

6.300 Signal Processing

Week 6, Lecture B:

System Abstraction (II): Impulse Response and Convolution

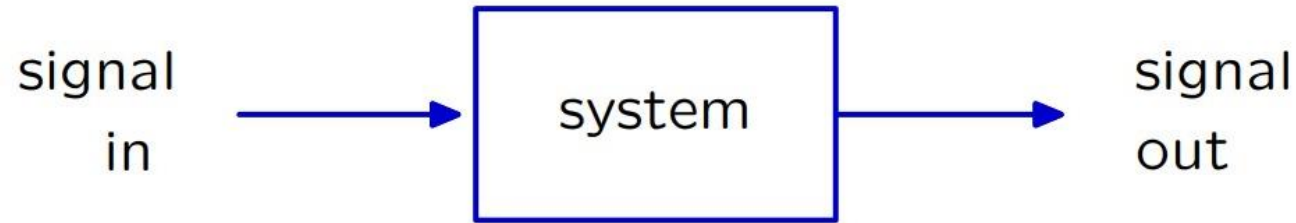
- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

Lecture slides are available on CATSOOP:

<https://sigproc.mit.edu/fall24>

Last Time: The System Abstraction

Represent a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.

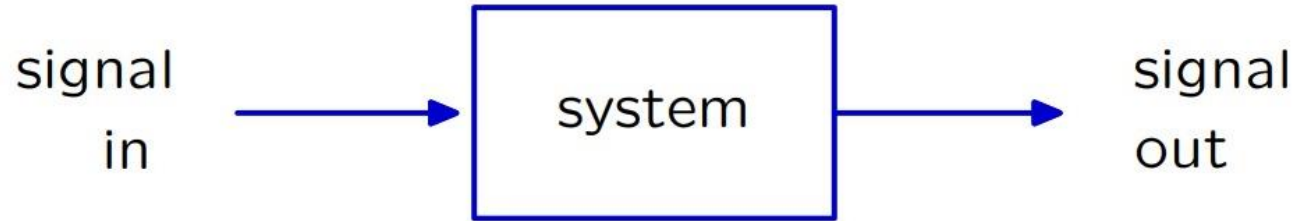


In this class, we will focus primarily on LTI (Linear, Time-Invariant) systems:

- **Linearity** (additivity and homogeneity)
- **Time invariance**

Such systems are both prevalent and mathematically tractable.

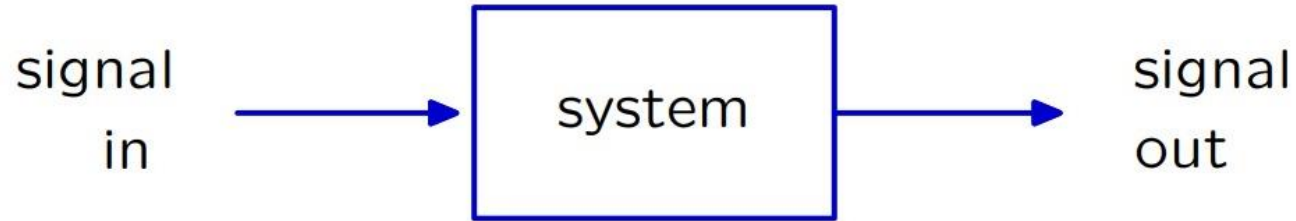
Multiple Representation of Systems



We can represent a system in the following three ways:

- **Difference (Differential) Equation**: represent system by algebraic constraints on samples
- **Convolution**: represent a system by its unit-sample response
- **Filter**: represent a system by its amplification or attenuation of frequency components

Today: Representing a System by its Unit-Sample Response



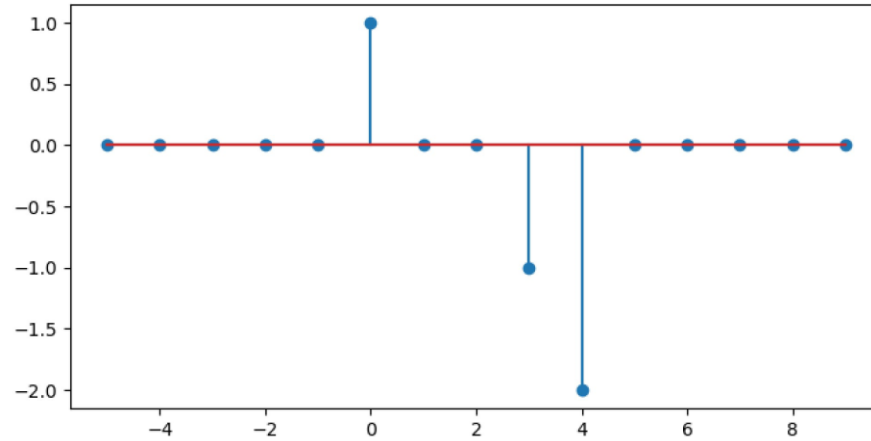
We can represent a system in the following three ways:

- **Difference (Differential) Equation:** represent system by algebraic constraints on samples
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its amplification or attenuation of frequency components

Representing a DT Signal as Sums of Delta's

Consider the following signal:

$$x[n] = \begin{cases} 1, & \text{if } n = 0 \\ -1, & \text{if } n = 3 \\ -2, & \text{if } n = 4 \\ 0, & \text{otherwise} \end{cases}$$



This signal can be represented as:

$$x[n] = \delta[n] - \delta[n - 3] - 2\delta[n - 4]$$

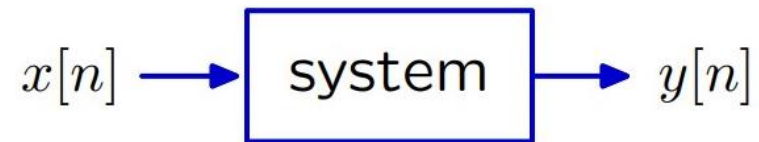
In general, we can represent a DT signal as a sum of scaled, shifted delta's:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m]$$

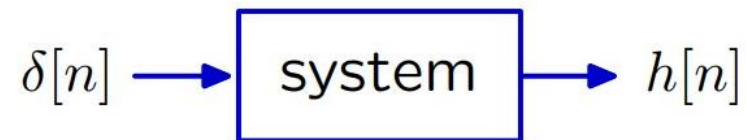
$$= \cdots + x[-1]\delta[n + 1] + x[0]\delta[n] + x[1]\delta[n - 1] + x[2]\delta[n - 2] + \cdots$$

Unit Sample Response

If a system is linear and time-invariant, its input-output relation is completely specified by the system's **unit sample response** $h[n]$.



The unit sample response $h[n]$ is the output of the system when the input is the unit sample signal $\delta[n]$.



The output for more complicated inputs can be computed by superposition of the unit sample response.

Superposition

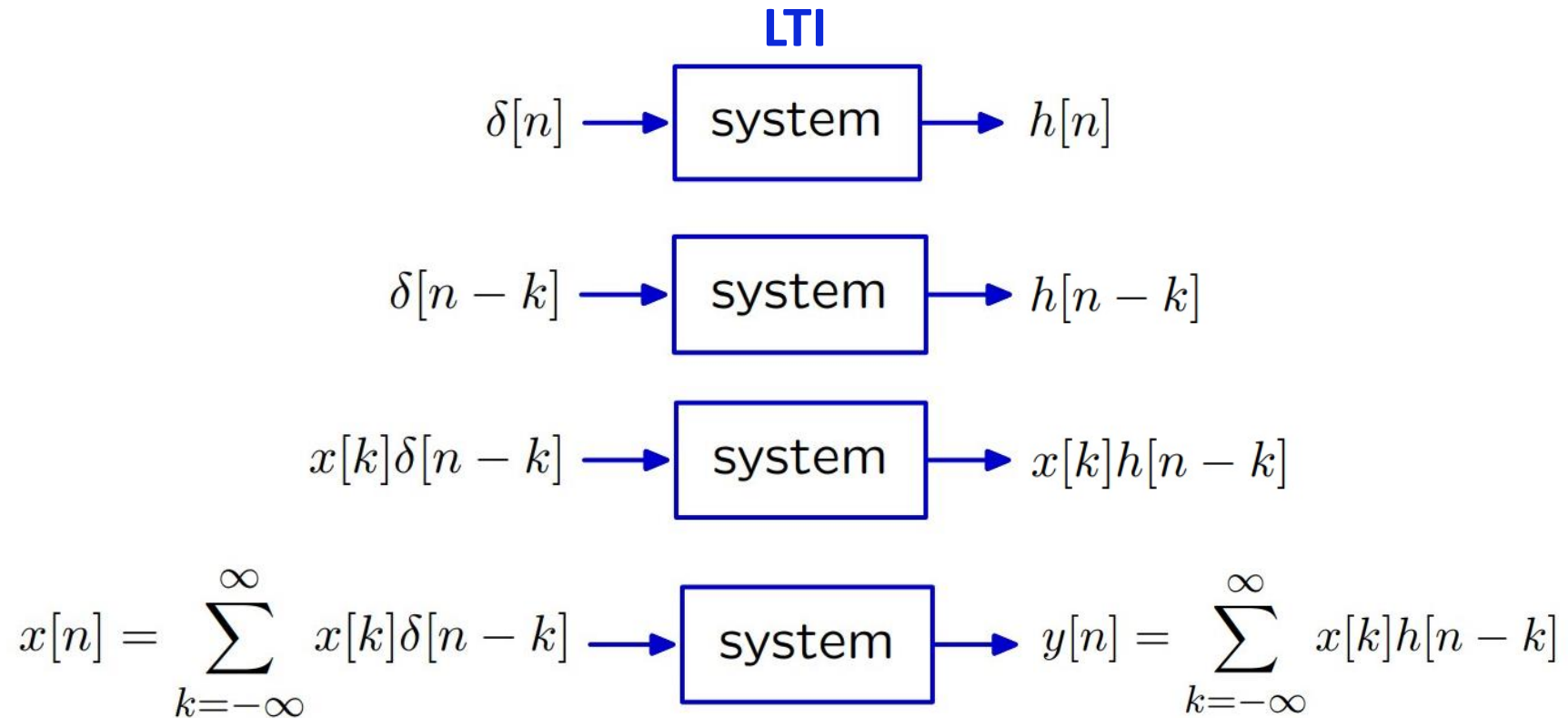
In general, we can represent a signal as a sum of scaled, shifted delta's:

$$\begin{aligned}x[n] &= \sum_{m=-\infty}^{\infty} x[m] \delta[n - m] \\ &= \dots + x[-1]\delta[n + 1] + x[0]\delta[n] + x[1]\delta[n - 1] + x[2]\delta[n - 2] + \dots\end{aligned}$$

If $h[\cdot]$ is the unit sample response of an LTI system, then the output of that system in response to this arbitrary input $x[\cdot]$ can be viewed as a sum of scaled, shifted unit sample responses:

$$\begin{aligned}y[n] &= \sum_{m=-\infty}^{\infty} x[m] h[n - m] \\ &= \dots + x[-1]h[n + 1] + x[0]h[n] + x[1]h[n - 1] + x[2]h[n - 2] + \dots\end{aligned}$$

Structure of Superposition



If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit sample responses.

Convolution

Response of an LTI system to an arbitrary input:



and the system's unit sample response is $h[\cdot]$,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] \equiv (x * h)[n]$$

This operation is called **convolution** (verb form: **convolve**).

Definition:

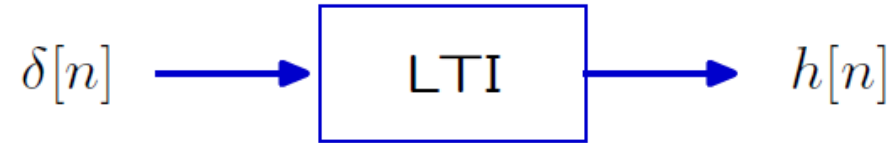
$$(x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m] h[n - m]$$

Note: It is customary (but confusing) to abbreviate this notation: $(x * h)[n] = x[n] * h[n]$
we avoid using that in 6.300.

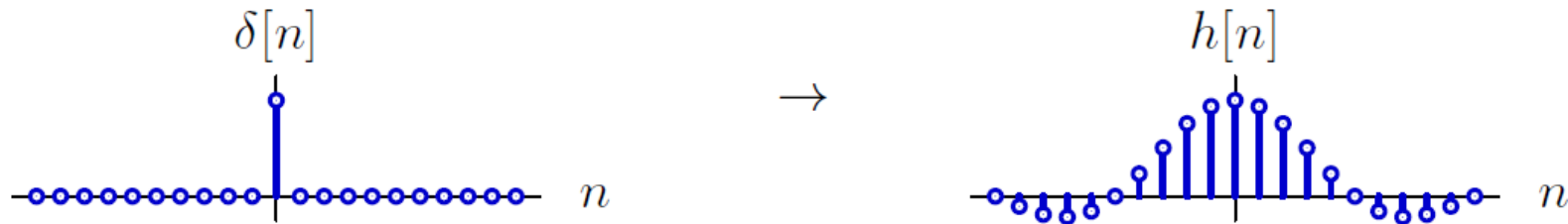
Convolution operates on signals, not samples. The symbols x and h represent DT signals. Convolution of x with h generates a new DT signal $x * h$.

Unit Sample Response

The unit-sample response is a **complete** description of an LTI system.



It can be used to determine the response to **any** other input.



Given $h[\cdot]$ one can compute the response $y[\cdot]$ to any input $x[\cdot]$:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

Ways to Compute Convolution Result

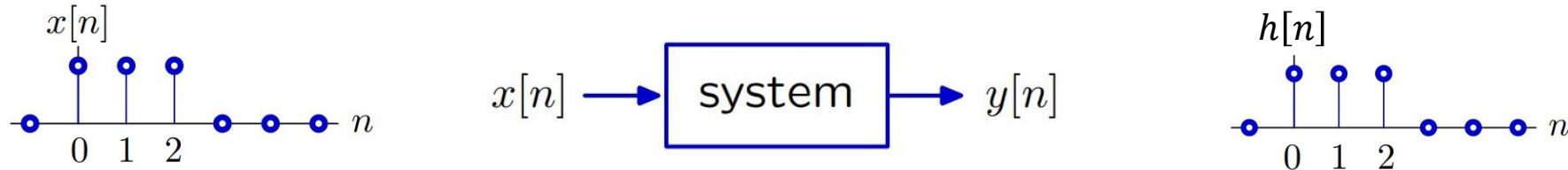
Knowing the unit sample response $h[\cdot]$ to a LTI system, there are different ways to compute the output $y[\cdot]$ to an arbitrary input $x[\cdot]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

1. Directly compute the superposition result (sum the scaled and shifted responses)
2. Compute the convolution result at each particular n : flip and shift

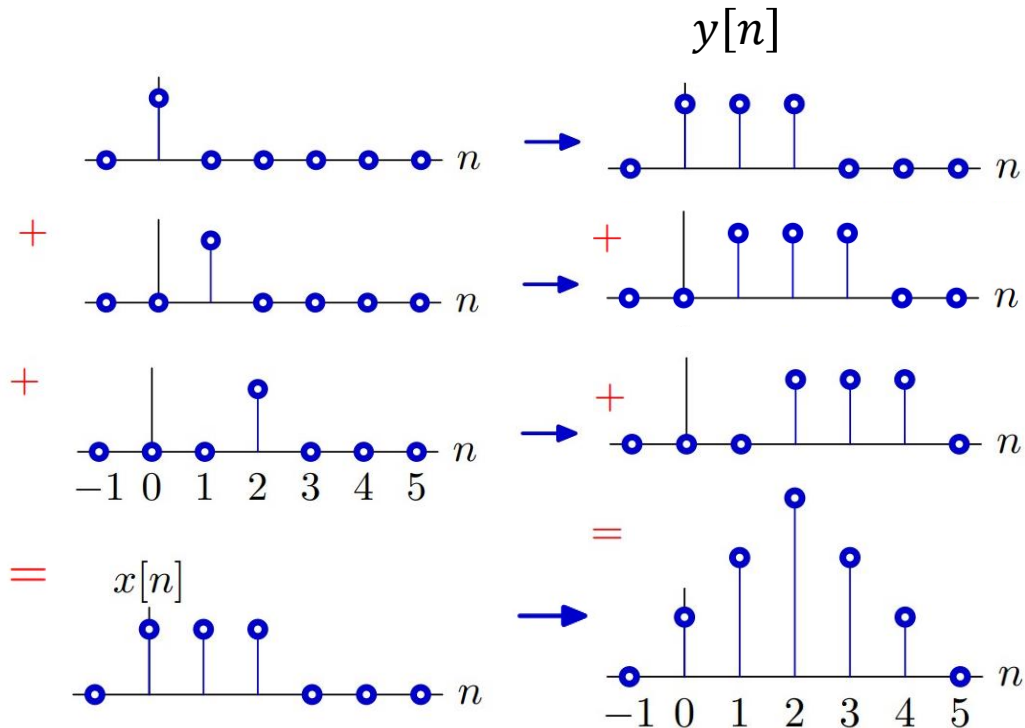
Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$, find the output $y[n]$ to this system with an input $x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$:



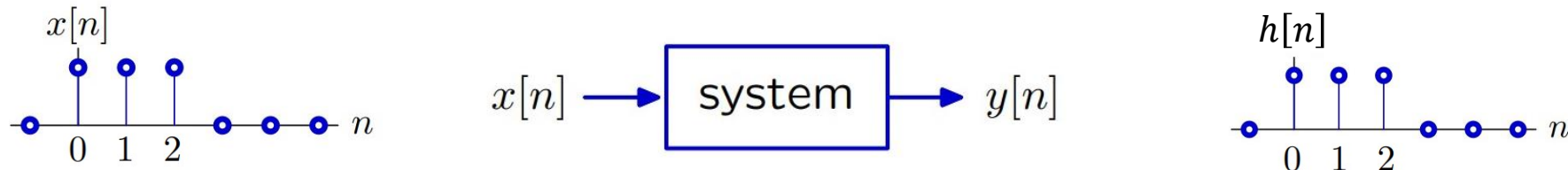
1. Directly compute the superposition result (sum the scaled and shifted responses):

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$



Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$, find the output $y[n]$ to this system with an input $x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$:

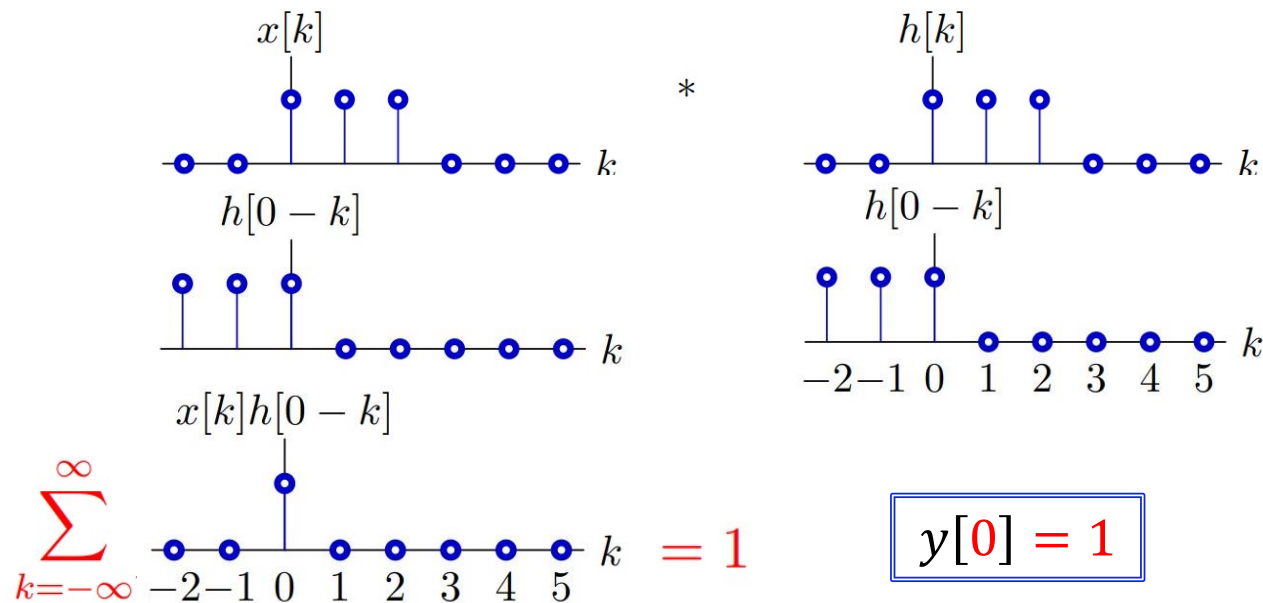


2. Compute the convolution result at each particular n : flip and shift

$$y[0] = \sum_{k=-\infty}^{\infty} x[k] h[0 - k]$$

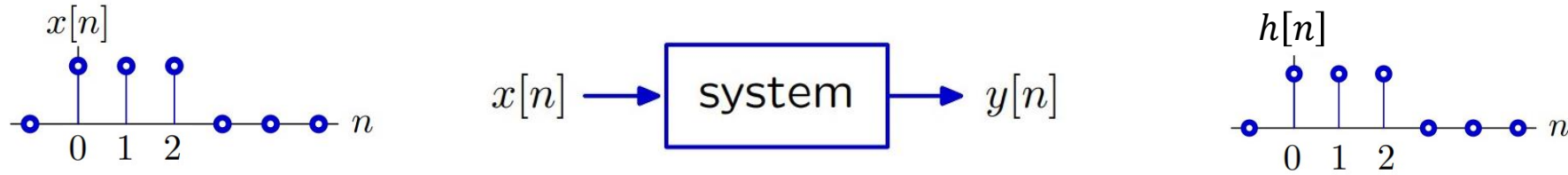
$x * h$:

- Flip $h[-k]$
- Shift $h[0 - k]$
- Multiply $x[k]h[0 - k]$ for every k
- Sum up for all k



Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$, find the output $y[n]$ to this system with an input $x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$:

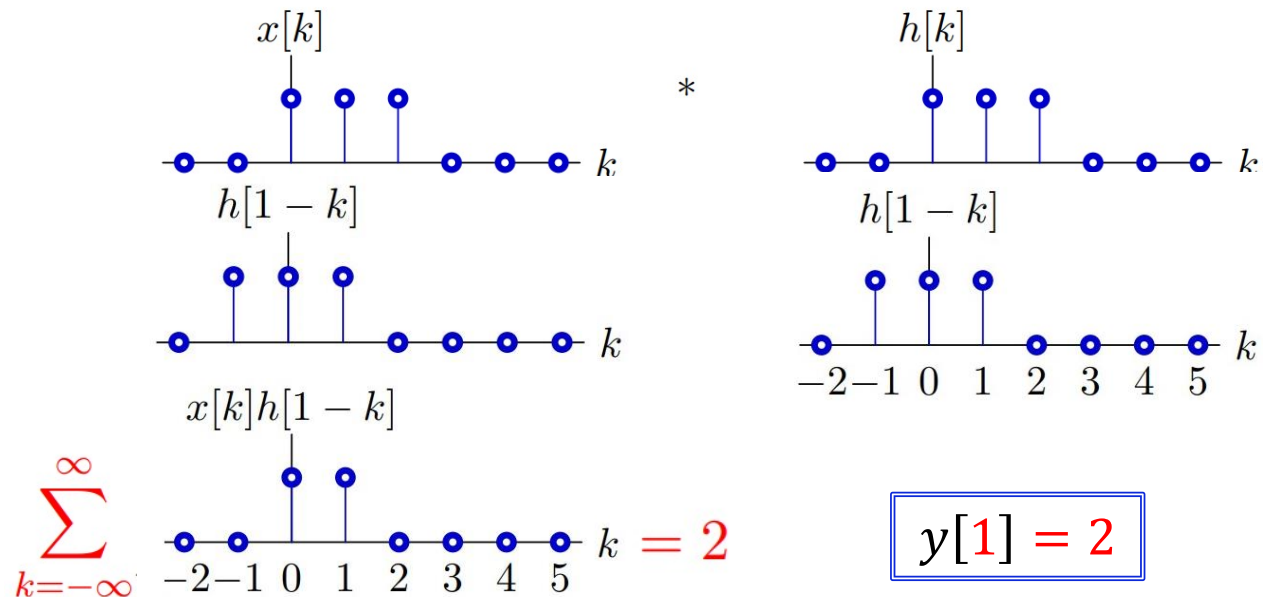


2. Directly compute the convolution at each particular n : flip and shift

$$y[1] = \sum_{k=-\infty}^{\infty} x[k] h[1 - k]$$

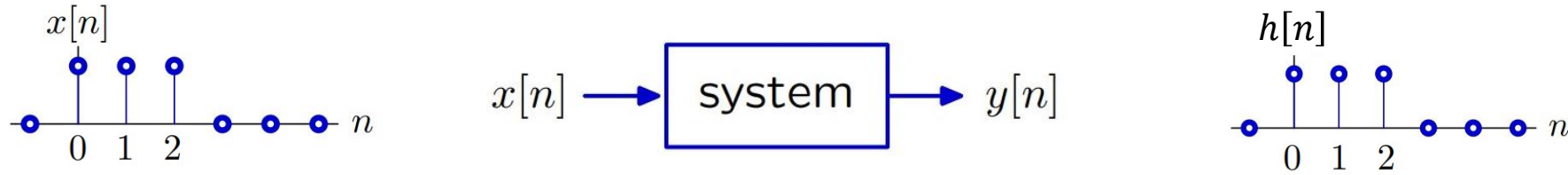
$x * h$:

- Flip $h[-k]$
- Shift $h[1 - k]$
- Multiply $x[k]h[1 - k]$ for every k
- Sum up for all k



Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$, find the output $y[n]$ to this system with an input $x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$:

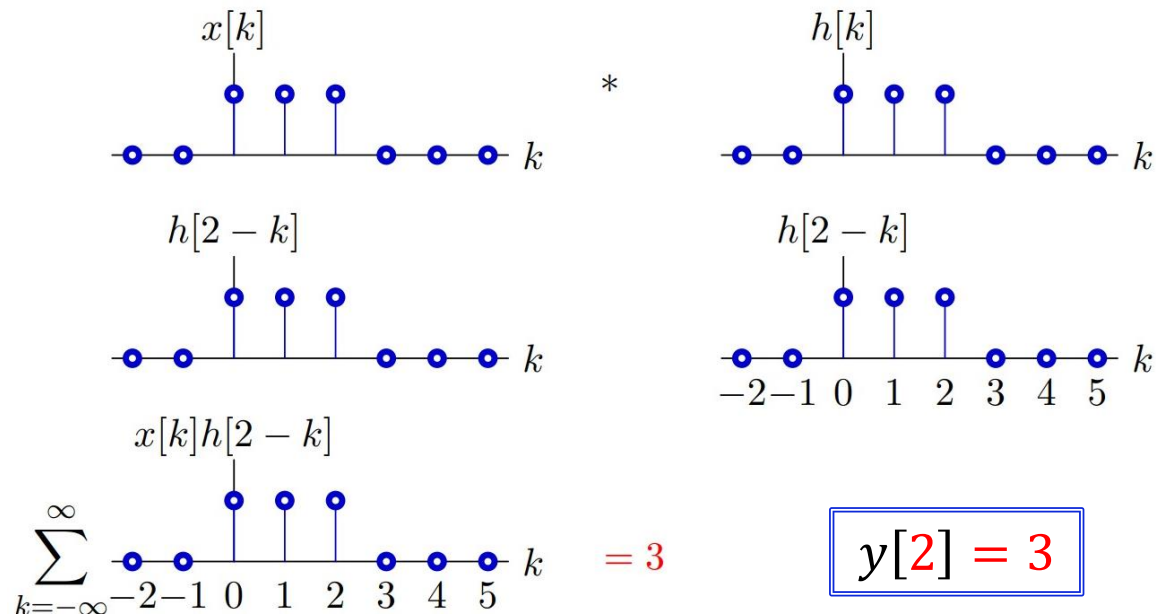


2. Directly compute the convolution at each particular n : flip and shift

$$y[2] = \sum_{k=-\infty}^{\infty} x[k] h[2 - k]$$

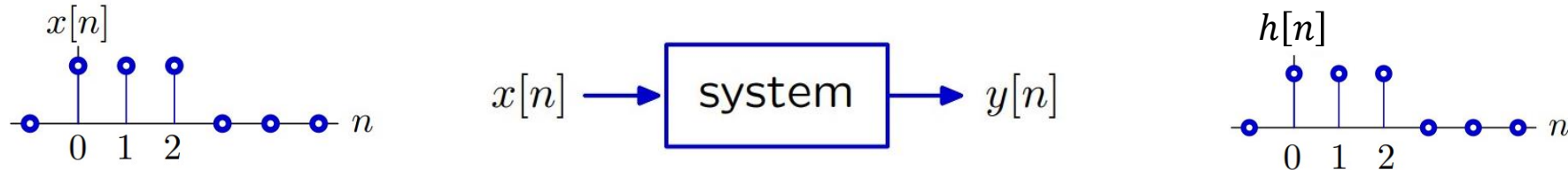
$x * h$:

- Flip $h[-k]$
- Shift $h[2 - k]$
- Multiply $x[k]h[2 - k]$ for every k
- Sum up for all k



Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$, find the output $y[n]$ to this system with an input $x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$:

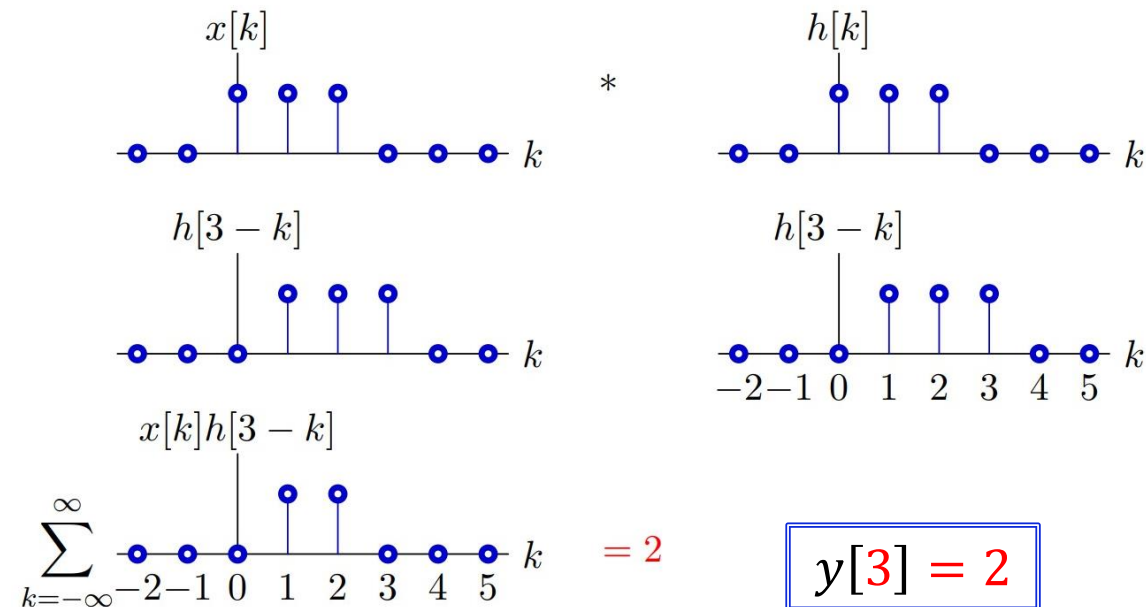


2. Directly compute the convolution at each particular n : flip and shift

$$y[3] = \sum_{k=-\infty}^{\infty} x[k] h[3 - k]$$

$x * h$:

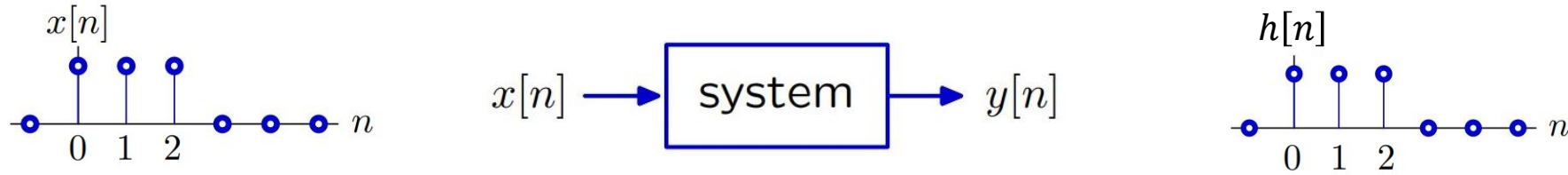
- Flip $h[-k]$
- Shift $h[3 - k]$
- Multiply $x[k]h[3 - k]$ for every k
- Sum up for all k



$$y[3] = 2$$

Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$, find the output $y[n]$ to this system with an input $x[n] = \delta[n] + \delta[n - 1] + \delta[n - 2]$:

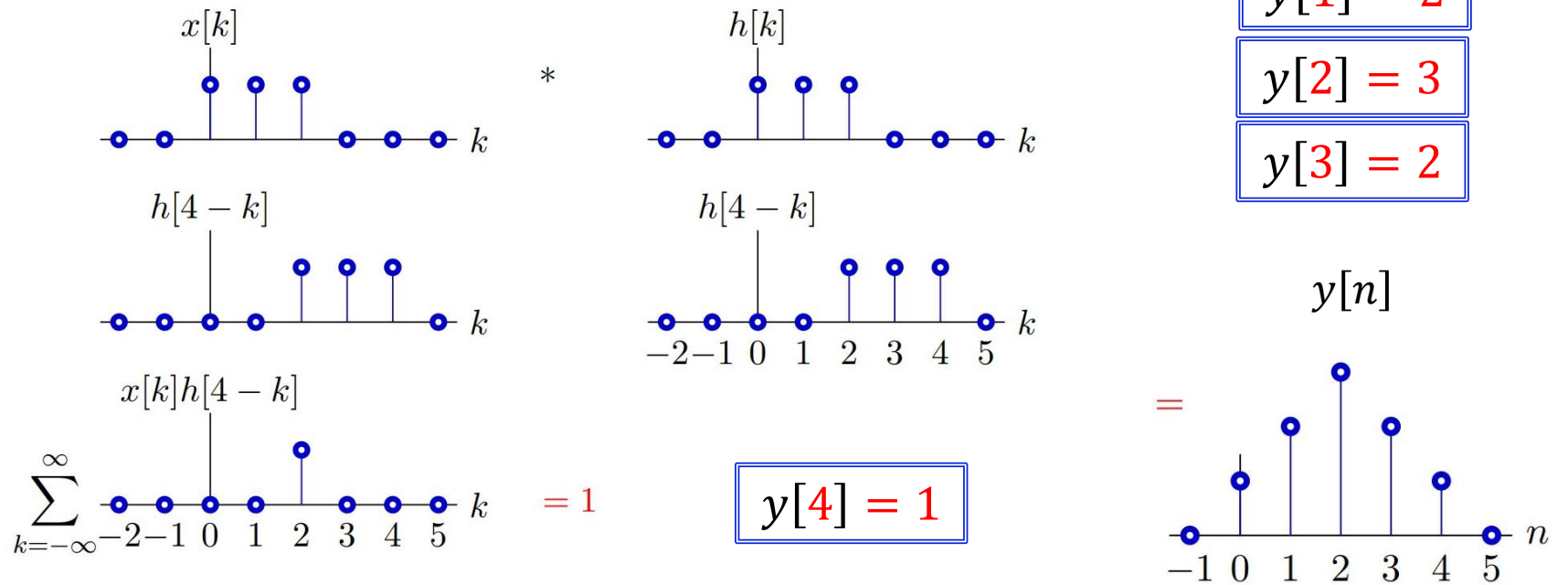


2. Directly compute the convolution at each particular n : flip and shift

$$y[4] = \sum_{k=-\infty}^{\infty} x[k] h[4 - k]$$

$x * h$:

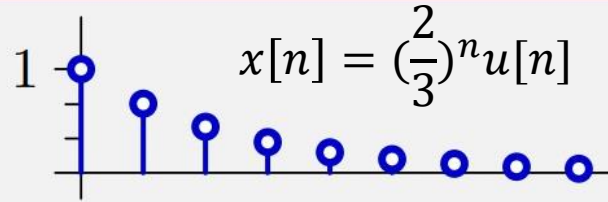
- Flip $h[-k]$
- Shift $h[4 - k]$
- Multiply $x[k]h[4 - k]$ for every k
- Sum up for all k



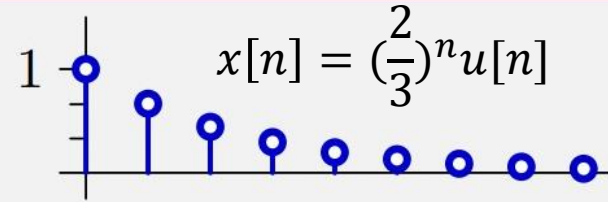
- $y[0] = 1$
- $y[1] = 2$
- $y[2] = 3$
- $y[3] = 2$

$$y[4] = 1$$

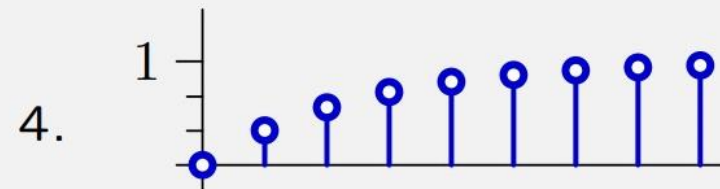
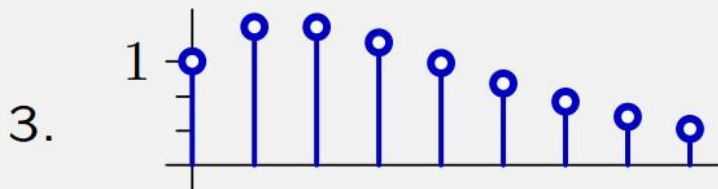
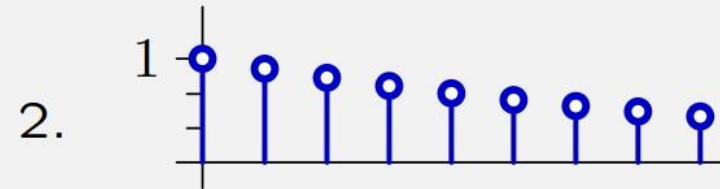
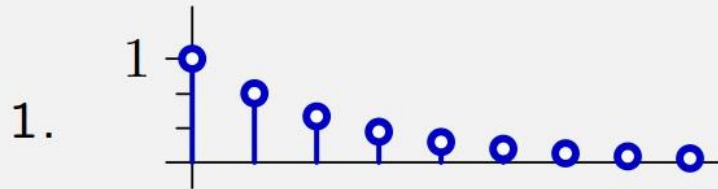
Check yourself



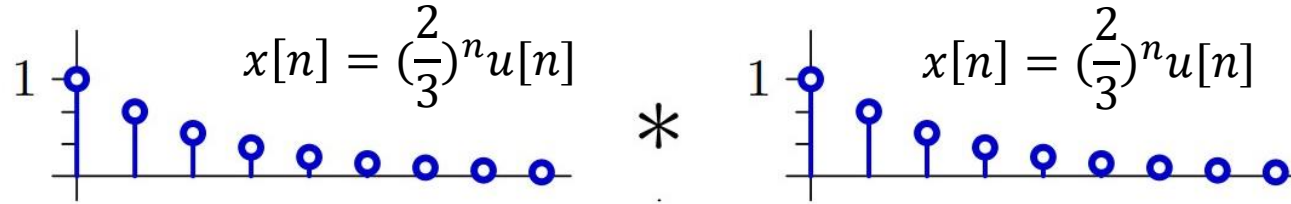
*



Which plot shows the result of the convolution above?



Check yourself



Direct calculation:

$$(x * x)[n] = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3} \right)^k u[k] \right) \left(\left(\frac{2}{3} \right)^{n-k} u[n-k] \right)$$

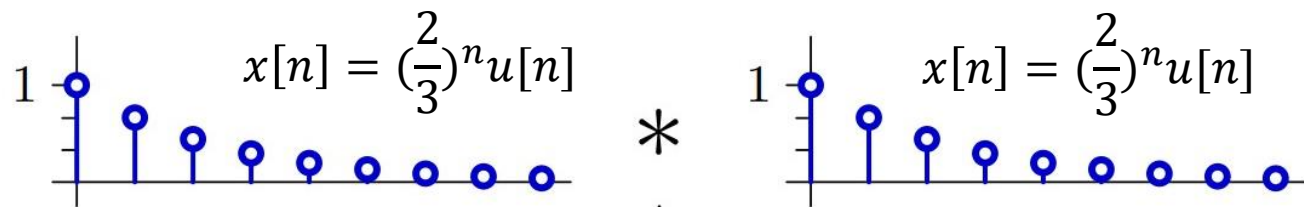
$$u[k] = 0 \text{ for } k < 0, \quad \text{if } n < 0, (x * x)[n] = 0$$
$$u[n-k] = 0 \text{ for } k > n$$

For $n \geq 0$:

$$(x * x)[n] = \sum_{k=0}^n \left(\left(\frac{2}{3} \right)^k \right) \left(\left(\frac{2}{3} \right)^{n-k} \right) = \sum_{k=0}^n \left(\left(\frac{2}{3} \right)^n \right) = \left(\frac{2}{3} \right)^n \sum_{k=0}^n 1 = (n+1) \left(\frac{2}{3} \right)^n$$

$$(x * x)[n] = (n+1) \left(\frac{2}{3} \right)^n \cdot u[n] = 1, \frac{4}{3}, \frac{4}{3}, \frac{32}{27}, \frac{80}{81}, \dots$$

Check yourself



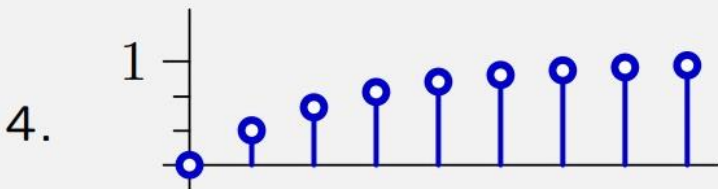
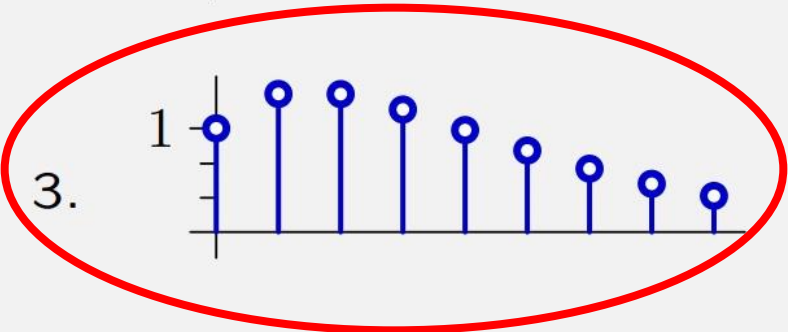
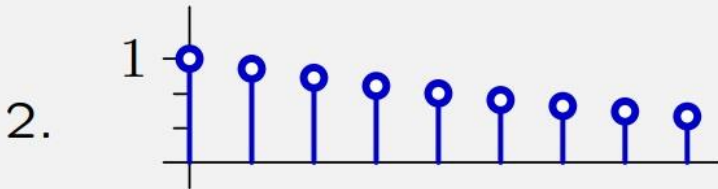
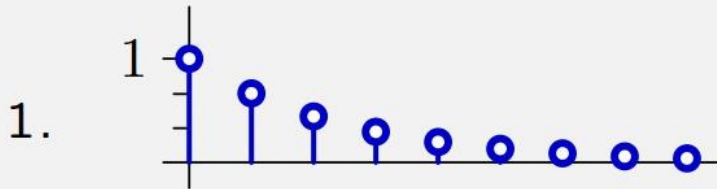
$n:$	0	1	2	3	4	5	
$x[0] \times h[n]:$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	\dots
$x[1] \times h[n-1]:$		$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	\dots
$x[2] \times h[n-2]:$			$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	\dots
$x[3] \times h[n-3]:$				$\frac{8}{27}$	$\frac{16}{81}$	$\frac{32}{243}$	\dots
$x[4] \times h[n-4]:$					$\frac{16}{81}$	$\frac{32}{243}$	\dots
$x[5] \times h[n-5]:$						$\frac{32}{243}$	\dots

$\sum x[m]h[n-m]:$	1	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{32}{27}$	$\frac{80}{81}$	$\frac{192}{243}$	\dots
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Check yourself



Which plot shows the result of the convolution above?

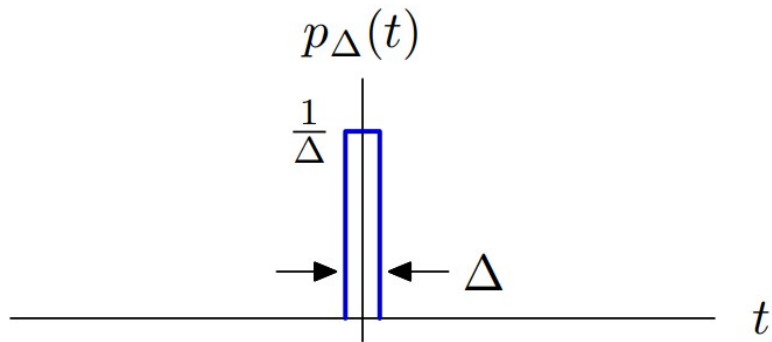


Continuous-Time LTI Systems

Superposition and convolution are of equal importance for CT systems.

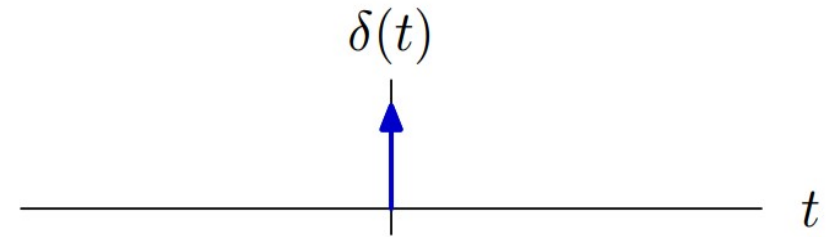
A CT LTI system is completely characterized by its impulse response, much as a DT LTI system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function $\delta(t)$ previously. It's defined in a limit as follows. Let $p_{\Delta}(t)$ represent a pulse of width Δ and height $1/\Delta$ so that its area is 1.



Then

$$\delta(t) = \lim_{\Delta \rightarrow 0} p_{\Delta}(t)$$

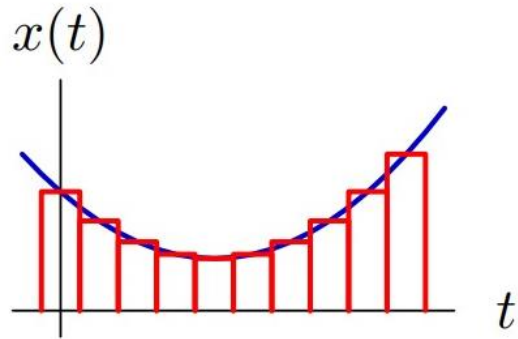


The impulse function can be used to break an arbitrary input $x(t)$ into time-based components, much as $\delta[k]$ is used for discrete-time signals.

Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_\Delta(t)$ (red).



$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) p_\Delta(t - k\Delta) \Delta \qquad x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m]$$

and the limit of $x_\Delta(t)$ as $\Delta \rightarrow 0$ will approximate $x(t)$.

$$\lim_{\Delta \rightarrow 0} x_\Delta(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta) p_\Delta(t - k\Delta) \Delta \rightarrow \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

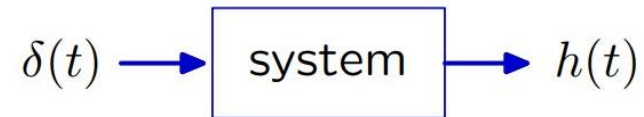
The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

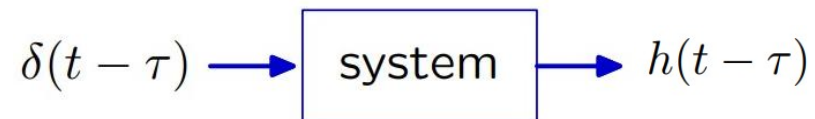
Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is completely specified by the system's impulse response $h(t)$.

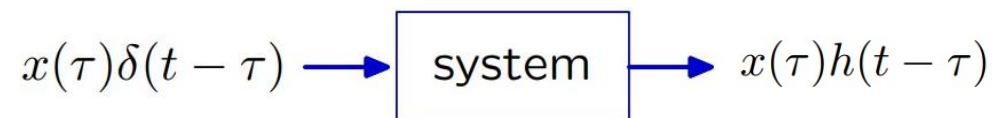
1. One can always find the impulse response of a LTI system.



2. Time invariance implies that shifting the input simply shifts the output.



3. Homogeneity implies that scaling the input simply scales the output.



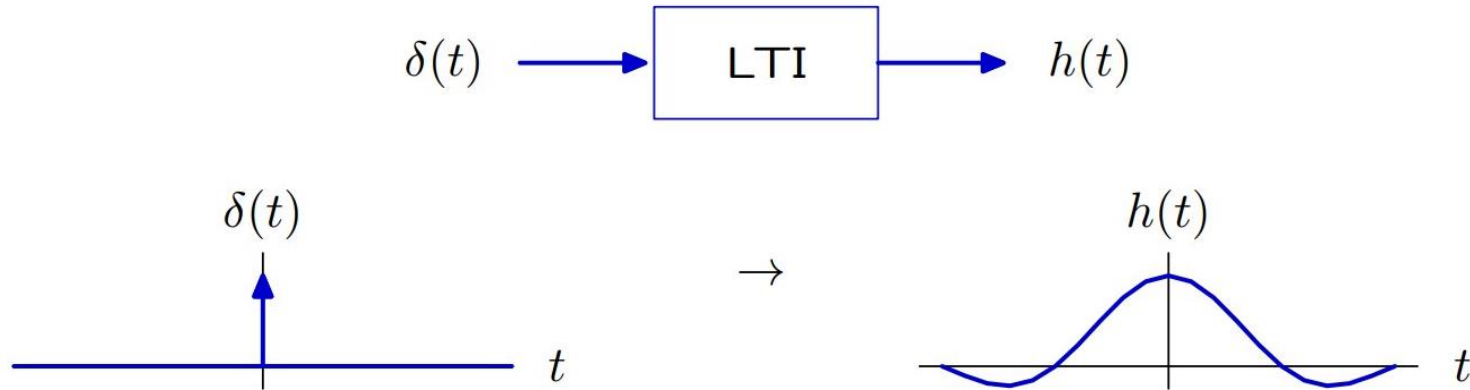
4. Additivity implies that the response to a sum is the sum of responses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \rightarrow \text{system} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \equiv (x * h)(t)$$

The output of an LTI system can always be found by convolving: $(x * h)(t)$.

Impulse Response

The impulse response is a complete description of a CT LTI system.



Given $h(t)$ one can compute the response to **any** arbitrary input signal $x(t)$.

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

Comparison of CT and DT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

$$\text{CT:} \quad y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

$$\text{DT:} \quad y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

Properties of Convolution(I) Commutativity:

$$(g * h)(t) = (h * g)(t)$$

$$(g * h)(t) \equiv \int_{-\infty}^{\infty} g(\tau) \cdot h(t - \tau) d\tau$$

let $\lambda = t - \tau$, then $\tau = t - \lambda$, $d\tau = -d\lambda$
for τ goes from $-\infty$ to ∞ , λ goes from ∞ to $-\infty$

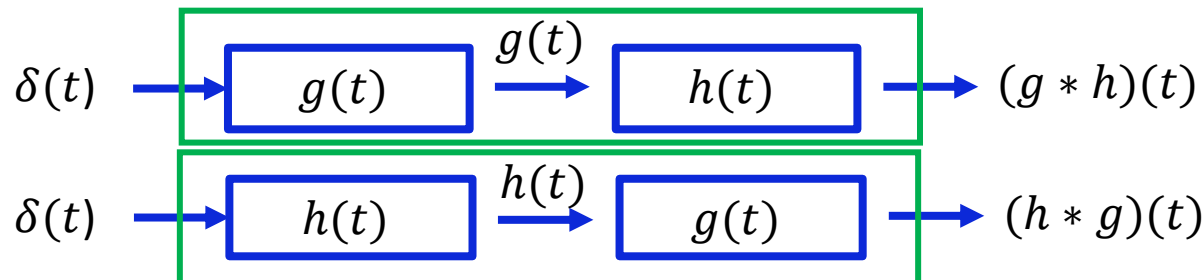
$$\begin{aligned}(g * h)(t) &= \int_{\infty}^{-\infty} g(t - \lambda) \cdot h(\lambda) (-d\lambda) \\ &= \int_{-\infty}^{\infty} g(t - \lambda) \cdot h(\lambda) d\lambda \\ &= (h * g)(t)\end{aligned}$$

$$(g * h)[n] = (h * g)[n]$$

$$(g * h)[n] = \sum_{k=-\infty}^{\infty} g[k]h[n - k]$$

let $m = n - k$, then $k = n - m$,
for k goes from $-\infty$ to ∞ , m goes from ∞ to $-\infty$

$$\begin{aligned}(g * h)[n] &= \sum_{m=\infty}^{-\infty} g[n - m]h[m] \\ &= \sum_{m=-\infty}^{\infty} g[n - m]h[m] = (h * g)[n]\end{aligned}$$



Properties of Convolution(II) Associativity:

$$((x * g) * h)(t) = (x * (g * h))(t)$$

$$((x * g) * h)(t) \equiv \int_{-\infty}^{\infty} h(t - \lambda) d\lambda \int_{-\infty}^{\infty} x(\tau) \cdot g(\lambda - \tau) d\tau$$

let $\mu = \lambda - \tau$, then $\lambda = \mu + \tau$, $d\lambda = d\mu$
for λ goes from $-\infty$ to ∞ , μ goes from $-\infty$ to ∞

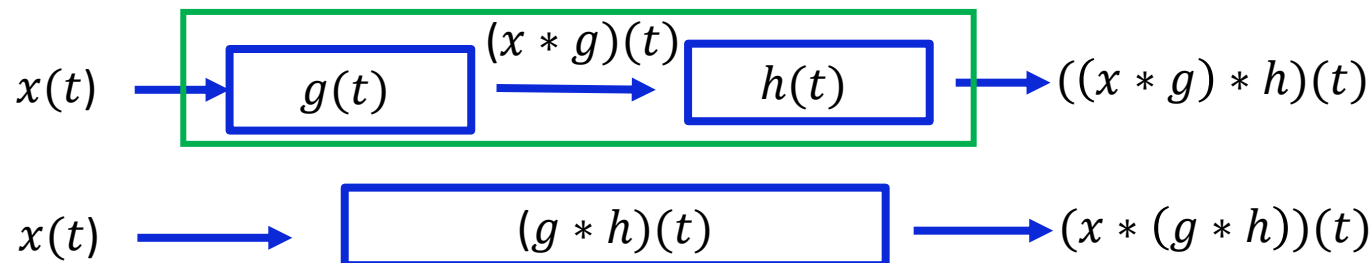
$$\begin{aligned} ((x * g) * h)(t) &= \int_{-\infty}^{\infty} x(\tau) d\tau \int_{-\infty}^{\infty} h(t - \tau - \mu) \cdot g(\mu) d\mu \\ &= \int_{-\infty}^{\infty} x(\tau) (g * h)(t - \tau) d\tau \\ &= (x * (g * h))(t) \end{aligned}$$

$$((x * g) * h)[n] = (x * (g * h))[n]$$

$$((x * g) * h)[n] = \sum_{m=-\infty}^{\infty} h[n - m] \sum_{k=-\infty}^{\infty} x[k] g[m - k]$$

let $l = m - k$, then $m = k + l$,
for m goes from $-\infty$ to ∞ , l goes from $-\infty$ to ∞

$$\begin{aligned} ((x * g) * h)[n] &= \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} h[n - k - l] g[l] \\ &= (x * (g * h))[n] \end{aligned}$$



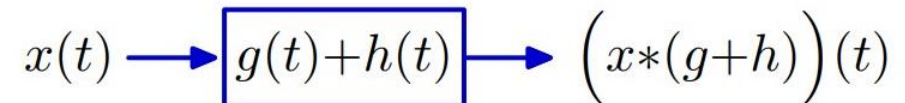
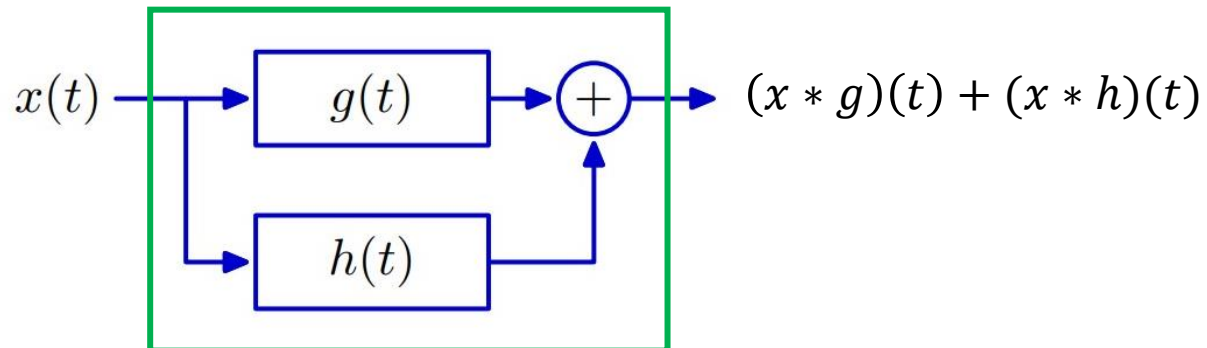
Properties of Convolution(III) Distributivity over addition

$$(x * (g + h))(t) = (x * g)(t) + (x * h)(t)$$

$$\begin{aligned}(x * (g + h))(t) &= \int_{-\infty}^{\infty} x(\tau) \cdot (g(t - \tau) + h(t - \tau)) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)g(t - \tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau \\ &= (x * g)(t) + (x * h)(t)\end{aligned}$$

$$(x * (g + h))[n] = (x * g)[n] + (x * h)[n]$$

$$\begin{aligned}(x * (g + h))[n] &= \sum_{k=-\infty}^{\infty} x[k] \cdot (g[n - k] + h[n - k]) \\ &= \sum_{k=-\infty}^{\infty} x[k] \cdot g[n - k] + \sum_{k=-\infty}^{\infty} x[k] \cdot h[n - k] \\ &= (x * g)[n] + (x * h)[n]\end{aligned}$$

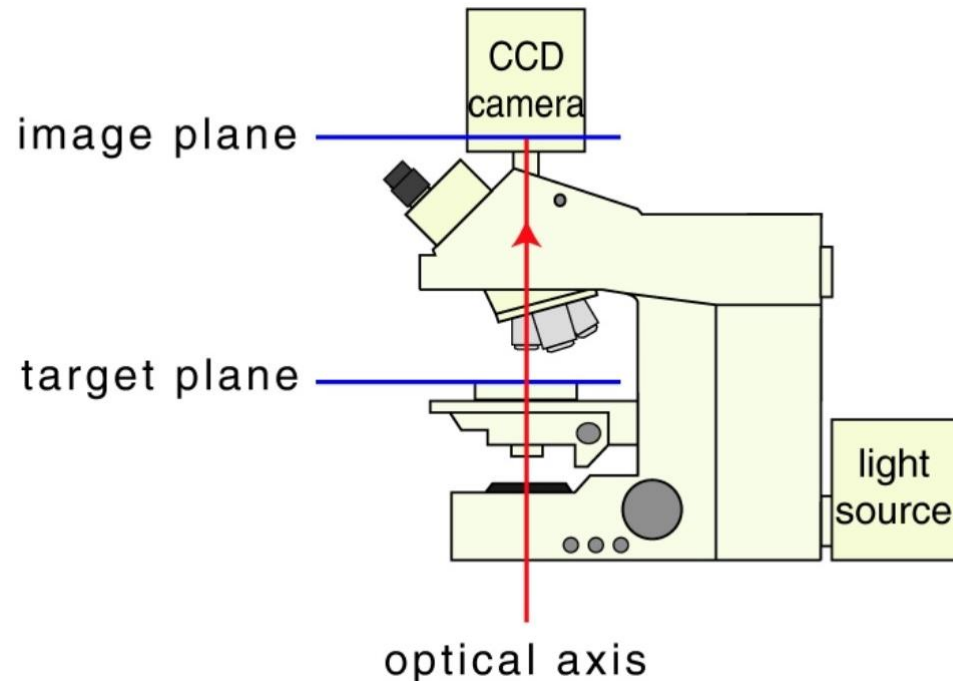


Applications of Convolution

Convolution is an important **conceptual tool**: it provides an important new way to think about the behaviors of systems.

Example systems: microscopes and telescopes.

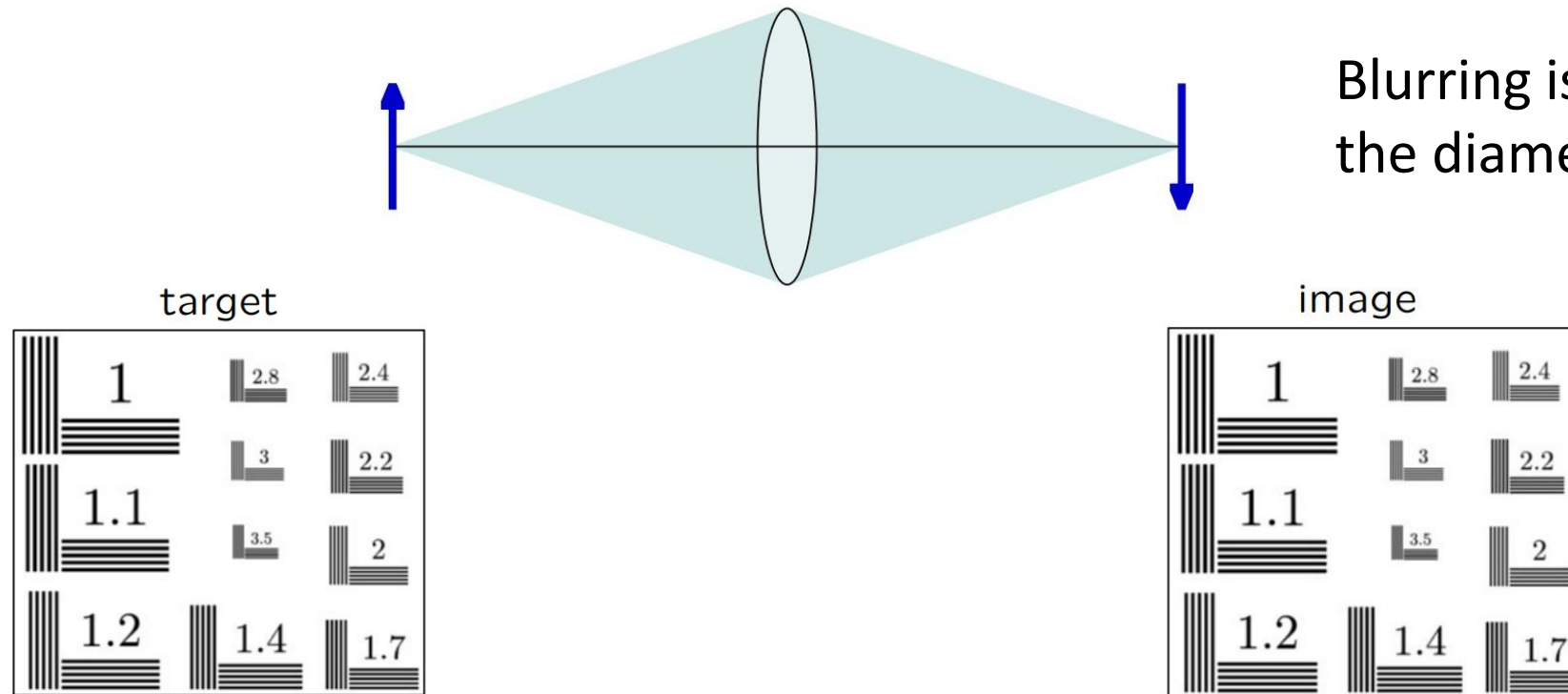
Microscopes:



Images from even the best microscopes are blurred.

Microscope

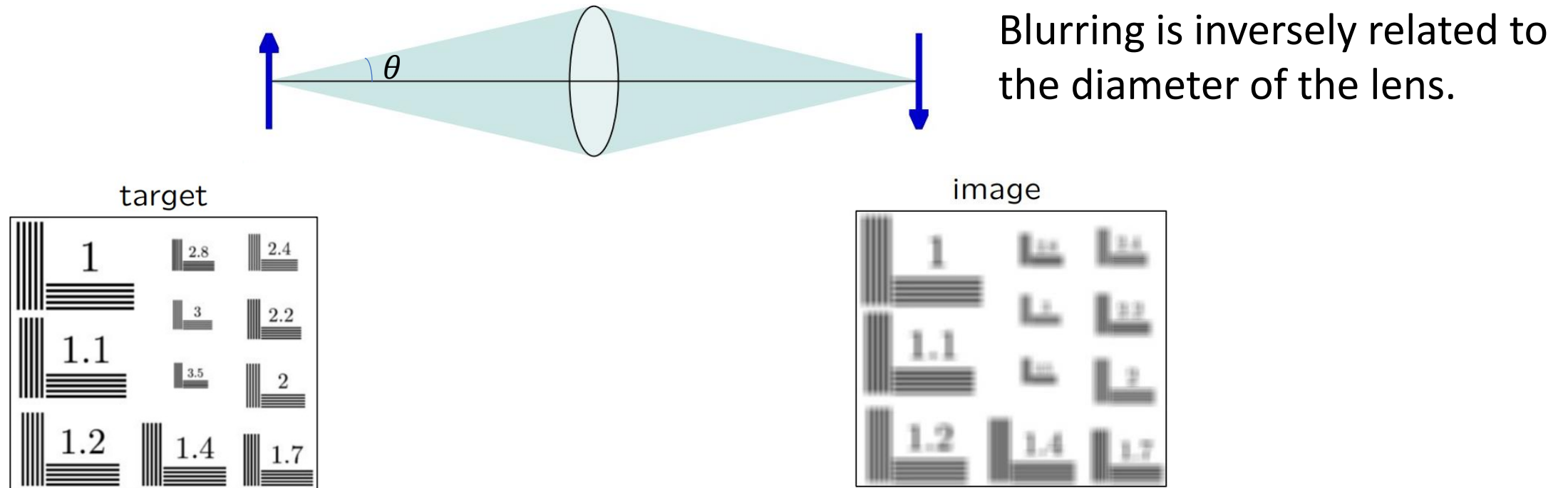
A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



Blurring is inversely related to the diameter of the lens.

Microscope

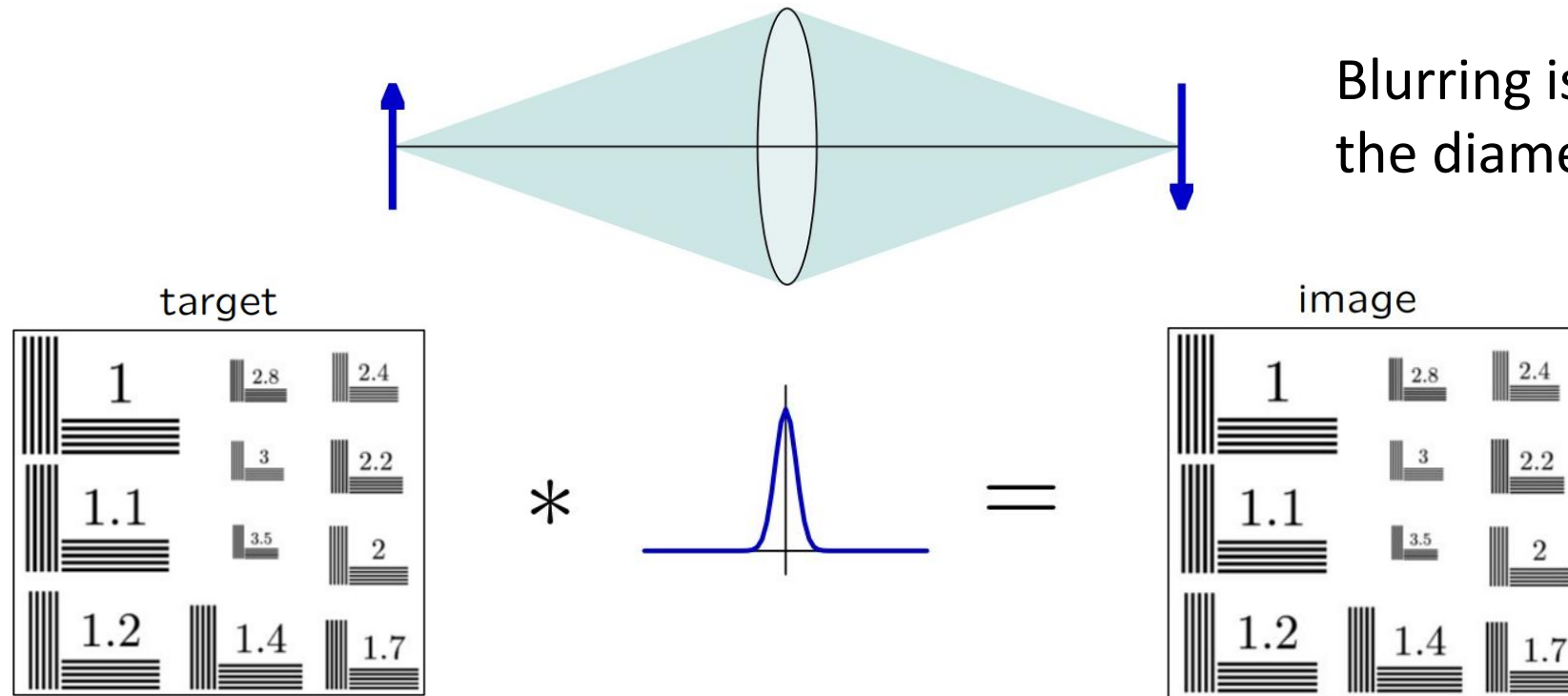
A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



Numerical Aperture: $NA = n \sin \theta$. n is refractive index

Microscope

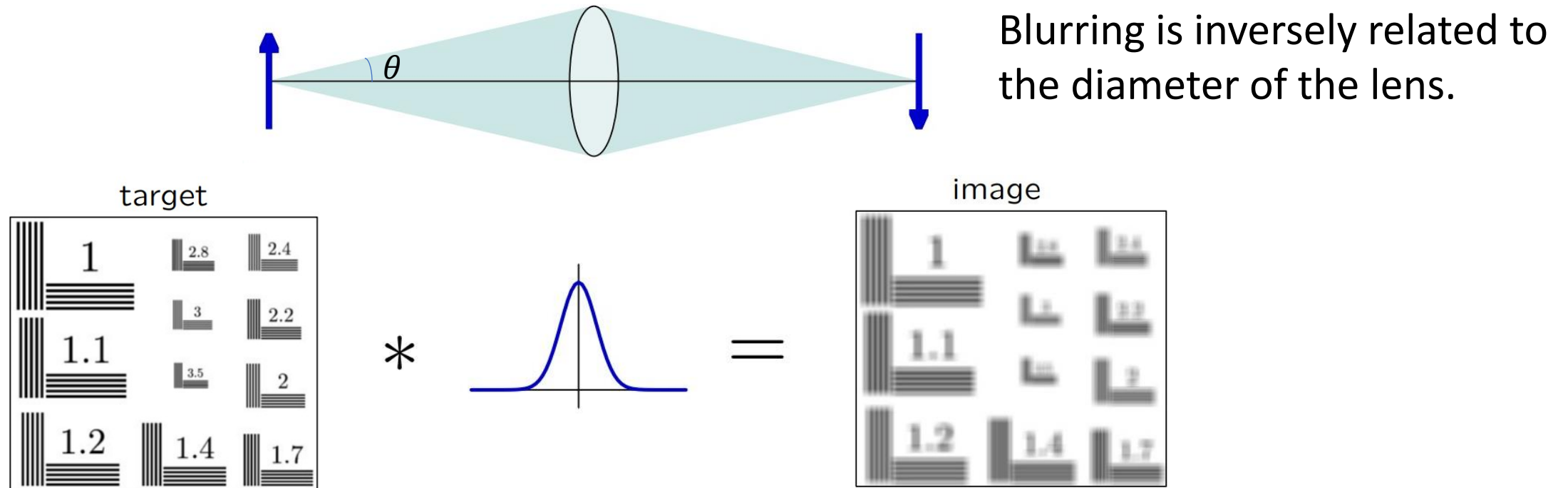
Blurring can be represented by convolving the image with the optical “point-spread-function” (3D impulse response).



Blurring is inversely related to the diameter of the lens.

Microscope

Blurring can be represented by convolving the image with the optical “point-spread-function” (3D impulse response).



Hubble Space Telescope

Hubble Space Telescope (1990-)

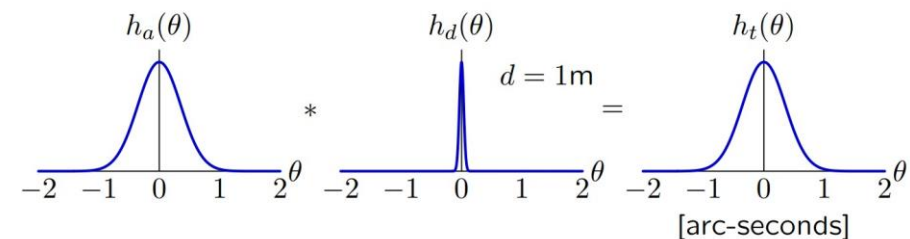
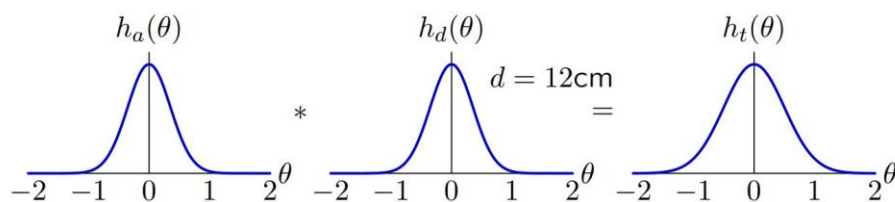
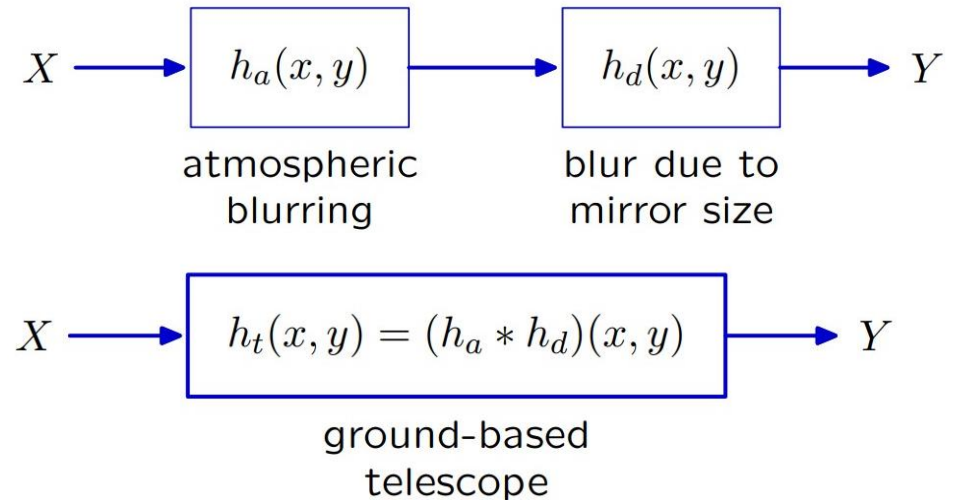


<https://hubblesite.org/>

Telescope blur can be represented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.

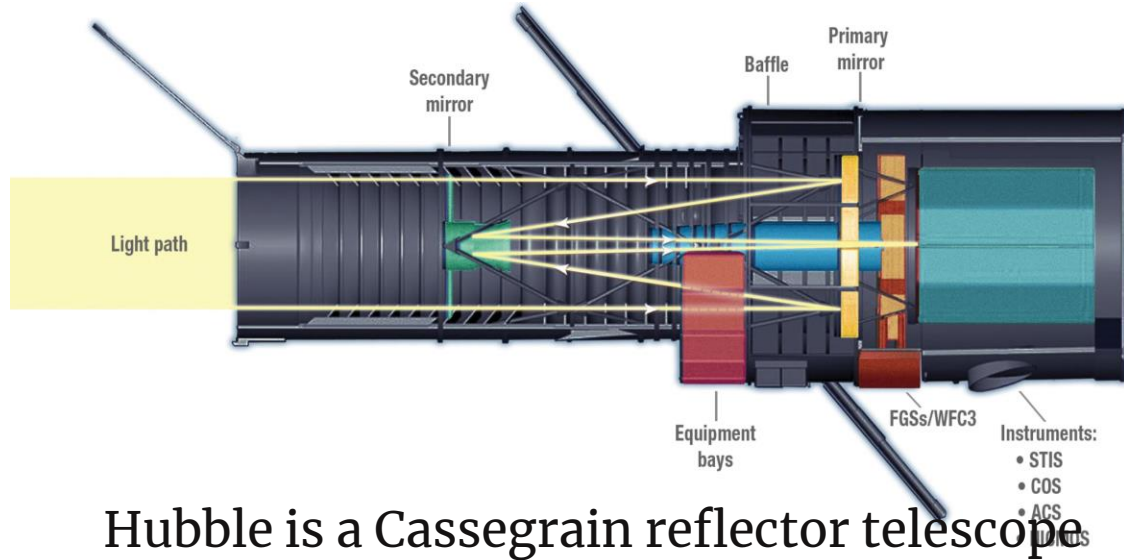
Why build a space telescope?

Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.



Hubble Space Telescope

The main optical components of the Hubble Space Telescope are two mirrors.

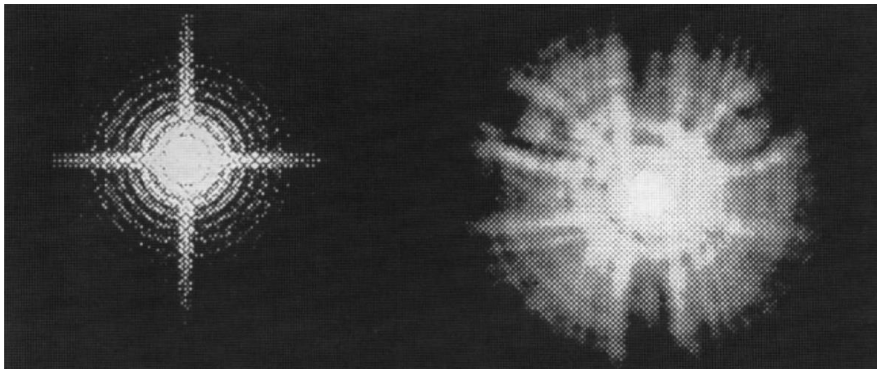


Hubble is a Cassegrain reflector telescope



Hubble's first pictures of distant stars (May 20, 1990) were more blurred than expected.

The diameter of the primary mirror is 2.4 meters. the outer edge of the mirror was ground too flat by a depth of 4 microns!



Scientists immediately began experimenting with algorithms to resolve and make use of Hubble's data and imagery, pushing forward image processing technology.

expected point-spread function

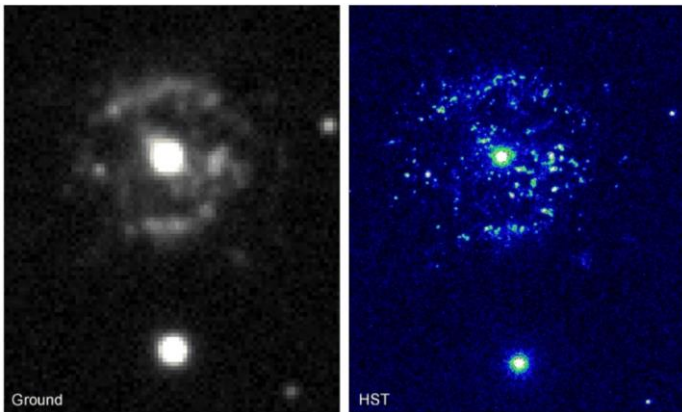
early Hubble image of distant star

Hubble Space Telescope

Corrective Optics Space Telescope Axial Replacement (COSTAR): eyeglasses for Hubble!

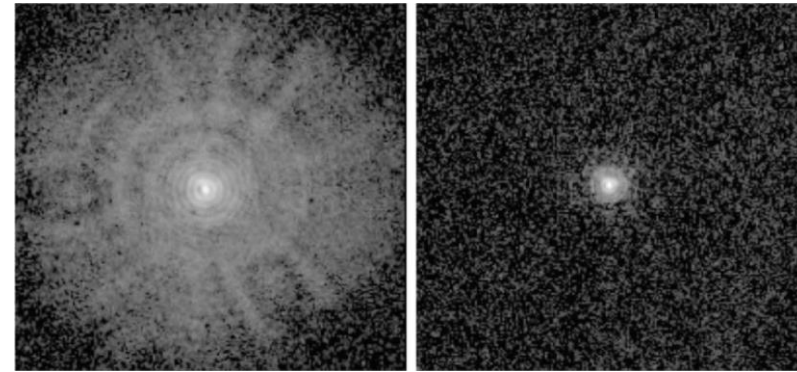


Astronauts completed major upgrades to the Hubble Space Telescope during a ten-day mission December 1993.



Images from ground-based telescope and Hubble.

Hubble images before and after COSTAR.



before

after



<https://hubblesite.org/>

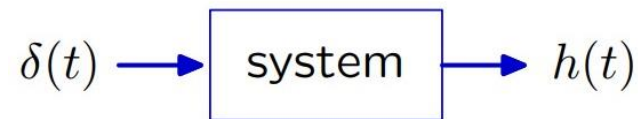
Summary

Convolution

$$\text{DT:} \quad y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

$$\text{CT:} \quad y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

The unit-sample/impulse response is a complete description of an LTI system.



One can find the response to an arbitrary input signal by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.