

# 6.300 Signal Processing

Week 7, Lecture B:

## System Abstraction (III): Frequency Response

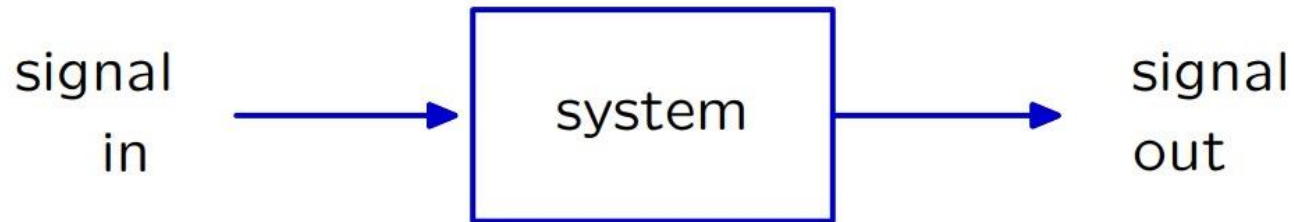
- Discrete-Time Frequency Response
- Continuous-Time Frequency Response

Lecture slides are available on CATSOOP:

<https://sigproc.mit.edu/fall24>

# The System Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



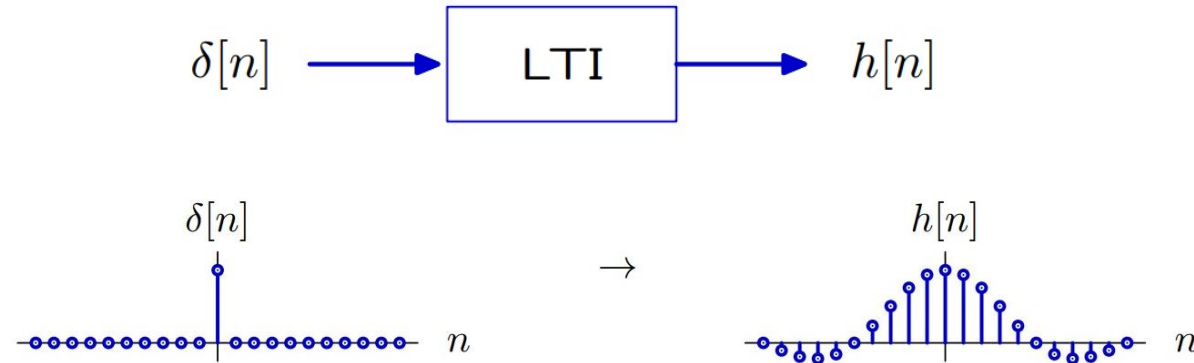
This abstraction is particularly powerful for **linear and time-invariant systems**, which are both **prevalent** and **mathematically tractable**.

We previously studied representations based on **difference/differential equations** and on **convolution**:

- **Difference/Differential Equation**: represent system by **algebraic constraints** on samples
- **Convolution**: represent a system by its **unit-sample/impulse response**
- **Filter**: represent a system as by its **frequency response** ← Today

# Represent a System by Its Unit-Sample-Response

Unit-sample response  $h[n]$  is a complete description of an LTI system.



The unit-sample signal is the **shortest** possible non-trivial DT signal.

It is the building block of any arbitrary DT signal  $x[n]$  :

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

The response to  $\delta[n]$  can be used to determine the response to any arbitrary input  $x[n]$ .

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

# Frequency Response

The frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to sinusoids.



The idea is to characterize a system by its response to signals of individual frequencies.

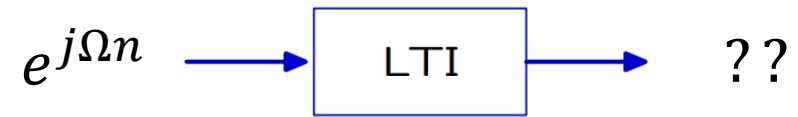
Since any arbitrary input can be represented by its frequency components, we can find the response to this arbitrary input from the system's frequency response.

Sinusoids differ from the unit-sample signal in important ways:

- eternal (longest possible signals) versus transient (shortest possible)
- comprises a single frequency versus a sum of all possible frequencies

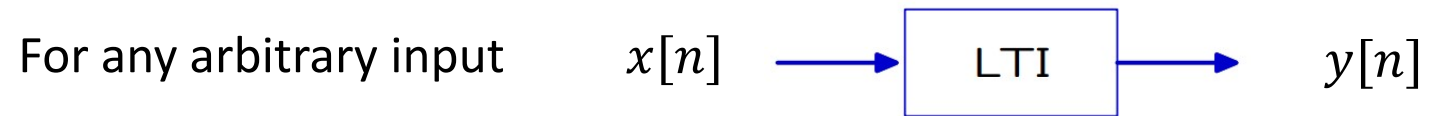
# Frequency Response

Using complex exponentials to characterize the frequency response.



# Frequency Response

The frequency response is a **complete** characterization of an LTI system.



# Frequency Representation of Convolution

For a LTI system,  $y[n] = (x * h)[n]$ . Find  $Y(\Omega)$ .

# Frequency Response

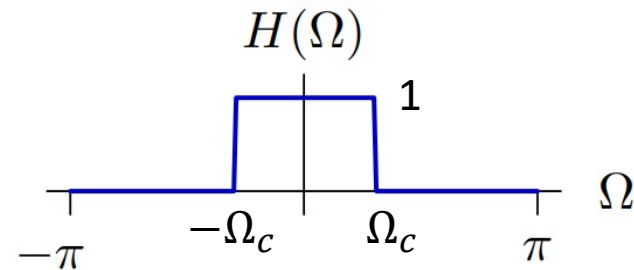
The frequency response can be the most insightful description of a system.

$$Y(\Omega) = H(\Omega)X(\Omega)$$

Each frequency component  $X(\Omega)$  is scaled by a factor  $H(\Omega)$ , which can be possibly complex. Multiplication of Fourier transforms can be regarded as **filtering**.

Example:

A low-pass filter passes frequencies near 0 and rejects those near  $\pi$ .



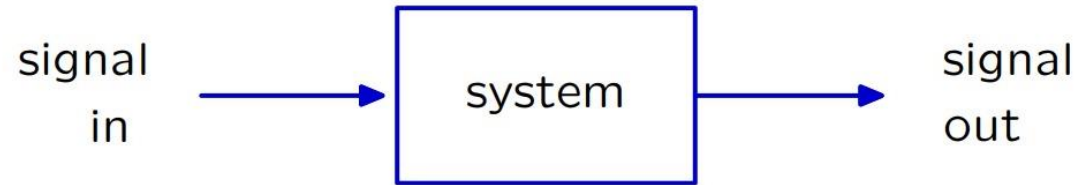
Very natural way to describe audio enhancements:

- bass-boost
- room equalizer
- tone control



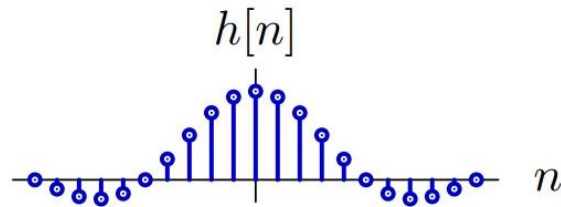
# System Abstraction

Three **complete** representations for linear, time-invariant systems.

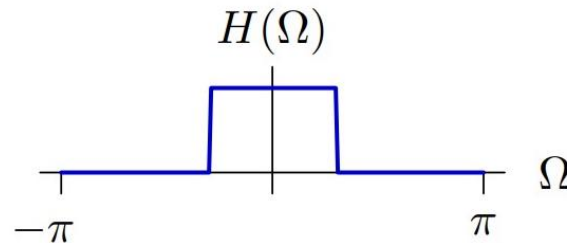


**Difference Equations:** relating output samples with input samples.

**Unit-Sample Response:** responses across time for a unit-sample input.



**Frequency Response:** responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **unit-sample response**!

# Example

Find the frequency response of a (causal) system described by the following difference equation:

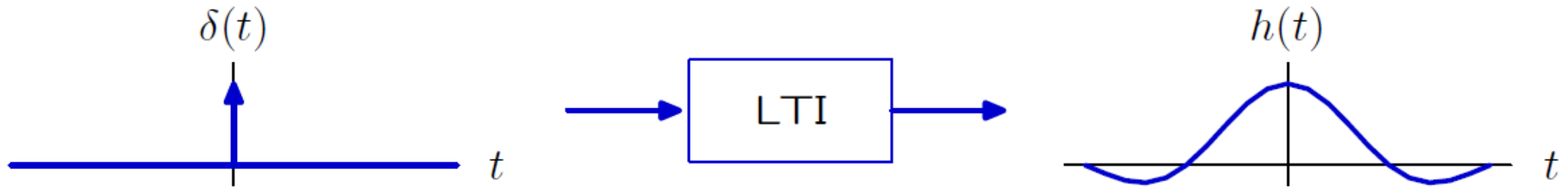
$$y[n] - \alpha y[n - 1] = x[n], \quad 0 < \alpha < 1$$

**What is your strategy? Mention at least one. Participation question for Lecture**

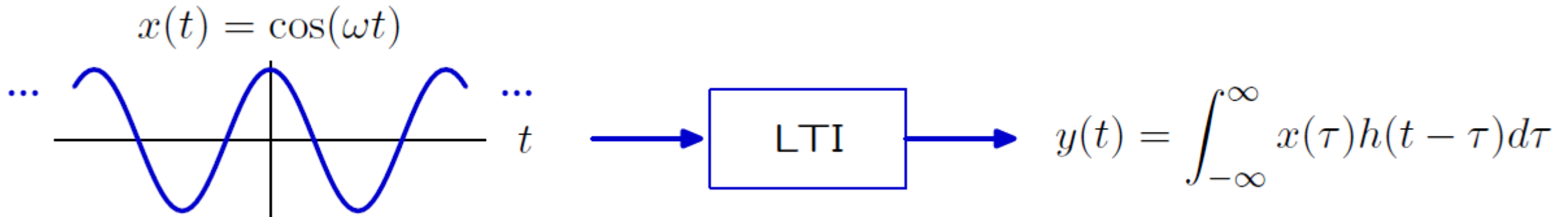
# Frequency Response of a Continuous-Time System

Use convolution to characterize the frequency response of a system.

The response of a CT LTI system to the Dirac delta function  $\delta(t)$  is the impulse response  $h(t)$ .



The response  $y(t)$  to a sinusoid  $x(t) = \cos(\omega t)$  is  $y(t) = (x * h)(t)$ .



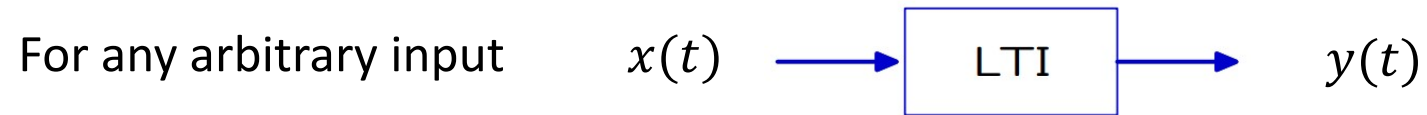
# Frequency Response

Using complex exponentials to characterize the frequency response.



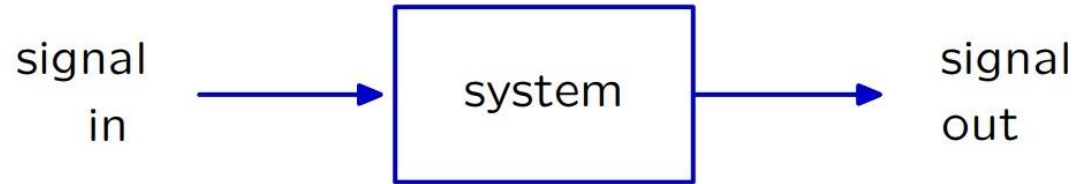
# Frequency Response

The frequency response is a **complete** characterization of an LTI system.



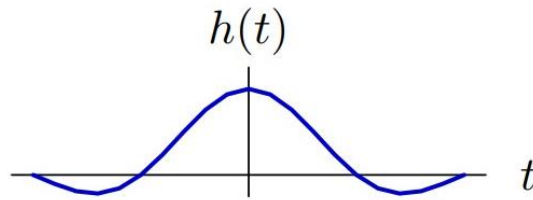
# CT System Abstraction

Three **complete** representations for linear, time-invariant systems.

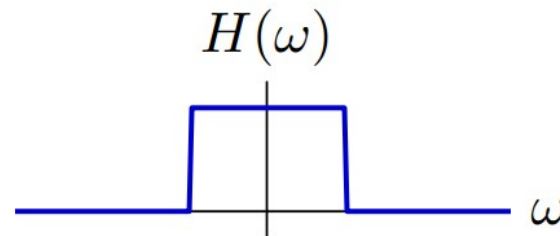


**Differential Equations:** relating output derivatives with input derivatives.

**Impulse Response:** responses across time for an impulse input.



**Frequency Response:** responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **impulse response**!

# Example

Find the frequency response of a system described by the following differential equation:

$$y(t) + \alpha \frac{dy(t)}{dt} = 2x(t), \alpha > 0$$

# Check yourself

Find the frequency response of a rectangular box averager:

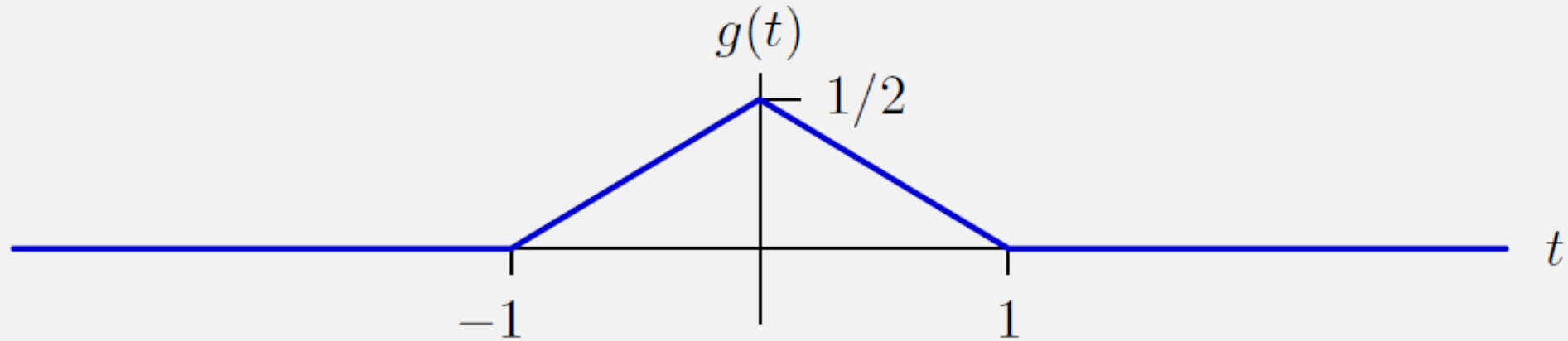
$$y(t) = \frac{1}{2} \int_{t-1}^{t+1} x(\tau) d\tau$$

(This CT averager is analogous to the three-point averager in DT.)



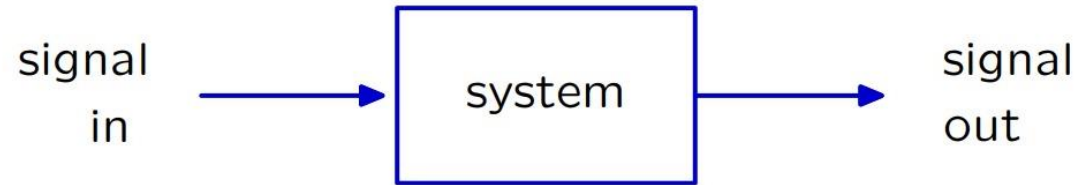
# Check yourself

Find the frequency response of a triangular averager:



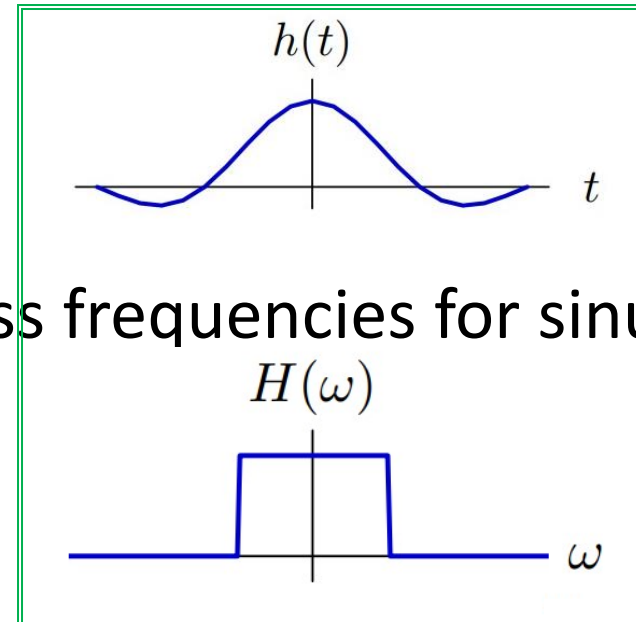
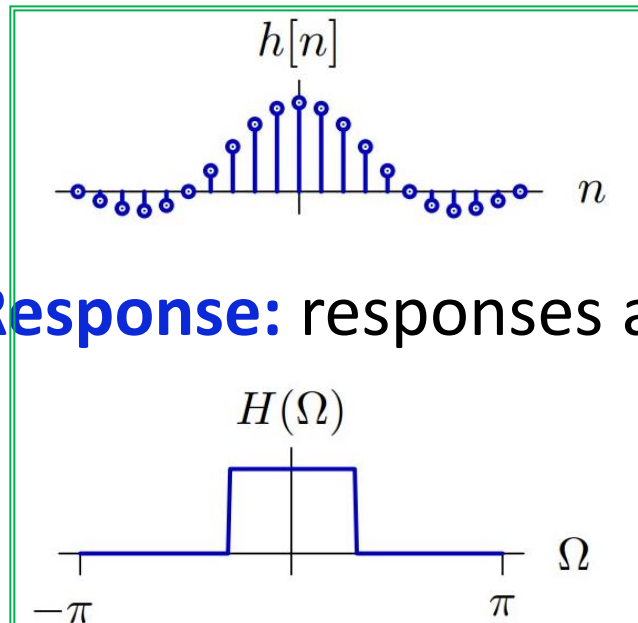
# Summary

Three **complete** representations for linear, time-invariant systems.



**Difference/Differential Equations:** relating output with input.

**Unit-Sample/Impulse Response:** responses across time for an impulse input.

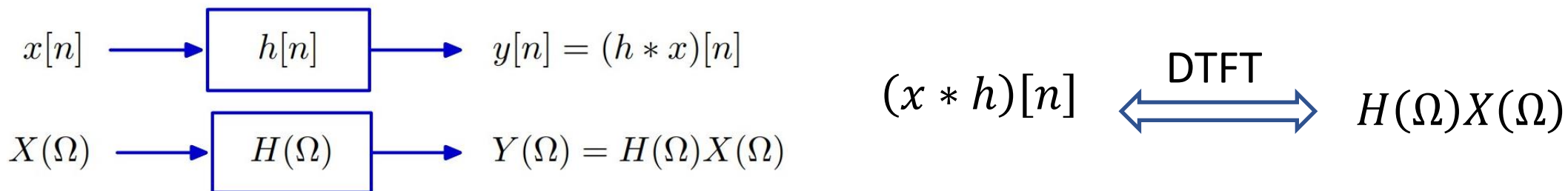


**Frequency Response:** responses across frequencies for sinusoidal inputs.

The **frequency response** is Fourier transform of **unit-sample/impulse response**!

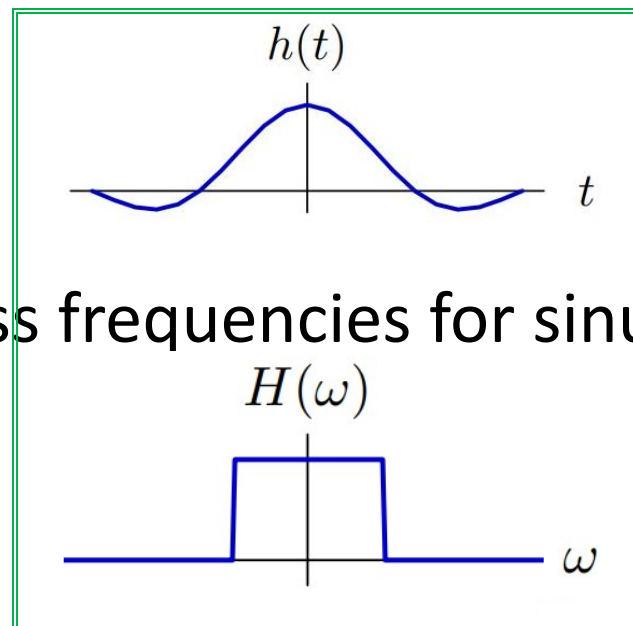
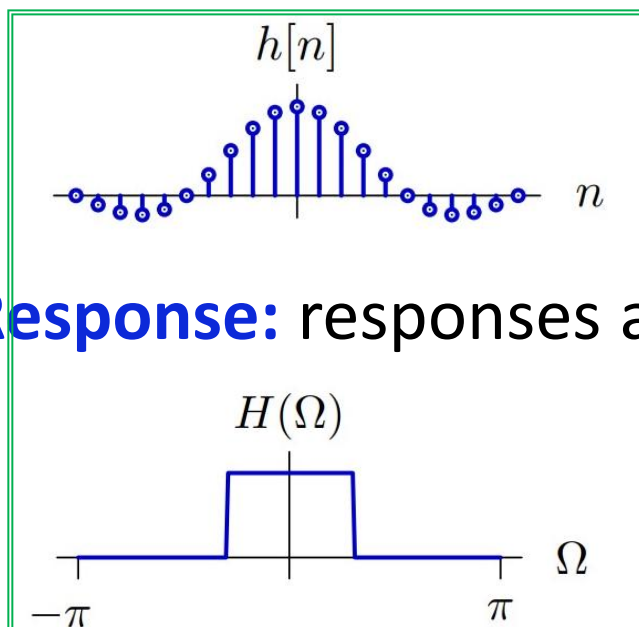
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