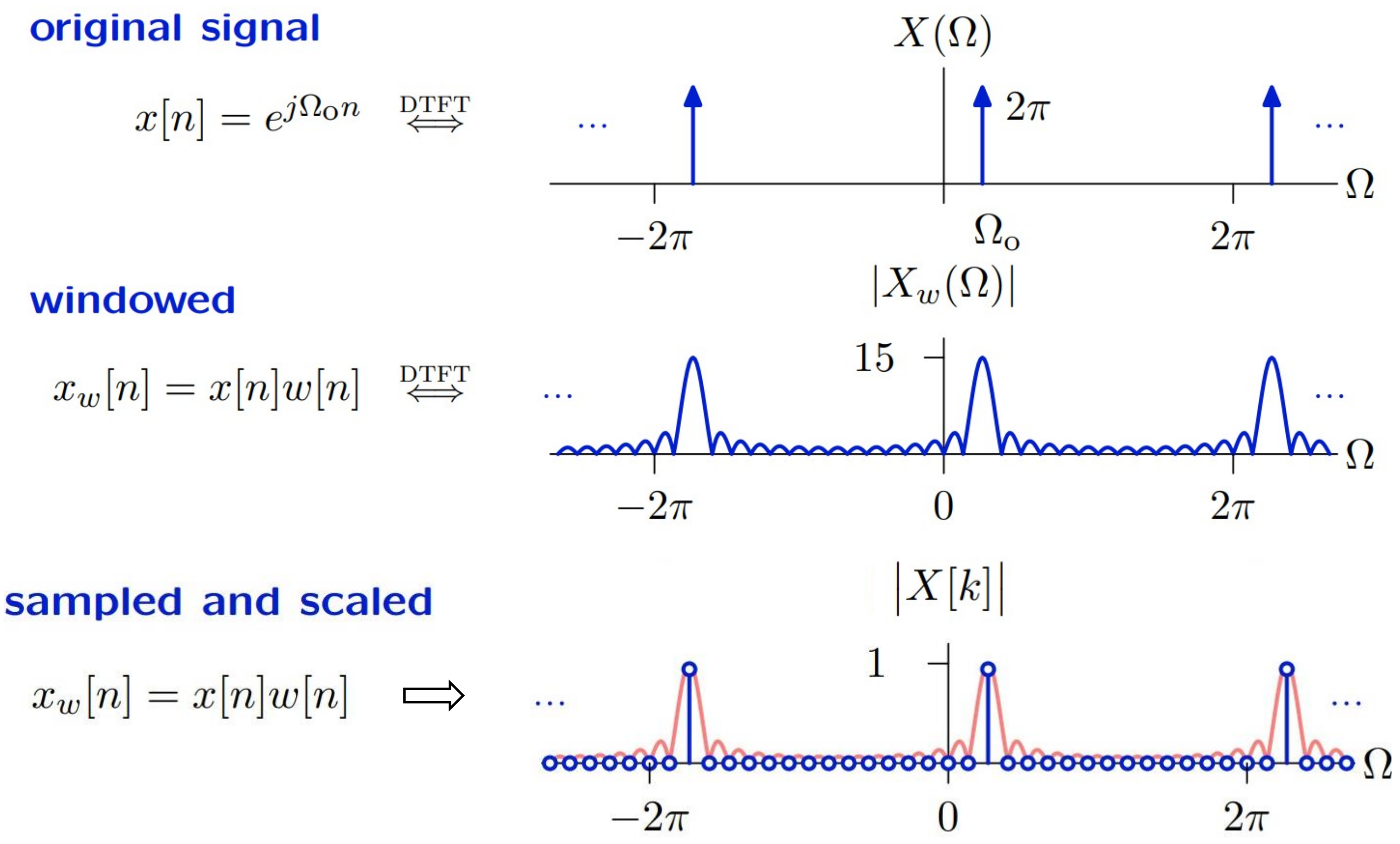
$$\Omega = \frac{2\pi k}{N}$$

One sample is taken at the peak, and the others fall on zeros.

Because the signal is periodic within the analysis window N.

Effect of Windowing on Fourier Representations

Considering the example of $x[n] = e^{j\Omega_0 n}$ and $x_w[n] = e^{j\Omega_0 n} w[n]$ with $\Omega_0 = \frac{4\pi}{15}$:



$$\Omega = \frac{2\pi k}{N}$$

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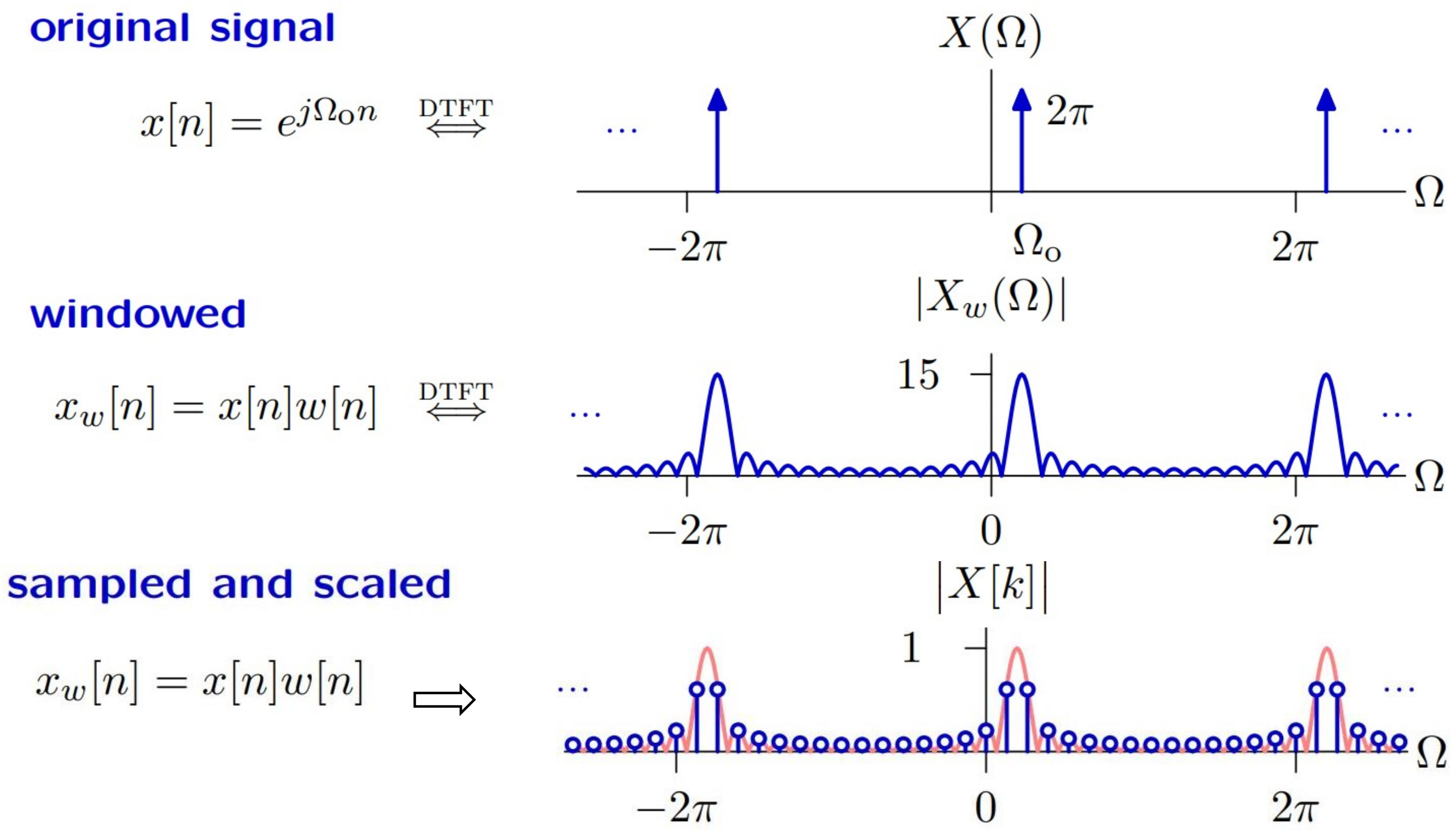
Effect of Windowing on Fourier Representations

Considering the example of $x[n] = e^{j\Omega_0 n}$ and $x_w[n] = e^{j\Omega_0 n} w[n]$ with $\Omega_0 = \frac{3\pi}{15}$:

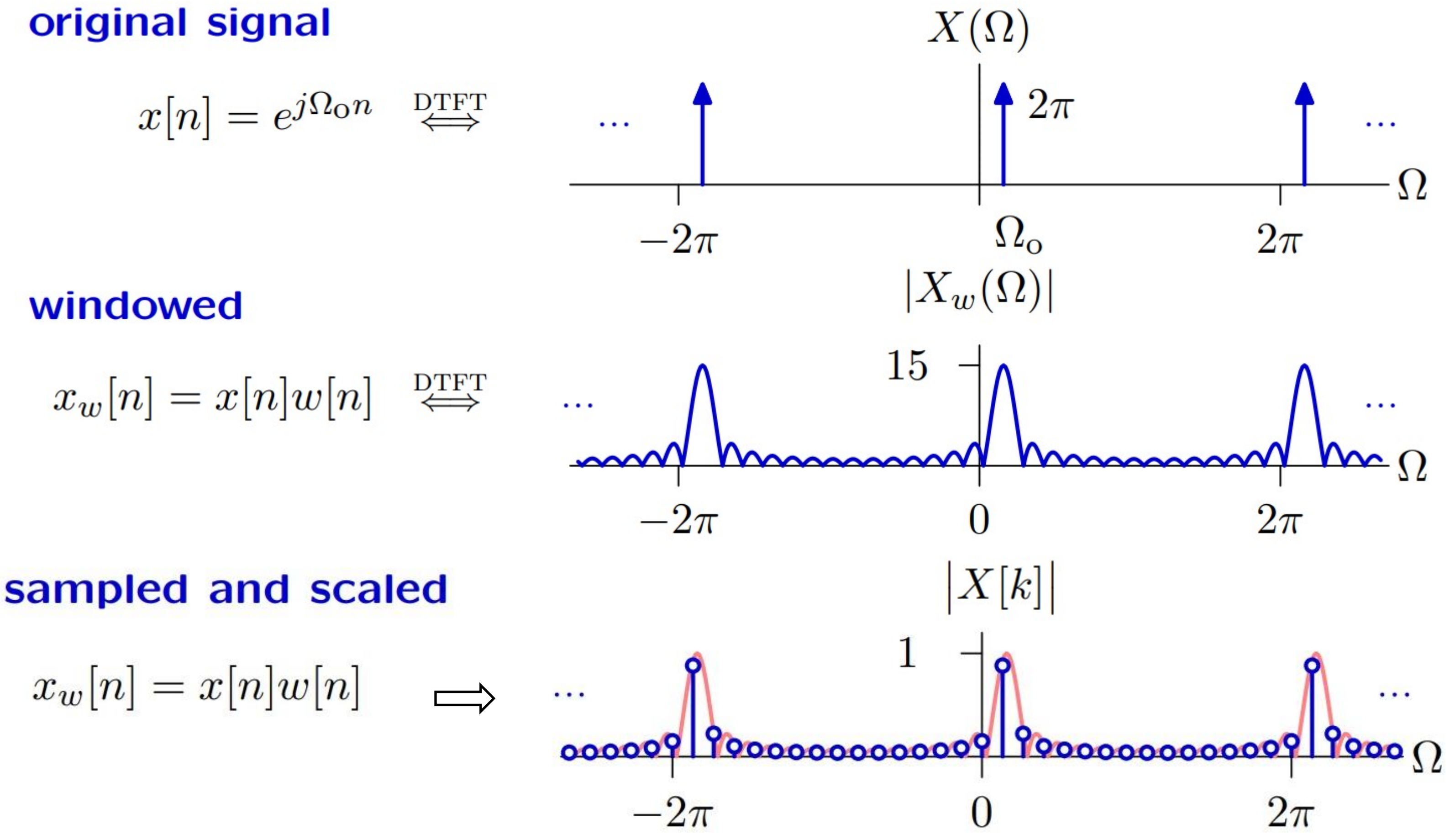


$$\Omega = \frac{2\pi k}{N}$$

Now none of the samples fall on zeros.

Effect of Windowing on Fourier Representations

Considering the example of $x[n] = e^{j\Omega_0 n}$ and $x_w[n] = e^{j\Omega_0 n} w[n]$ with $\Omega_0 = \frac{2.4\pi}{15}$:



$$\Omega = \frac{2\pi k}{N}$$

Generally, the relation between the samples is complicated.

Summary

Today we introduced a new Fourier representation for DT signals:

the Discrete Fourier Transform (DFT).

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n}$$

Synthesis equation

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n}$$

Analysis equation

The DFT has a number of features that make it particular convenient

- It is not limited to periodic signals.
- It has discrete frequency (k instead of Ω) and finite length: convenient for numerical computation.

Applying an analysis window introduces spectra smearing.